Design and Verify CPS with a Constraint Satisfaction Problem (CSP) Approach

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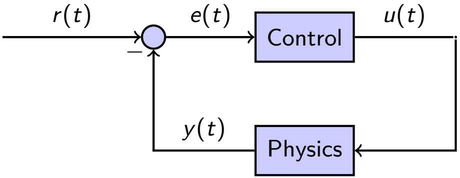
A small cyber-physical system: closed-loop control



- **Physics** is usually defined by non-linear differential equations (with parameters)

$$\dot{\mathbf{x}} = f(\mathbf{x}(t), u(t), \mathbf{p}) \quad , \qquad \mathbf{y}(t) = g(\mathbf{x}(t))$$

- **Control** may be a continuous-time PI algorithm

$$e(t) = r(t) - y(t) \quad , \qquad u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

What is designing a controller?

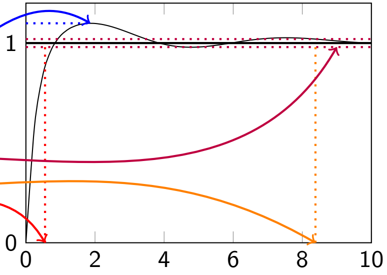
Find values for K_p and K_i such that a **given specification** is satisfied.

Specification of PID Controllers

PID controller: requirements based on closed-loop response

We observe the output of the plant

- Overshoot: Less than 10%
- Steady-state error: Less than 2%
- Settling time: Less than 10s
- Rise time: Less than 2s



Note: such properties come from the **asymptotic behavior** of the closed-loop system.

Classical method to study/verify closed-loop systems

Numerical simulations but

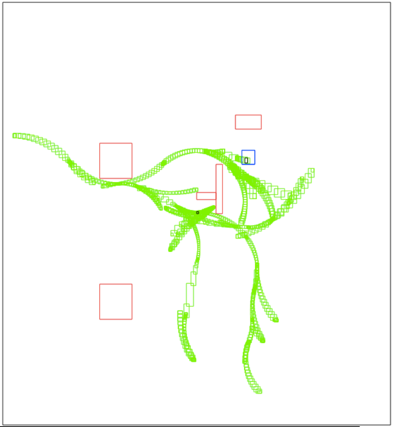
- do not take into account that models are only an approximation;
- produce approximate results.

and not adapted to deal with uncertainties

Synthesis and Verification methods for/of cyber-physical systems

Some requirements

- Shall deal with **discrete-time**, **continuous-time** parts and **their interactions**
- Shall **take into account uncertainties**: model, data, resolution methods
- Shall consider **temporal properties**



Example of properties (coming from box-RRT[1])

- system stays in safe zone ($\forall t$) or finishes in goal zone ($\exists t$)
- system avoids obstacle ($\exists t$)

for **different quantification's** of initial state-space ($\forall \mathbf{x}$ or $\exists \mathbf{x}$), parameters, etc.

[1] Pepy *et al.* Reliable robust path planning, Journal of AMCS, 2009

Set-based simulation

Definition

numerical simulation methods implemented with interval analysis methods

Goals

takes into account various uncertainties (bounded) or approximations to produce rigorous results

Example

A simple nonlinear dynamics of a car

$$\dot{v} = \frac{-50.0v - 0.4v^2}{m} \qquad \text{with} \quad m \in [990, 1010] \quad \text{and} \quad v(0) \in [10, 11]$$

One Implementation **DynIBEX**: a combination of **CSP solver** (IBEX[1]) with **validated numerical integration methods** based on **Runge-Kutta**

[1] Gilles Chabert (EMN) et al. `http://www.ibex-lib.org`

Constraint Satisfaction Problems

Basics of interval analysis

- **Interval arithmetic** (defined also for: sin, cos, etc.):

$$
\begin{aligned}
[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \\
[\underline{x}, \overline{x}] * [\underline{y}, \overline{y}] &= [\min\{\underline{x} * \underline{y}, \underline{x} * \overline{y}, \overline{x} * \underline{y}, \overline{x} * \overline{y}\}, \\
&\qquad \max\{\underline{x} * \underline{y}, \underline{x} * \overline{y}, \overline{x} * \underline{y}, \overline{x} * \overline{y}\}]
\end{aligned}
$$

- Let an **inclusion function** $[f] : \mathbb{IR} \to \mathbb{IR}$ for $f : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\{f(a) \mid \exists a \in [I]\} \subseteq [f]([I])$$

with $a \in \mathbb{R}$ and $I \in \mathbb{IR}$.

Example of inclusion function: Natural inclusion

$[x] = [1, 2], \quad [y] = [-1, 3]$, and $\quad f(x, y) = xy + x$

$$[f]\big([x], [y]\big) := [x] * [y] + [x]$$

$$= [1, 2] * [-1, 3] + [1, 2] = [-2, 6] + [1, 2] = [-1, 8]$$

Numerical Constraint Satisfaction Problems

NCSP

A NCSP $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ is defined as follows:

- $\mathcal{V} := \{v_1, \dots, v_n\}$ is a finite set of variables which can also be represented by the vector $\mathbf{v}$;
- $\mathcal{D} := \{[v_1], \dots, [v_n]\}$ is a set of intervals such that $[v_i]$ contains all possible values of v_i. It can be represented by a box $[\mathbf{v}]$ gathering all $[v_i]$;
- $\mathcal{C} := \{c_1, \dots, c_m\}$ is a set of constraints of the form $c_i(\mathbf{v}) \equiv f_i(\mathbf{v}) = 0$ or $c_i(\mathbf{v}) \equiv g_i(\mathbf{v}) \leqslant 0$, with $f_i : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ for $1 \leqslant i \leqslant m$.
  Note: Constraints $\mathcal{C}$ are interpreted as a conjunction of equalities and inequalities.

Remark: The solution of a NCSP is a valuation of $\mathbf{v}$ ranging in $[\mathbf{v}]$ and satisfying the constraints $\mathcal{C}$.

Example

- $\mathcal{V} = \{x\}$
- $\mathcal{D}_x = \big\{[1, 10]\big\}$ $\quad\Longrightarrow\quad x \in [1, 1.09861]$
- $\mathcal{C} = \big\{x \exp(x) \leqslant 3\big\}$

Remark: if $[\mathbf{v}] = \emptyset$ then the problem is not satistafiable

Interval constraints and contractor

Interval constraint

Given a inclusion function $[f]$, a box $[\mathbf{z}]$, we look for a box $[\mathbf{x}]$, s.t.

$$[f]([\mathbf{x}]) \subseteq [\mathbf{z}]$$

Remark: if $[\mathbf{x}] = \emptyset$ then the problem is unsafisiable

A simple resolution algorithm

```
put [x] in a list X
while X is not empty
   take [x] in X
   if [f]([x]) is included in [z] then keep [x] in S as a solution
   else if width([x]) < tol then split [x], put [x1] and [x2] in X
```

Contractor

A contractor $\mathcal{C}_{[f],[\mathbf{z}]}$ associated to constraint $[f]([\mathbf{x}]) \subseteq [\mathbf{z}]$ such that

- Reduction:

$$\mathcal{C}_{[f],[\mathbf{z}]}\left([\mathbf{x}]\right) \subseteq [\mathbf{x}]$$

- Soundness:

$$[f]\left([\mathbf{x}]\right) \cap [\mathbf{z}] = [f]\left(\mathcal{C}_{[f],[\mathbf{z}]}\left([\mathbf{x}]\right)\right) \cap [\mathbf{z}]$$

Note: several contractor algorithms exist, *e.g.*, FwdBwd, 3BCID, etc.

Contractor: example FwdBwd

Example

- $\mathcal{V} = \{x, y, z\}$
- $\mathcal{D} = \big\{[1,2], [-1,3], [0,1]\big\}$
- $\mathcal{C} = \big\{x + y = z\big\}$

Forward evaluation

- $[z] = [z] \cap ([x] + [y])$
  as $[x] + [y] = [1,2] + [-1,3] = [0,5] \quad \Rightarrow [z] = [0,1] \cap [0,5]$ No improvement yet

Backward evaluation

- $[y] = [y] \cap ([z] - [x]) = [-1,3] \cap [-2,0] = [-1,0]$ Refinement of $[y]$
- $[x] = [x] \cap ([z] - [y]) = [1,2] \cap [0,2] = [1,2]$ No refinement of $[x]$

Remark: this process can be iterated until a fixpoint is reached

Remark: the order of constraints is important for a fast convergence

IBEX in one slide

- Easy definition of functions
- Numerical constraints
- Pruning methods
- Interval evaluation of functions

```
#include "ibex.h"

using namespace std;
using namespace ibex;

int main() {

  Variable x;
  Function f (x, x*exp(x));

  NumConstraint c1(x, f(x) <= 3.0);

  CtcFwdBwd contractor(c1);

  IntervalVector box(1);
  box[0]=Interval(1,10);

  cout << "f" << box << " = " << f.eval(box) << endl;
  contractor.contract(box);
  cout << "after contraction box = " << box << endl;
}
```

IBEX is also a parametric solver of constraints, an optimizer, etc.

Validated numerical integration

Initial Value Problem of Ordinary Differential Equations

Consider an IVP for ODE, over the time interval $[0, T]$

$$\dot{\mathbf{y}} = f(\mathbf{y}) \quad \text{with} \quad \mathbf{y}(0) = \mathbf{y}_0$$

IVP has a unique solution $\mathbf{y}(t; \mathbf{y}_0)$ if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in $\mathbf{y}$
but for our purpose we suppose f smooth enough, *i.e.*, of class C^k

Goal of numerical integration

- Compute a sequence of time instants: $t_0 = 0 < t_1 < \cdots < t_n = T$
- Compute a sequence of values: $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$ such that

$$\forall i \in [0, n], \quad \mathbf{y}_i \approx \mathbf{y}(t_i; \mathbf{y}_0) \ .$$

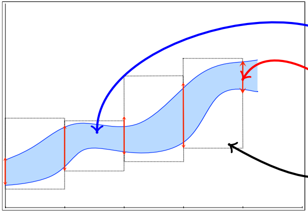
Validated solution of IVP for ODE

Goal of validated numerical integration

- Compute a sequence of time instants: $t_0 = 0 < t_1 < \cdots < t_n = T$
- Compute a sequence of values: $[\mathbf{y}_0], [\mathbf{y}_1], \ldots, [\mathbf{y}_n]$ such that

$$\forall i \in [0, n], \quad [\mathbf{y}_i] \ni \mathbf{y}(t_i; \mathbf{y}_0) \ .$$

A two-step approach



- Exact solution of $\dot{\mathbf{y}} = f(\mathbf{y}(t))$ with $\mathbf{y}(0) \in \mathcal{Y}_0$
- Safe approximation at discrete time instants
- Safe approximation between time instants

State of the Art on Validated Numerical Integration

Taylor methods

They have been developed since 60's (Moore, Lohner, Makino and Berz, Corliss and Rhim, Neher *et al.*, Jackson and Nedialkov, etc.)

- prove the existence and uniqueness: **high order interval Picard-Lindelöf**
- works very well on various kinds of problems:
  - **non stiff** and **moderately stiff** linear and non-linear systems,
  - with **thin uncertainties on initial conditions**
  - with (a writing process) **thin uncertainties on parameters**
- **very efficient** with automatic differentiation techniques
- **wrapping effect fighting**: interval centered form and QR decomposition
- **many software**: AWA, COSY infinity, VNODE-LP, CAPD, etc.

Some extensions

- Taylor polynomial with Hermite-Obreskov (Jackson and Nedialkov)
- Taylor polynomial in Chebyshev basis (T. Dzetkulic)
- etc.

New validated methods, why?

Numerical solutions of IVP for ODEs are produced by

- Adams-Bashworth/Moulton methods
- BDF methods
- Runge-Kutta methods
- etc.

each of these methods is adapted to a particular class of ODEs

Runge-Kutta methods

- have **strong stability** properties for various kinds of problems (A-stable, L-stable, algebraic stability, etc.)
- may **preserve quadratic algebraic invariant** (symplectic methods)
- can produce **continuous output** (polynomial approximation of $\mathbf{y}(t; \mathbf{y}_0)$)

Can we benefit these properties in validated computations?

History on Interval Runge-Kutta methods

- Andrzej Marciniak *et al.* work on this topic since 1999

  > *"The form of $\psi(t, y(t))$ is very complicated and cannot be written in a general form for an arbitrary p"*

  The implementation OOIRK is not freely available and limited number of methods.

- Hartmann and Petras, ICIAM 1999
  No more information than an abstract of 5 lines.

- Bouissou and Martel, SCAN 2006 (only RK4 method)
  Implementation GRKLib is not available

- Bouissou, Chapoutot and Djoudi, NFM 2013 (any explicit RK)
  Implementation is not available

- Alexandre dit Sandretto and Chapoutot, 2016 (any explicit and implicit RK)
  implementation DynIBEX is open-source, combine with IBEX

Examples of Runge-Kutta methods

Single-step fixed step-size explicit Runge-Kutta method

e.g. explicit Trapzoidal method (or Heun's method)[2] is defined by:

$$\mathbf{k}_1 = f(t_\ell, \mathbf{y}_\ell) \ , \qquad \mathbf{k}_2 = f(t_\ell + 1h, \mathbf{y}_\ell + h1\mathbf{k}_1)$$

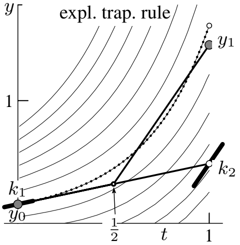
$$\mathbf{y}_{i+1} = \mathbf{y}_\ell + h\left(\frac{1}{2}\mathbf{k}_1 + \frac{1}{2}\mathbf{k}_2\right)$$

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Intuition

- $\dot{y} = t^2 + y^2$
- $y_0 = 0.46$
- $h = 1.0$

dotted line is the exact solution.

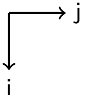


[2]example coming from "Geometric Numerical Integration", Hairer, Lubich and Wanner.

Runge-Kutta methods

s-stage Runge-Kutta methods are described by a Butcher tableau

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
\vdots & \vdots & \vdots & & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s \\
 & b'_1 & b'_2 & \cdots & b'_s \quad \text{(optional)}
\end{array}
$$



which induces the following algorithm

$$
\mathbf{k}_i = f\left(t_\ell + c_i h_\ell, \quad \mathbf{y}_\ell + h_\ell \sum_{j=1}^{s} a_{ij}\mathbf{k}_j\right), \qquad \mathbf{y}_{\ell+1} = \mathbf{y}_\ell + h_\ell \sum_{i=1}^{s} b_i \mathbf{k}_i
$$

- **Explicit** method (ERK) if $a_{ij} = 0$ is $i \leqslant j$
- **Diagonal Implicit** method (DIRK) if $a_{ij} = 0$ is $i \leqslant j$ and at least one $a_{ii} \neq 0$
- **Implicit** method (IRK) otherwise

Validated Runge-Kutta methods

A validated algorithm

$$[\mathbf{y}_{\ell+1}] = [\mathrm{RK}]\,(h, [\mathbf{y}_\ell]) + \mathrm{LTE} \ .$$

Challenges

1. Computing with sets of values (intervals) taking into account dependency problem and wrapping effect;
2. Bounding the approximation error of Runge-Kutta formula.

Our approach

- **Problem 1** is solved using **affine arithmetic** (an extension of interval) replacing centered form and QR decomposition
- **Problem 2** is solved by bounding the **Local Truncation Error** (LTE) of Runge-Kutta methods based on **B-series** and **Order condition**.

Order condition states that a method of Runge-Kutta family is of order p **iff**

- the Taylor expansion of the exact solution
- and the Taylor expansion of the numerical methods

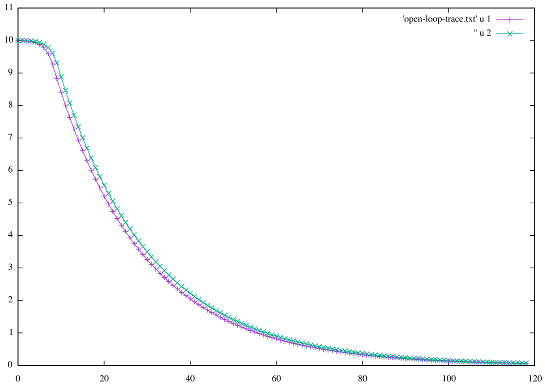
have the same $p+1$ first coefficients.

Simulation of an open loop system

A simple dynamics of a car

$$\dot{y} = \frac{-50.0y - 0.4y^2}{m} \qquad \text{with} \quad m \in [990, 1010]$$

Simulation for 100 seconds with $y(0) = 10$



The last step is $y(100) = [0.0591842, 0.0656237]$

Simulation of an open loop system

- ODE definition
- IVP definition
- Parametric simulation engine

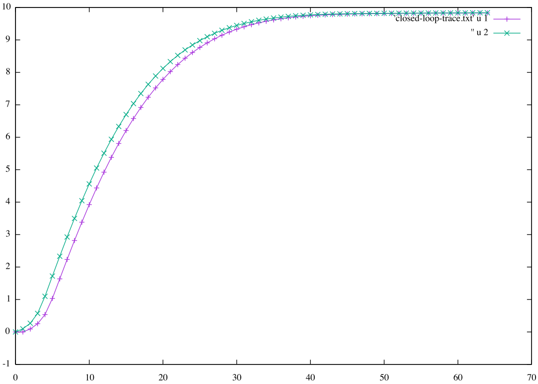
```
int main(){

  const int n = 1;
  Variable y(n);

  IntervalVector state(n);
  state[0] = 10.0;

  // Dynamique d'une voiture avec incertitude sur sa
masse
  Function ydot(y, ( -50.0 * y[0] - 0.4 * y[0] * y[0])
                 / Interval (990, 1010));
  ivp_ode vdp = ivp_ode(ydot, 0.0, state);

  // Integration numerique ensembliste
  simulation simu = simulation(&vdp, 100, RK4, 1e-5);
  simu.run_simulation();

  //For an export in order to plot
  simu.export1d_yn("export-open-loop.txt", 0);

  return 0;
}
```

Simulation of a closed-loop system

A simple dynamics of a car with a PI controller

$$\begin{pmatrix} \dot{y} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{k_p(10.0-y)+k_i w-50.0y-0.4y^2}{m} \\ 10.0-y \end{pmatrix} \quad \text{with} \quad m \in [990, 1010], k_p = 1440, k_i = 35$$

Simulation for 10 seconds with $y(0) = w(0) = 0$



The last step is $y(10) = [9.83413, 9.83715]$

Simulation of a closed-loop system

```
#include "ibex.h"

using namespace ibex;

int main(){

  const int n = 2;
  Variable y(n);

  IntervalVector state(n);
  state[0] = 0.0;
  state[1] = 0.0;

  // Dynamique d'une voiture avec incertitude sur sa masse + PI
  Function ydot(y, Return ((1440.0 * (10.0 - y[0]) + 35.0 * y[1]  - y[0] * (50.0  + 0.4 * y[0]))
                         / Interval (990, 1010),
                         10.0 - y[0]));
  ivp_ode vdp = ivp_ode(ydot, 0.0, state);

  // Integration numerique ensembliste
  simulation simu = simulation(&vdp, 10.0, RK4, 1e-7);
  simu.run_simulation();

  simu.export1d_yn("export-closed-loop.txt", 0);

  return 0;
}
```

Simulation of an hybrid closed-loop system

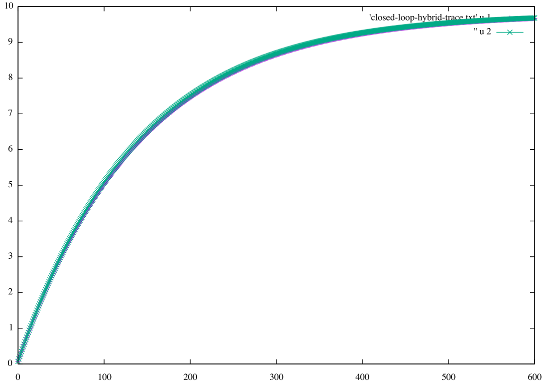
A simple dynamics of a car with a discrete PI controller

$$\dot{y} = \frac{u(k) - 50.0y - 0.4y^2}{m} \qquad \text{with} \quad m \in [990, 1010]$$

$$i(t_k) = i(t_{k-1}) + h(c - y(t_k)) \qquad \text{with} \quad h = 0.005$$

$$u(t_k) = k_p(c - y(t_k)) + k_i i(t_k) \qquad \text{with} \quad k_p = 1400, k_i = 35$$

Simulation for 3 seconds with $y(0) = 0$ and $c = 10$



Simulation of an hybrid closed-loop system

```
#include "ibex.h"

using namespace ibex;
using namespace std;

int main(){
  const int n = 2; Variable y(n);
  Affine2Vector state(n);
  state[0] = 0.0; state[1] = 0.0;

  double t = 0; const double sampling = 0.005;
  Affine2 integral(0.0);

  while (t < 3.0) {
    Affine2 goal(10.0);
    Affine2 error = goal - state[0];

    // Controleur PI discret
    integral = integral + sampling * error;
    Affine2 u = 1400.0 * error + 35.0 * integral;
    state[1] = u;

    // Dynamique d'une voiture avec incertitude sur sa masse
    Function ydot(y, Return((y[1] - 50.0 * y[0] - 0.4 * y[0] * y[0])
                        / Interval (990, 1010), Interval(0.0)));
    ivp_ode vdp = ivp_ode(ydot, 0.0, state);

    // Integration numerique ensembliste
    simulation simu = simulation(&vdp, sampling, RK4, 1e-6);
    simu.run_simulation();

    // Mise a jour du temps et des etats
    state = simu.get_last(); t += sampling;
  }
```

- Manual handling of discrete-time evolution

Differential constraint satisfaction problems

Goal: Extension of CSP to deal with ODEs

Our goal: add differential constraints into CSP framework.

State of the Art on CSP + ODE

- J. Cruz in 2003 introduces ODE into CSP framework by adding a differential problems combined with NSCP
- A. Goldsztejn *et al.* in 2010 extended CSP with ODE by only using **solution operator** of ODE

This work pursues the work of Goldsztejn *et al.* by providing a free open-source implementation: **DynIBEX**

Main idea is to add some constraints on the results of validated numerical integration.

Quantified Constraint Satisfaction Differential Problems

$$S \equiv \dot{\mathbf{y}} = f(\mathbf{y}(t), \mathbf{p})$$

QCSDP

Let S be a differential system and $t_{\text{end}} \in \mathbb{R}_+$ the time limit. A QCSDP is a NCSP defined by

- a set of variables $\mathcal{V}$ *including* t, a vector $\mathbf{y}_0$, $\mathbf{p}$
  We represent these variables by the vector $\mathbf{v}$;
- an initial domain $\mathcal{D}$ containing *at least* $[0, t_{\text{end}}]$, $\mathcal{Y}_0$, and $\mathcal{P}$;
- a set of constraints $\mathcal{C} = \{c_1, \ldots, c_e\}$ composed of predicates over sets, that is, constraints of the form

$$c_i \equiv Q\mathbf{v} \in \mathcal{D}_i . f_i(\mathbf{v}) \diamond \mathcal{A}, \qquad \forall 1 \leqslant i \leqslant e$$

  with $Q \in \{\exists, \forall\}$, $f_i : \wp(\mathbb{R}^{|\mathcal{V}|}) \to \wp(\mathbb{R}^q)$ stands for non-linear arithmetic expressions defined over variables $\mathbf{v}$ and solution of differential system S, $\mathbf{y}(t; \mathbf{y}_0, \mathbf{p}, \mathbf{u}) \equiv \mathbf{y}(\mathbf{v})$, $\diamond \in \{\subseteq, \cap_\emptyset\}$ and $\mathcal{A} \subseteq \mathbb{R}^q$ where $q > 0$.

Note: we follow the same approach that Goldsztejn et al.[3]

[3] Including ODE Based Constraints in the Standard CP Framework, CP10

Box-QCSDP as abstraction of QCSDP

Box-QCSDP

Let S be a differential system and $t_{\text{end}} \in \mathbb{R}_+$ the time limit A Box-QCSDP is defined by

- a set of variables $\mathcal{V}$ including *at least* t, a vector $\mathbf{y}_0$, $\mathbf{p}$, $\mathbf{u}$
  We represent these variables by the vector $\mathbf{v}$;
- an initial box $[\mathbf{d}]$ containing *at least* $[0, t_{\text{end}}]$, $[\mathbf{y}_0]$, $[\mathbf{u}]$, and $[\mathbf{p}]$;
- a set of interval constraints $\mathcal{C} = \{c_1, \ldots, c_e\}$ composed of predicates over sets, that is, constraints of the form

$$c_i \equiv Q\mathbf{v} \in [\mathbf{d}_i].[f_i](\mathbf{v}) \diamond \alpha(\mathcal{A}), \qquad \forall 1 \leqslant i \leqslant e$$

  with $Q \in \{\exists, \forall\}$, $[f_i] : \mathbb{IR}^{|\mathcal{V}|} \to \mathbb{IR}^q$ stands for non-linear arithmetic expressions defined over variables $\mathbf{v}$ and interval enclosure solution $[\mathbf{y}](t; \mathbf{y}_0, \mathbf{p}, \mathbf{u}) \equiv [\mathbf{y}](\mathbf{v})$, $\diamond \in \{\subseteq, \cap_\emptyset\}$ and $\alpha \in \{\text{Hull}, \text{Int}\}$

A simple resolution algorithm

1. Solve ODE with validated numerical integration
2. Solve constraints using standard NSCP techniques

Box-QCSDP as abstraction of QCSDP

Abstraction using boxes is not so straightforward to preserve soundness, each possible constraints must be studied !

			$\alpha(\mathcal{A})$	
			$\mathrm{Int}(\mathcal{A})$	$\mathrm{Hull}(\mathcal{A})$
$\alpha(g)$	$[g]$	$\subset$	`true`	?
		$\supset$	`false`	?
		$\cap_{=\emptyset}$	?	`true`
		$\cap_{\neq\emptyset}$	?	`false`

Legend

- **?:** no result implies guaranteed result on original formula
- **true:** abstract formula valid then the original one valid,

$$[g](\mathbf{v}) \subset \mathrm{Int}(\mathcal{A}) \Rightarrow g(\mathbf{v}) \subset \mathcal{A}$$

- **false:** abstract formula not valid then the original one not valid,

$$\neg([g](\mathbf{v}) \cap_{\neq\emptyset} \mathrm{Hull}(\mathcal{A})) \Rightarrow \neg(g(\mathbf{v}) \cap_{\neq\emptyset} \mathcal{A})$$

DynIBEX: a Box-QCSDP solver with restrictions

Solving arbitrary quantified constraints is hard!

We focus on particular problems of robotics involving quantifiers

- Robust controller synthesis: $\exists \mathbf{u}$, $\forall \mathbf{p}$, $\forall \mathbf{y}_0$ + temporal constraints
- Parameter synthesis: $\exists \mathbf{p}$, $\forall \mathbf{u}$, $\forall \mathbf{y}_0$ + temporal constraints
- etc.

We also defined a set of temporal constraints useful to analyze/design robotic application.

Verbal property	QCSDP translation
Stay in $\mathcal{A}$	$\forall t \in [0, t_{\text{end}}],\ [\mathbf{y}](t, \mathbf{v}') \subseteq \text{Int}(\mathcal{A})$
In $\mathcal{A}$ at τ	$\exists t \in [0, t_{\text{end}}],\ [\mathbf{y}](t, \mathbf{v}') \subseteq \text{Int}(\mathcal{A})$
Has crossed $\mathcal{A}$*	$\exists t \in [0, t_{\text{end}}],\ [\mathbf{y}](t, \mathbf{v}') \cap \text{Hull}(\mathcal{A}) \neq \emptyset$
Go out $\mathcal{A}$	$\exists t \in [0, t_{\text{end}}],\ [\mathbf{y}](t, \mathbf{v}') \cap \text{Hull}(\mathcal{A}) = \emptyset$
Has reached $\mathcal{A}$*	$[\mathbf{y}](t_{\text{end}}, \mathbf{v}') \cap \text{Hull}(\mathcal{A}) \neq \emptyset$
Finished in $\mathcal{A}$	$[\mathbf{y}](t_{\text{end}}, \mathbf{v}') \subseteq \text{Int}(\mathcal{A})$

*: shall be used in negative form

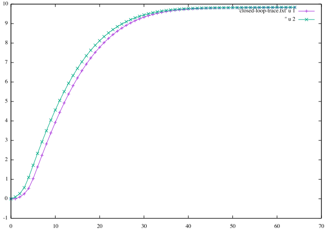
Simulation of a closed-loop system with safety

A simple dynamics of a car with a PI controller

$$\begin{pmatrix} \dot{y} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{k_p(10.0-y)+k_i w-50.0y-0.4y^2}{m} \\ 10.0-y \end{pmatrix} \quad \text{with} \quad m \in [990, 1010], k_p = 1440, k_i = 35$$

and **a safety propriety**

$$\forall t, y(t) \in [0, 11]$$



Failure

$$y([0, 0.0066443]) \in [-0.00143723, 0.0966555]$$

Simulation of a closed-loop system with safety property

```
#include "ibex.h"

using namespace ibex;

int main(){
  const int n = 2;
  Variable y(n);

  IntervalVector state(n);
  state[0] = 0.0; state[1] = 0.0;

  // Dynamique d'une voiture avec incertitude sur sa masse + PI
  Function ydot(y, Return ((1440.0 * (10.0 - y[0]) + 35.0 * y[1]  - y[0] * (50.0  + 0.4 * y[0]))
                         / Interval (990, 1010),
                         10.0 - y[0]));
  ivp_ode vdp = ivp_ode(ydot, 0.0, state);

  simulation simu = simulation(&vdp, 10.0, RK4, 1e-6);
  simu.run_simulation();

  // verification de surete
  IntervalVector safe(n);
  safe[0] = Interval(0.0, 11.0);
  bool flag = simu.stayed_in (safe);
  if (!flag) {
    std::cerr << "error safety violation" << std::endl;
  }

  return 0;
}
```

Some experiments

Experiment 1 – Tuning PI controller [SYNCOP'15]

A cruise control system two formulations:

- uncertain linear dynamics;

$$\dot{v} = \frac{u - bv}{m}$$

- uncertain non-linear dynamics

$$\dot{v} = \frac{u - bv - 0.5\rho CdAv^2}{m}$$

with

- m the mass of the vehicle
- u the control force defined by a PI controller
- bv is the rolling resistance
- $F_{\text{drag}} = 0.5\rho CdAv^2$ is the aerodynamic drag (ρ the air density, CdA the drag coefficient depending of the vehicle area)

Experiment 1 – Settings and algorithm

Embedding the PI Controller into the differential equations:

- $u = K_p(v_{set} - v) + K_i \int (v_{set} - v)ds$ with v_{set} the desired speed
- Transforming $\text{int}_{\text{err}} = \int (v_{set} - v)ds$ into differential form

$$\frac{\text{int}_{\text{err}}}{dt} = v_{set} - v$$

$$\dot{v} = \frac{K_p(v_{set} - v) + K_i \text{int}_{\text{err}} - bv}{m}$$

Main steps of the algorithm

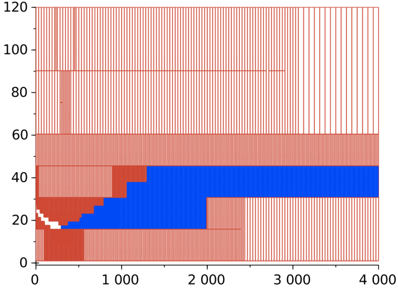
- Pick an interval values for K_p and K_i
- **Simulate** the closed-loop systems with K_p and K_i
  - if `specification is not satisfied`: **bisect** (up to minimal size) intervals and run simulation with smaller intervals
  - if `specification is satisfied` try other values of K_p and K_i

Experiment 1 – Paving results

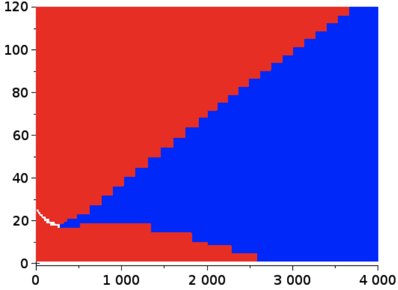
Result of paving for both cases with

- $K_p \in [1, 4000]$ and $K_i \in [1, 120]$
- $v_{\text{set}} = 10$, $t_{\text{end}} = 15$, $\alpha = 2\%$ and $\epsilon = 0.2$ and minimal size=1
- constraints: $y(t_{end}) \in [r - \alpha\%, r + \alpha\%]$ and $\dot{y}(t_{end}) \in [-\epsilon, \epsilon]$

Linear case (CPU $\approx$ 10 minutes)



Non-linear case (CPU $\approx$ 80 minutes)

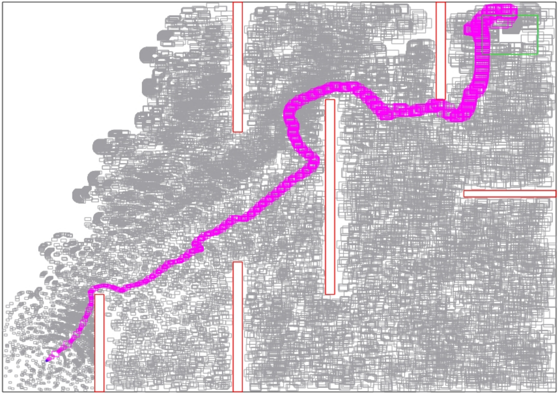


Experiment 2 – Robust path planer

Enhancement of Box-RRT (Pepy *et al.*) with

- dedicated control law
- cost function to minimize distance (Box-RRT*)

$$\exists K > 0 \text{ and } \mathbf{u} \in \mathbb{U} \text{ such that}$$
$$\forall \mathbf{s}_0 \in \mathbb{S}_{\text{init}},\ \forall\ \mathbf{s}(K\Delta t; \mathbf{s}_0) \in \mathbb{S}_{\text{goal}} \text{ and } \forall t \in [0, K\Delta t],\ \mathbf{s}(t; \mathbf{s}_0) \in \mathbb{S}_{\text{free}},$$



Conclusion

DynIBEX is one **ingredient** of verification tools for cyber-physical systems.
It can **handle uncertainties**, can **reason on sets of trajectories**.

Also applied on

- Controller synthesis of sampled switched systems [SNR'16]
- Parameter tuning in the design of mobile robots [MORSE'16]

Enhanced with

- methods to solve algebraic-differential equations [Reliable Computing'16]
- a contractor approach [SWIM'16]
- a Box-QCSDP framework [IRC'17]

Future work (a piece of)

- Pursue and improve cooperation with IBEX language
- Improve algorithm of validated numerical integration (*e.g.*, sensitivity)
- SMT modulo ODE