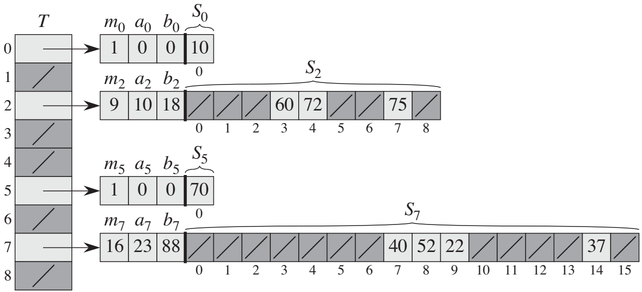
Universal Hash Functions and Perfect Hashing



Universal Hashing

- If a malicious adversary chooses the keys to be hashed by some fixed hash function, he can choose n keys x_i such that they all hash to the same value

$$H(x_i) = h \quad \text{for} \quad i = 1, 2, \ldots, n$$

- This implies that the hash table will have $\Theta(n)$ retrieval time
- Any fixed hash function would have this worst-case behavior
- The only effective way to improve the situation is to choose the hash function *randomly* in a way that is independent of the keys
- This approach is called *universal hashing*
- It can yield provably good performance in the average, not matter which keys the adversary chooses

Universal Hashing

- In universal hashing, we select the hash function at random from a carefully designed class of hash functions
- Randomization guarantees that no single input will evoke worst-case behavior
- This selection is done at the beginning of each execution
- Therefore, the algorithm can behave differently on each execution, even for the same input
- This will guarantee good average-case performance
- Of course, poor performance will occur when the selected hash function hashes the keys poorly
- However, the probability of this situation is small, and is the same for any set of keys of the same size

Universal Hash Functions

- Let $\mathcal{H}$ be a finite collection of hash functions that map a given universe U of keys into the range $\{0, 1, 2, \ldots, m-1\}$
- The set $\mathcal{H}$ is said to be universal if for each pair of keys $x, y \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(x) = h(y)$ is at most $|\mathcal{H}|/m$
- In other words, with a hash function randomly chosen from $\mathcal{H}$, the chance of collision between x and y is $1/m$
- This is the same chance of collision if $h(x)$ and $h(y)$ were randomly and independently chosen from the set $\{0, 1, 2, \ldots, m-1\}$

Average Case Behavior

- Suppose a hash function h is randomly chosen from a universal collection of hash functions
- It is used to hash n keys into a table T of size m, with chaining as the collision resolution method
- Let α be the load factor, defined as $\alpha = n/m$
- If x is not in the table, the expected length of the list that the key x hashes into is at most α
- If x is in the table, the expected length of the list that contains the key x is at most $1 + \alpha$

Average Case Behavior

- Consider a pair of keys x and y; due to the definition of the universal hashing, the probability that they collide is

$$P[h(x) = h(y)] \leq \frac{1}{m}$$

- Let the random variable R_{xy} take the value of 1 when $h(x) = h(y)$ and 0 otherwise; the expected value of R_{xy} is

$$E[R_{xy}] = \frac{1}{m}$$

- Let the random variable S_x be the number of keys other than x that hash to the same slot as x, given as

$$S_x = \sum_{\substack{y \in T \\ y \neq x}} R_{xy}$$

Average Case Behavior

- Therefore, we have

$$E[S_x] \;=\; E[\sum_{\substack{y\in T\\ y\neq x}} R_{xy}\,] \;=\; \sum_{\substack{y\in T\\ y\neq x}} E[R_{xy}] \;\leq\; \sum_{\substack{y\in T\\ y\neq x}} \frac{1}{m}$$

- If $x \notin T$, then the list length is equal to S_x and

$$|\{y : y \in T \text{ and } y \neq x\}| = n$$

  and thus the expected list length $E[S_x] \leq n/m = \alpha$
- If $x \in T$, then because x appears in the list $T[h(x)]$ and the count does not include x, we have the list length as $S_x + 1$ and

$$|\{y : y \in T \text{ and } y \neq x\}| = n - 1$$

  and thus the expected list length

$$E[S_x] + 1 \leq (n-1)/m + 1 = 1 + \alpha - 1/m < 1 + \alpha$$

Average Case Behavior

Theorem

Using universal hashing and collision resolution by chaining in an initially empty table with m slots, it takes $\Theta(n)$ time to handle any sequence of n Insert, Find, and Delete operations containing $O(m)$ Insert operations.

- The number of Insert operations is $O(m)$, thus we have $n = O(m)$ which implies $\alpha = O(1)$
- The Insert and Delete operations take constant time, and the expected time for Find operation is $O(1)$ since the expected length of the list is at most α
- Therefore, the expected time for the entire sequence of n operations is $O(n)$ since each operation takes $\Omega(1)$, the bound $\Theta(n)$ is obtained

Designing Universal Hash Functions

- We will give 3 constructions and show them that they are universal
- The first construction is based on linear congruential arithmetic with two distinct moduli: p and m, where p is a prime
- The second construction uses a random 0-1 matrix and mod 2 arithmetic
- The third method is based the dot-product modulo m

Construction of $\mathcal{H}_{p,m}$

- Select a prime p that is large enough so that every possible key is in the range 0 to $p-1$
- Let $\mathcal{Z}_p = \{0, 1, 2, \ldots, p-1\}$ and $\mathcal{Z}_p^* = \{1, 2, \ldots, p-1\}$
- The size of the universe of the keys is p which is larger than the hash table size m, i.e., $p > m$
- Consider the integer $a \in \mathcal{Z}_p^*$ and $b \in \mathcal{Z}_p$
- Define the hash function family as

$$h_{a,b}(x) = (a \cdot x + b \bmod p) \bmod m$$

- The class of hash functions is defined as

$$\mathcal{H}_{p,m} = \{h_{a,b} \mid a \in \mathcal{Z}_p^* \text{ and } b \in \mathcal{Z}_p\}$$

Properties of $\mathcal{H}_{p,m}$

- An Example: $p = 17$ and $m = 6$, we have $h_{3,4}(8) = 5$ since

$$
\begin{aligned}
h_{3,4}(8) &= ((3 \cdot 8 + 4) \bmod 17) \bmod 6 \\
&= (28 \bmod 17) \bmod 6 \\
&= 11 \bmod 6 \\
&= 5
\end{aligned}
$$

- Each hash function $h_{a,b}$ maps $\mathcal{Z}_p$ to $\mathcal{Z}_m$: the keys are in the range 0 to $p - 1$, while the hash values are from 0 to $m - 1$
- This family has the nice property that the table size m is arbitrary, not necessarily a prime
- There are $p - 1$ choices of a and p choices of b, and thus, there are $p(p - 1)$ hash functions

Proving Universality of $\mathcal{H}_{p,m}$

Theorem

The class $\mathcal{H}_{p,m}$ of hash functions is universal.

- Consider two distinct keys x and y from $\mathcal{Z}_p$, so that $x \neq y$
- For a given hash function $h_{a,b}$, first compute

$$\begin{aligned} r &= (a \cdot x + b) \bmod p \\ s &= (a \cdot y + b) \bmod p \end{aligned}$$

- $r - s = a(x - y)$ is nonzero since $x \neq y$ and $a \neq 0$, and p is prime
- Therefore, if $x \neq y$, we will always have $r \neq s$
- There will not be collision on the "mod p level"

Proving Universality of $\mathcal{H}_{p,m}$

- Moreover, each possible $p(p-1)$ pair of (a, b) with $a \neq 0$ yields a different pair (r, s) since

$$\begin{aligned} a &= (r-s)(x-y)^{-1} \bmod p \\ b &= (r-ax) \bmod p \end{aligned}$$

- There are $p(p-1)$ possible pairs (r, s) with $r \neq s$, and thus, there is a one-to-one correspondence between pairs (a, b) with $a \neq 0$ and pairs (r, s) with $r \neq s$

Proving Universality of $\mathcal{H}_{p,m}$

- Thus, for any given pairs of inputs x and y, if we pick (a, b) uniformly at random from $\mathcal{Z}_p^* \times \mathcal{Z}_p$, the resulting pair is equally likely to be any pair of distinct values modulo p
- The probability that distinct keys x and y collide is equal to the probability $r = s \pmod{m}$ when r and s are randomly chosen as distinct values modulo p
- Furthermore, the probability that s collides with r when reduced modulo m is at most $1/m$, and therefore

$$P[h_{a,b}(x) = h_{a,b}(y)] \ \leq \ 1/m$$

so that $\mathcal{H}_{p,m}$ is universal

Construction of the Matrix Method

- Assume that the keys are u bits long: $x = (x_{u-1} \cdots x_1 x_0)$
- The hash table size as a power of two, as $m = 2^b$, and the hash values $z = h(x)$ are b-bit integers: $z = (z_{b-1} \cdots z_1 z_0)$
- The hash function h is computed using a 0-1 random matrix of dimension $b \times u$, denoted as A
- The hash operation $h(x)$ takes the key x expressed as a u-bit binary number and multiplies with the matrix A to obtain the b-bit hash
- All computations are done in mod 2: the Galois field GF(2)

Properties of the Matrix Method

- An Example: Let $u = 4$ and $b = 3$, therefore, the keys are 4-bit long $x = (x_3 x_2 x_1 x_0)$ and the hash values are 3-bit long $z = (z_2 z_1 z_0)$
- The random 0-1 matrix is of size $b \times u = 3 \times 4$
- Taking A as below, the computation of $z = h(x)$ is performed using

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Let $x = (x_3 x_2 x_1 x_0) = (0101)$, we obtain $(z_2 z_1 z_0) = (011)$ as

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0 \\ 0+0+1+0 \\ 1+0+1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Proving Universality

Theorem

For $x \neq y$, $P[h(x) = h(y)] = 1/m = 2^{-b}$, therefore the class of matrix hash functions with a randomly selected 0-1 matrices is universal.

- Take an arbitrary x and y
- They must differ in at least one bit position
- Assume that x and y differ in the ith bit, i.e., they are given as $(x_{u-1} \cdots x_i \cdots x_1 x_0)$ and $(y_{u-1} \cdots y_i \cdots y_1 y_0)$ such that $x_i \neq y_i$
- WLOG, assume $x_i = 0$ and $y_i = 1$
- Now choose the entire A matrix except its ith column

Proving Universality

- Since this is the column that multiplies the ith bit x or y, the hash values $h(x)$ and $h(y)$ are the same, except the contribution of the ith column of A is not included yet
- The length of ith column is b, and there are 2^b different choices for this column
- Every time we change a bit in this column, we flip the corresponding bit in $h(y)$ since $y_i = 1$
- There are exactly one in 2^b chance that $h(x) = h(y)$
- Therefore, the hash function is universal

Proving Universality

- Consider $x = (x_3 x_2 x_1 x_0) = (0101)$ and $y = (y_3 y_2 y_1 y_0) = (1101)$ so that x and y differ only in the 3rd bit $x_3 \neq y_3$

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \mathbf{0} \end{bmatrix} \text{ and } \begin{bmatrix} z'_0 \\ z'_1 \\ z'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1) + \mathbf{0} \cdot 0 \\ (0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1) + \mathbf{1} \cdot 0 \\ (1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1) + \mathbf{0} \cdot 0 \end{bmatrix}$$

$$\begin{bmatrix} z'_0 \\ z'_1 \\ z'_2 \end{bmatrix} = \begin{bmatrix} (1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1) + \mathbf{0} \cdot 1 \\ (0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1) + \mathbf{1} \cdot 1 \\ (1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1) + \mathbf{0} \cdot 1 \end{bmatrix}$$

Proving Universality

- The contribution of the first three columns of the A matrix to the hash value is the same, and the difference occurs in the contribution of the last column

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (1) + \mathbf{0} \cdot 0 \\ (1) + \mathbf{1} \cdot 0 \\ (1) + \mathbf{0} \cdot 0 \end{bmatrix} \text{ and } \begin{bmatrix} z_0' \\ z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} (1) + \mathbf{0} \cdot 1 \\ (1) + \mathbf{1} \cdot 1 \\ (1) + \mathbf{0} \cdot 1 \end{bmatrix}$$

- As we use A matrices each of which is different in the last column (there are 8 such columns), we obtain different $[z_0', z_1', z_2']^T$ vectors

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} z_0' \\ z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} (1) + \mathbf{0} \cdot 1 \\ (1) + \mathbf{1} \cdot 1 \\ (1) + \mathbf{0} \cdot 1 \end{bmatrix}$$

Proving Universality

- Only in 0 case in which the last column is $[0,0,0]^T$, we will obtain $[z_0', z_1', z_2']^T = [z_0, z_1, z_2]^T$, which is the case when the last column of A is selected as $[0,0,0]^T$

$$\begin{bmatrix}(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{0}\cdot 1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{1}\cdot 1\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{0}\cdot 1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{1}\cdot 1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{0}\cdot 1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{0}\cdot 1\\(1)+\mathbf{1}\cdot 1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{0}\cdot 1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}\;;\;\begin{bmatrix}(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{1}\cdot 1\\(1)+\mathbf{1}\cdot 1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

- Therefore $h(x) = h(y)$ only in 1 out of 8 cases
- There are exactly one in 2^b chance that $h(x) = h(y)$

Construction of the Dot-Product Mod m Method

- Let m be prime
- Decompose the key x into $r+1$ digits each with the value in the set $\mathcal{Z}_m = \{0, 1, 2, \ldots, m-1\}$
- We have $x = (x_r x_{r-1} \cdots x_1 x_0)$ with $x_i \in \mathcal{Z}_m$
- Let $a = (a_r a_{r-1} \cdots a_1 a_0)$ be a random vector such that $a_i \in \mathcal{Z}_m$
- Define the hash function family as

$$h_a(x) = \sum_{i=0}^{r} a_i x_i \pmod{m}$$

- The size of $\mathcal{H}$ is m^{r+1}

Proving Universality

Theorem

The set $\mathcal{H} = \{h_a\}$ is universal.

- Let $x = (x_r \cdots x_1 x_0)$ and $y = (y_r \cdots y_1 y_0)$ be two distinct keys
- Thus, they differ in at least one digit position, WLOG position 0
- For how many $h_a \in \mathcal{H}$ do x and y collide?
- The equality $h(x) = h(y)$ implies

$$\sum_{i=0}^{r} a_i x_i = \sum_{i=0}^{r} a_i y_i \pmod{m}$$

Proving Universality

- Equivalently we have

$$
\begin{aligned}
\sum_{i=0}^{r} a_i(x_i - y_i) &= 0 \pmod{m} \\
a_0(x_0 - y_0) + \sum_{i=1}^{r} a_i(x_i - y_i) &= 0 \pmod{m} \\
a_0(x_0 - y_0) &= -\sum_{i=1}^{r} a_i(x_i - y_i) \pmod{m}
\end{aligned}
$$

Proving Universality

- Since $x_0 \neq y_0$ and m is prime, the inverse $(x_0 - y_0)^{-1} \pmod{m}$ exists, which implies

$$a_0 = -(x_0 - y_0)^{-1} \left[\sum_{i=1}^{r} a_i(x_i - y_i) \right] \pmod{m}$$

- Thus, for any choices of $a_1, a_2, \ldots, a_r$, exactly one choice of a_0 causes x and y collide
- How many h_a functions cause x and y collide?

Proving Universality

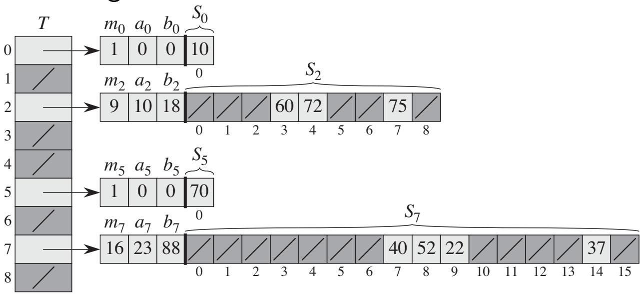
- There are m choices for each of $a_1, a_2, \ldots, a_r$ but once they are chosen, there is only once choice of a_0 that causes x and y collide
- Therefore, the number of hash functions that causes x and y collide is

$$m^r \cdot 1 = m^r = \frac{m^{r+1}}{m} = \frac{|\mathcal{H}|}{m}$$

that makes $\mathcal{H}$ a universal hash function family

Perfect Hashing

- A hashing technique is called **perfect hashing** if $O(1)$ memory accesses are required to perform a search in the **worst case**
- To create a perfect hashing, we use two levels of hashing, with universal hashing at each level



Perfect Hashing

- The first level is the same as hashing with chaining: we hash n keys into m slots using a hash function h from a family of universal hash functions
- However, instead of making a linked list of keys hashing to slot j, we use a **secondary hash table** S_j with an associate hash function h_j
- By choosing the hash functions h_j carefully, we can guarantee that there are **no collisions at the secondary level**
- In order to guarantee that there are no collisions on the secondary level, we need to let the size m_j of the hash table S_j be the square of the number n_j of keys hashing to slop j

Perfect Hashing

- Consider the key set $K = \{10, 22, 37, 40, 52, 60, 70, 72, 74\}$
- The first level hash function is

$$h(k) = (ak + b \bmod p) \bmod m$$

with parameters $(m, a, b, p) = (9, 3, 42, 101)$, where m is the table size

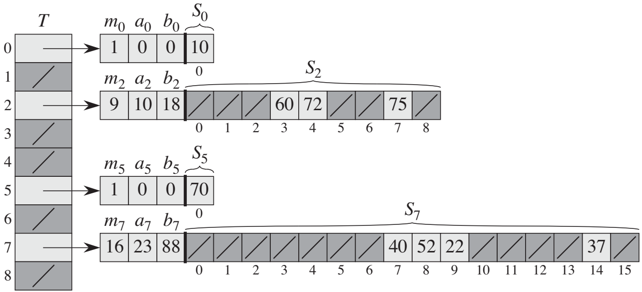
- For example, $h(75)$ is computed as

$$\begin{aligned} h(75) &= (3 \cdot 75 + 42 \bmod 101) \bmod 9 \\ &= (267 \bmod 101) \bmod 9 \\ &= 65 \bmod 9 \\ &= 2 \end{aligned}$$

Perfect Hashing

- A secondary hash table S_j stores all keys hashing to slot j
- The size of hash table S_j is $m_j = n_j^2$, where n_j is the number of keys hashing to slot j
- The associated hash function of S_j is

$$h_j(k) = (a_j k + b_j \bmod p) \bmod m_j$$

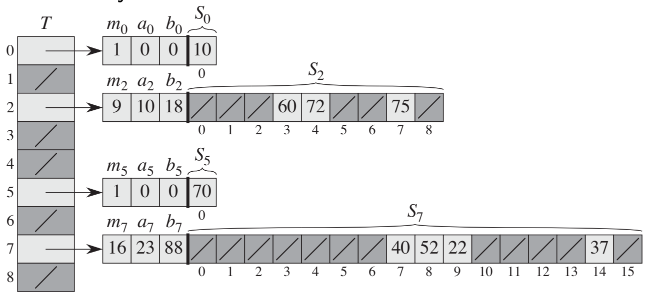


Perfect Hashing

- On the second level, we use the hash function belonging to Slot 2, which has the parameters $(m_2, a_2, b_2) = (9, 10, 18)$ and the same prime $p = 101$, therefore, we compute $h_2(75)$ as

$$\begin{aligned} h_2(75) &= (10 \cdot 75 + 18 \bmod 101) \bmod 9 \\ &= 7 \end{aligned}$$

and place the key 75 in the 7th cell of the Slot 2 table



Perfect Hashing Properties

- If we store n keys in a hash table of size $m = n^2$ using a universal hash function, then the probability of collision is $1/2$
- There are $C(n, 2)$ pairs of different pairs of keys
- The probability that a pair collides is $1/m$, if h is chosen from $\mathcal{H}$
- Let X be the number of collisions, since $m = n^2$, the expected value of X is

$$E[X] = C(n, 2) \cdot \frac{1}{n^2} = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} < \frac{1}{2}$$

Perfect Hashing Properties

- Since we choose $m = n^2$, a hash function h chosen at random from $\mathcal{H}$ is more likely not to have collisions
- Given a **static** set of n keys, it is easy to find a collision-free hash function h
- When n is large, a hash table of size $m = n^2$ is excessive
- However, in the two-level approach we only hash the entries in each slot
- On the first level the hash function h hashes n keys into $m = n$ slots
- Then, if n_j keys hash to slot jm we use the secondary hash table of size $m_j = n_j^2$ to provide a collision-free constant-time lookup

Perfect Hashing Storage Requirement

- In the first level table size is $m = n$, and therefore, the amount of the memory used is $O(n)$ for the primary hash table
- In the secondary hash tables, each hash table S_j is of size n_j^2
- To compute the total memory used in the secondary tables, we need to know the expected sum of the squares of the number of keys n_j that hash to slot j, which turns out to be

$$E\left[\sum_{j=0}^{m-1} m_j\right] = E\left[\sum_{j=0}^{m-1} n_j^2\right] < 2n$$

- Therefore, the total secondary storage is also $O(n)$