Part I

Breadth First Search

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

- **DFS** good for exploring graph structure
- **BFS** good for exploring *distances*

Queues

A **queue** is a list of elements which supports the following operations

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

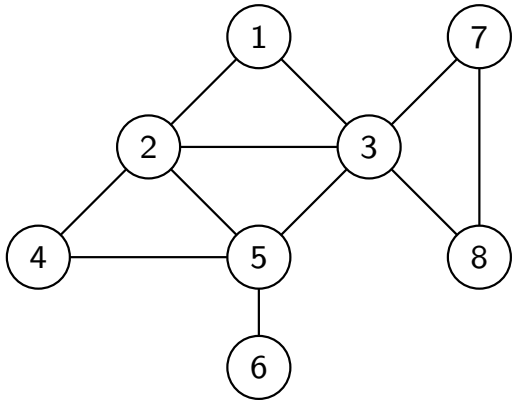
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

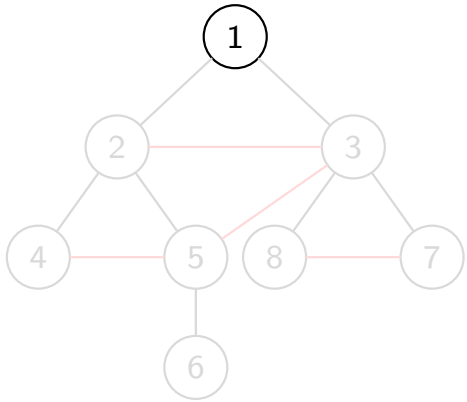
BFS(s)

```
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

BFS(s) *runs in* $O(n + m)$ *time.*





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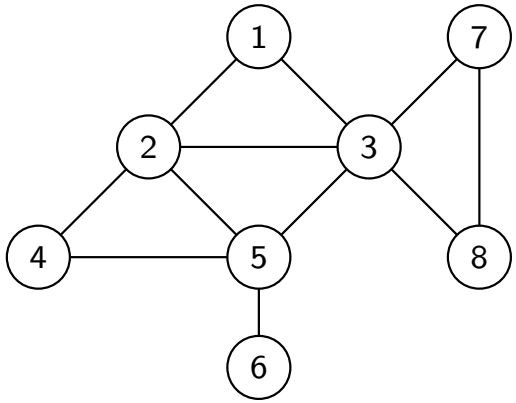
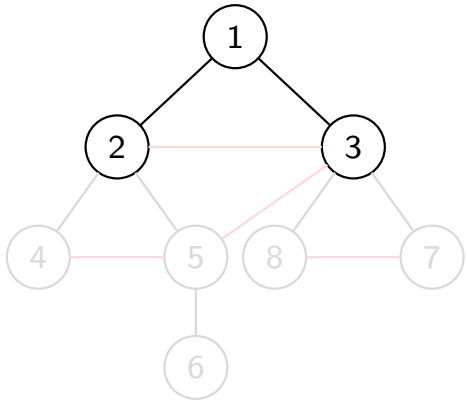
9. []

BFS tree is the set of black edges.





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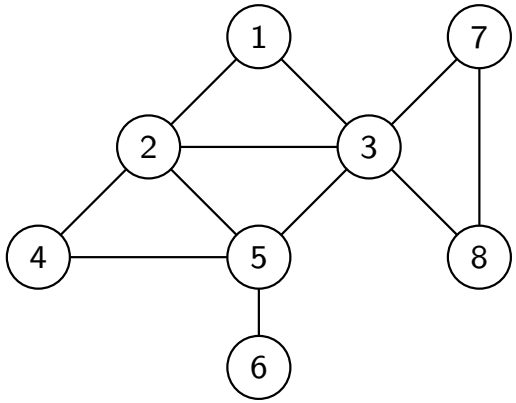

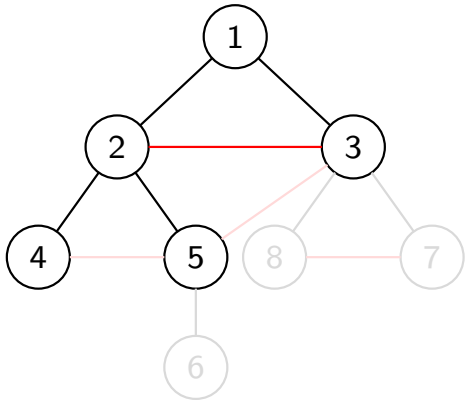

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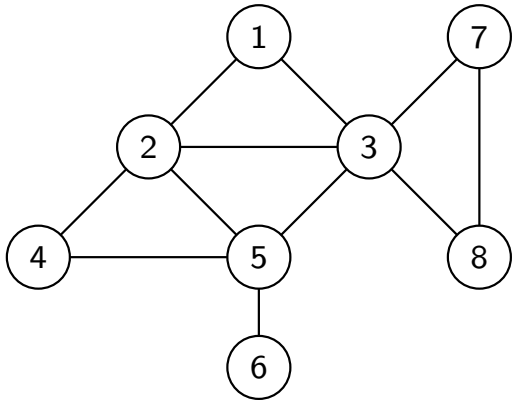

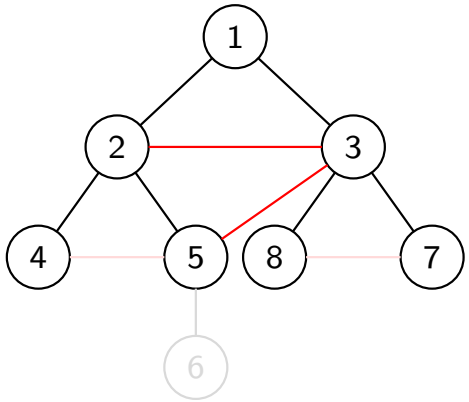

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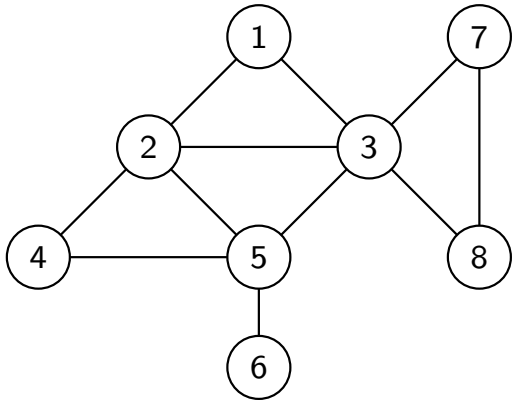

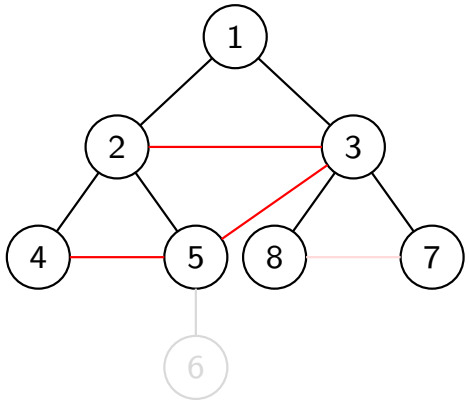

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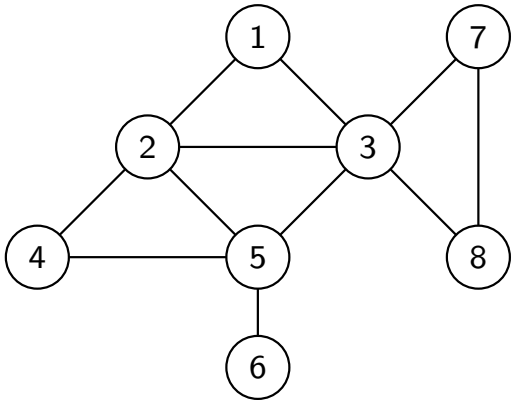

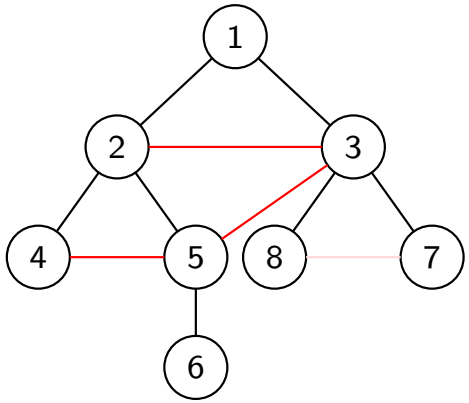

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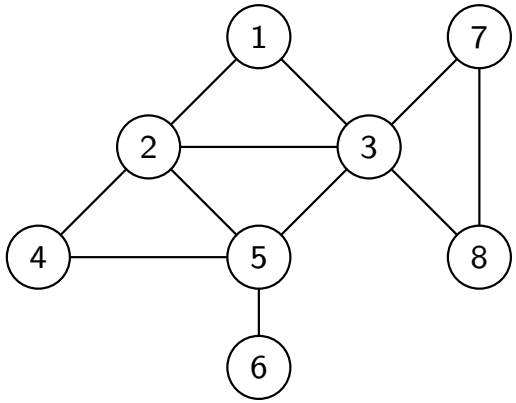

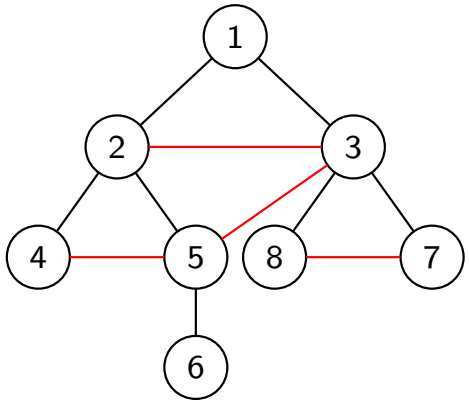

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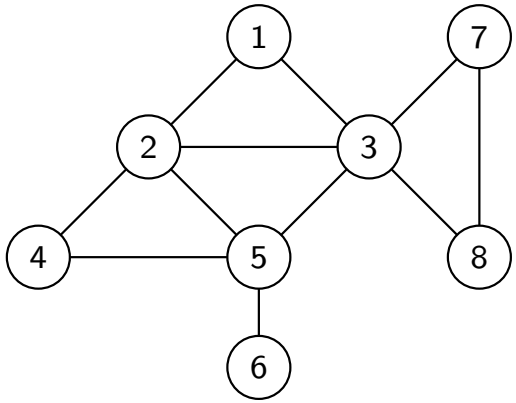

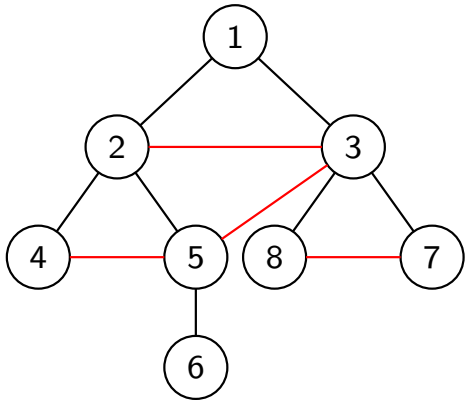

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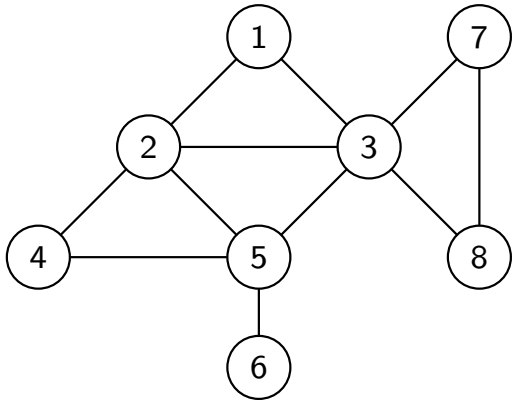

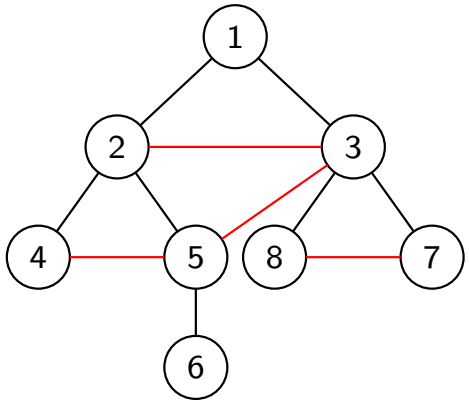

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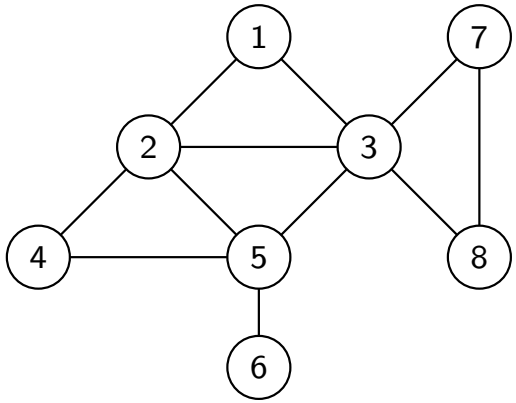

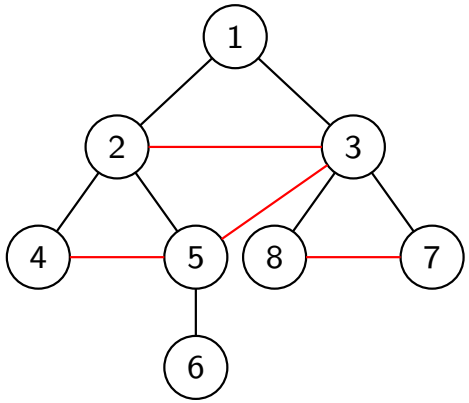

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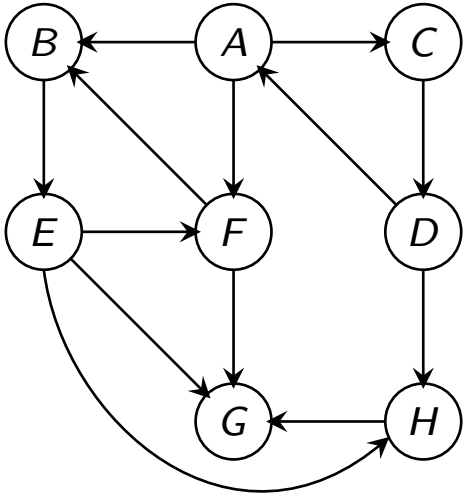
B
A
C
E
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H

```
BFS(s)
    Mark all vertices as unvisited and for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Proposition

The following properties hold upon termination of **BFS(s)**

(A) The search tree contains exactly the set of vertices in the connected component of **s**.

(B) If $\mathbf{dist(u) < dist(v)}$ *then* **u** *is visited before* **v**.

(C) For every vertex **u**, $\mathbf{dist(u)}$ *is indeed the length of shortest path from* **s** *to* **u**.

(D) If **u, v** *are in connected component of* **s** *and* $\mathbf{e = \{u, v\}}$ *is an edge of* **G**, *then either* **e** *is an edge in the search tree, or* $\mathbf{|dist(u) - dist(v)| \leq 1}$.

Proof.

Exercise. □

Proposition

The following properties hold upon termination of **BFS(s)**:

(A) *The search tree contains exactly the set of vertices reachable from* **s**

(B) *If* $\mathbf{dist(u) < dist(v)}$ *then* **u** *is visited before* **v**

(C) *For every vertex* **u**, $\mathbf{dist(u)}$ *is indeed the length of shortest path from* **s** *to* **u**

(D) *If* **u** *is reachable from* **s** *and* $\mathbf{e = (u, v)}$ *is an edge of* **G**, *then either* **e** *is an edge in the search tree, or* $\mathbf{dist(v) - dist(u) \leq 1}$. *Not necessarily the case that* $\mathbf{dist(u) - dist(v) \leq 1}$.

Proof.

Exercise. □

BFSLayers(s):

Mark all vertices as unvisited and initialize T to be empty

Mark s as visited and set $L_0 = \{s\}$

$i = 0$

while L_i is not empty **do**

    initialize L_{i+1} to be an empty list

    for each u in L_i **do**

        for each edge $(u, v) \in Adj(u)$ **do**

        if v is not visited

            mark v as visited

            add (u, v) to tree T

            add v to L_{i+1}

    $i = i + 1$

Running time: $O(n + m)$

```
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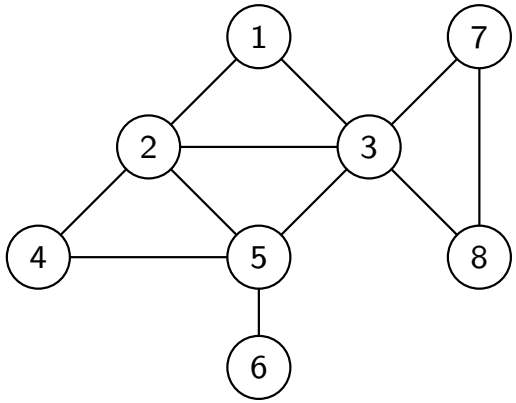
1
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Proposition

The following properties hold on termination of **BFSLayers**(**s**).

- **BFSLayers**(**s**) *outputs a* **BFS** *tree*
- $\mathbf{L_i}$ *is the set of vertices at distance exactly* **i** *from* **s**
- *If* **G** *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
  - **tree** *edge between two* consecutive *layers*
  - *non-tree* **forward**/**backward** *edge between two consecutive layers*
  - *non-tree* **cross-edge** *with both* **u**, **v** *in same layer*
  - $\Longrightarrow$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Proposition

The following properties hold on termination of **BFSLayers(s)**, *if* **G** *is directed.*

For each edge $\mathbf{e = (u, v)}$ *is one of four types:*

- *a* **tree** *edge between consecutive layers,* $\mathbf{u \in L_i, v \in L_{i+1}}$ *for some* $\mathbf{i \geq 0}$
- *a non-tree* **forward** *edge between consecutive layers*
- *a non-tree* **backward** *edge*
- *a* **cross-edge** *with both* **u, v** *in same layer*

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Input A (undirected or directed) graph $\mathbf{G = (V, E)}$ with edge lengths (or costs). For edge $\mathbf{e = (u, v)}$, $\boldsymbol{\ell(e) = \ell(u, v)}$ is its length.

- Given nodes $\mathbf{s, t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
- Given node $\mathbf{s}$ find shortest path from $\mathbf{s}$ to all other nodes.
- Find shortest paths for all pairs of nodes.

Many applications!

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Lengths

Single-Source Shortest Path Problems

Input A (undirected or directed) graph $\mathbf{G = (V, E)}$ with non-negative edge lengths. For edge $\mathbf{e = (u, v)}$, $\boldsymbol{\ell(e) = \ell(u, v)}$ is its length.

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- Restrict attention to directed graphs
- Undirected graph problem can be reduced to directed graph problem - how?
  - Given undirected graph $\mathbf{G}$, create a new directed graph $\mathbf{G'}$ by replacing each edge $\{\mathbf{u, v}\}$ in $\mathbf{G}$ by $\mathbf{(u, v)}$ and $\mathbf{(v, u)}$ in $\mathbf{G'}$.
  - set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$

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  - set $\boldsymbol{\ell(u, v) = \ell(v, u) = \ell(\{u, v\})}$

Special case: All edge lengths are 1.

- Run **BFS**(s) to get shortest path distances from s to all other nodes.
- $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all e?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on e

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

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Special case: All edge lengths are $\mathbf{1}$.

- Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
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Why does **BFS** work?

BFS(s) explores nodes in increasing distance from **s**

Lemma

Let **G** *be a directed graph with non-negative edge lengths. Let* $\mathbf{dist(s, v)}$ *denote the shortest path length from* **s** *to* **v**. *If* $\mathbf{s = v_0 \to v_1 \to v_2 \to \ldots \to v_k}$ *is a shortest path from* **s** *to* $\mathbf{v_k}$ *then for* $\mathbf{1 \leq i < k}$*:*

- $\mathbf{s = v_0 \to v_1 \to v_2 \to \ldots \to v_i}$ *is a shortest path from* **s** *to* $\mathbf{v_i}$
- $\mathbf{dist(s, v_i) \leq dist(s, v_k)}$.

Proof.

Suppose not. Then for some $\mathbf{i < k}$ there is a path $\mathbf{P'}$ from **s** to $\mathbf{v_i}$ of length strictly less than that of $\mathbf{s = v_0 \to v_1 \to \ldots \to v_i}$. Then $\mathbf{P'}$ concatenated with $\mathbf{v_i \to v_{i+1} \ldots \to v_k}$ contains a strictly shorter

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- $\mathbf{dist(s, v_i) \leq dist(s, v_k)}$.

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter

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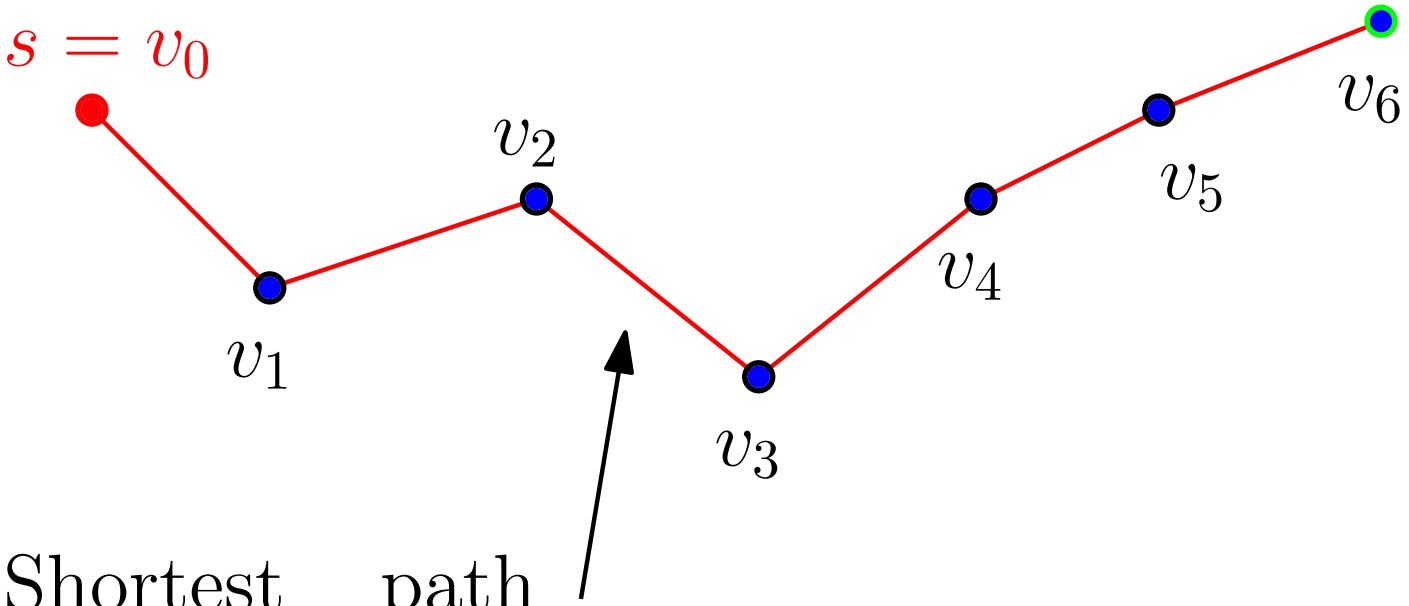
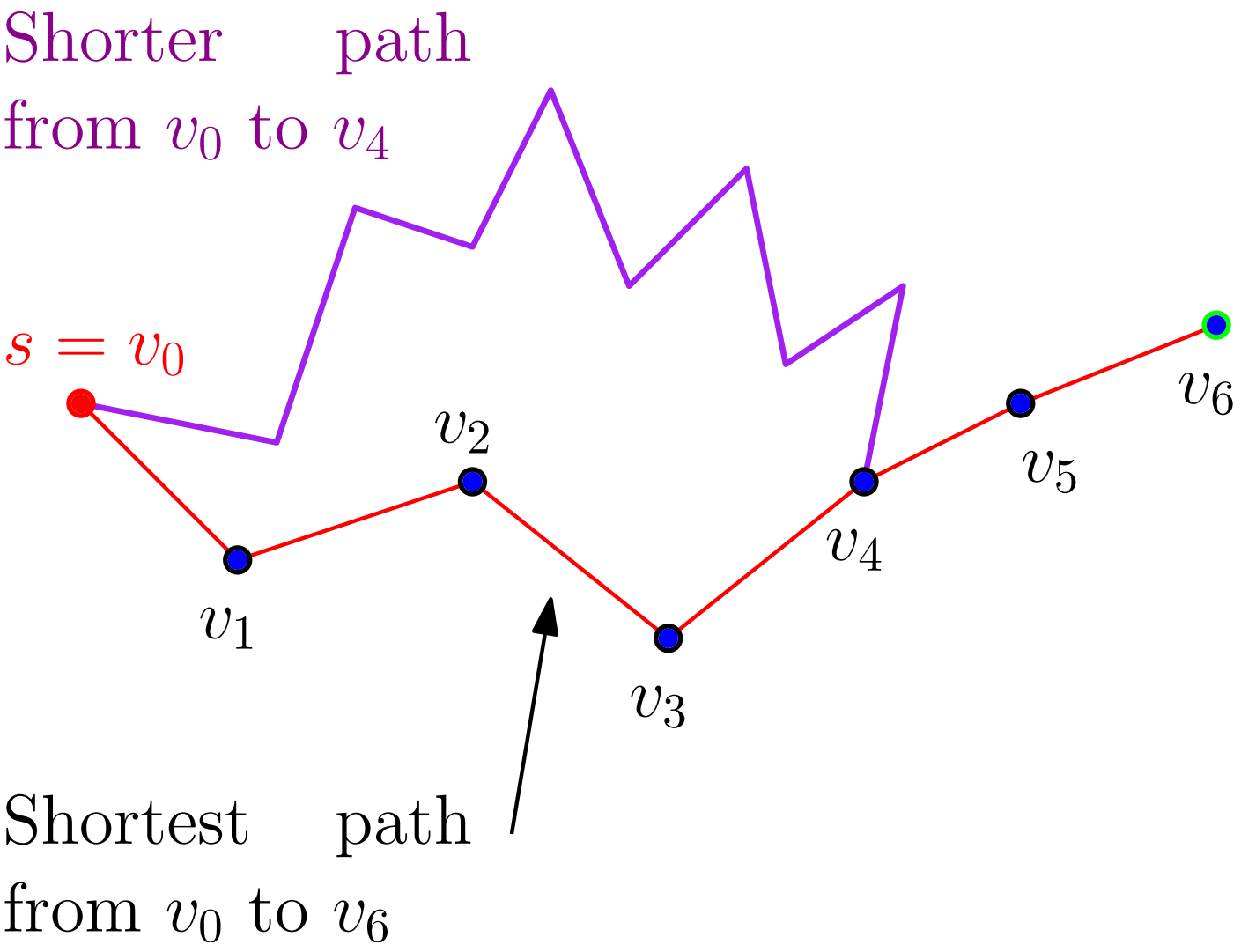
Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter path to v_k than $s = v_0 \to v_1 \ldots \to v_k$. □



Shortest path from v_0 to v_6


Shorter path
from v_0 to v_4
$s = v_0$
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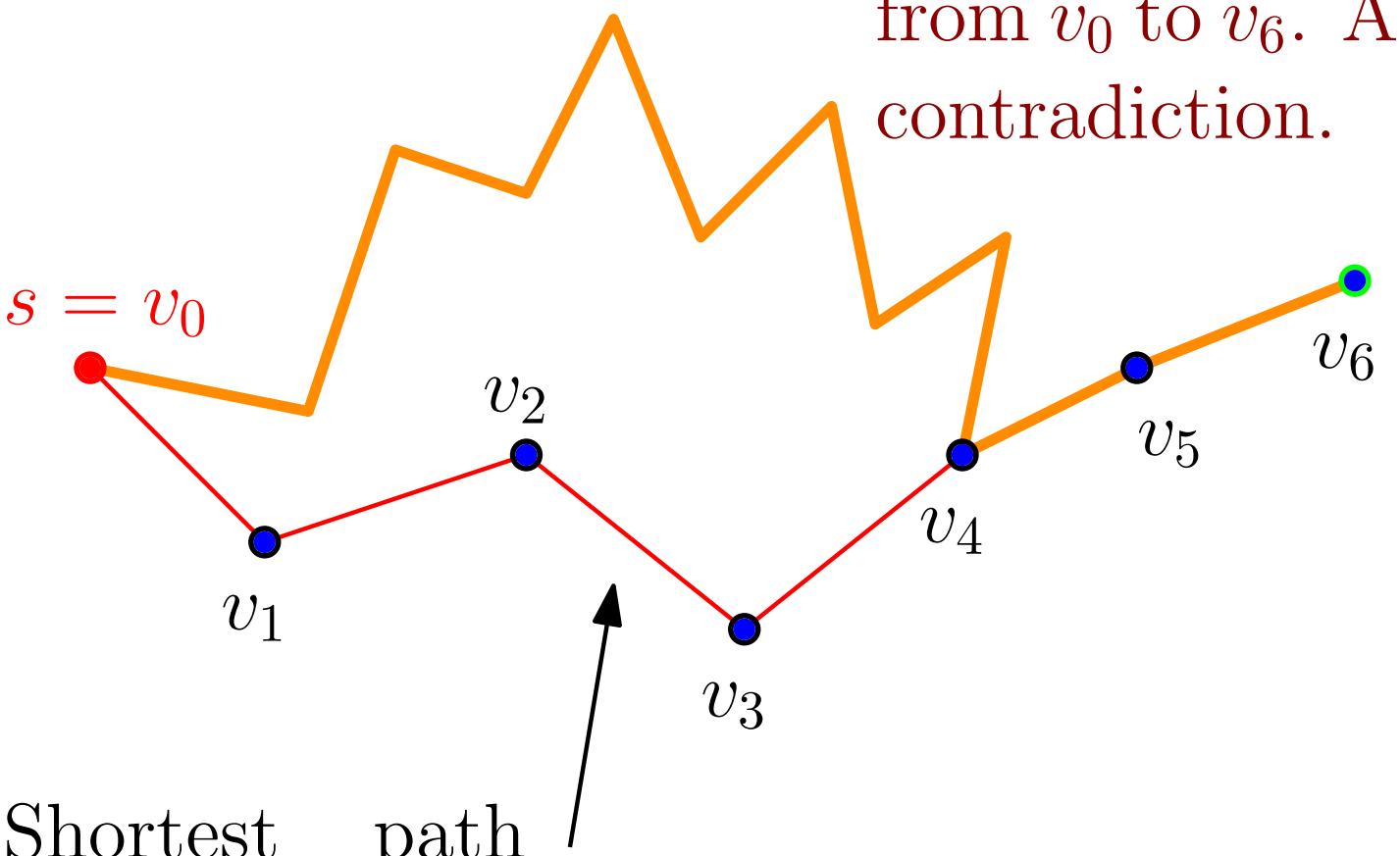
v_5
v_6
Shortest path
from v_0 to v_6


A shorter path from v_0 to v_6. A contradiction.
$s = v_0$
v_1
v_2
v_3
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v_5
v_6
Shortest path from v_0 to v_6

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

Initialize for each node v, $\mathbf{dist(s, v) = \infty}$
Initialize $S = \emptyset$,
for $i = 1$ to $|V|$ **do**
    (* Invariant: S contains the $i - 1$ closest nodes to s *)
    Among nodes in $V \setminus S$, find the node v that is the
        ith closest to s
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    $S = S \cup \{v\}$

How can we implement the step in the for loop?

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- **S** contains the **i − 1** closest nodes to **s**
- Want to find the **i**th closest node from **V − S**.

What do we know about the **i**th closest node?

Claim

*Let **P** be a shortest path from **s** to **v** where **v** is the **i**th closest node. Then, all intermediate nodes in **P** belong to **S**.*

Proof.

If **P** had an intermediate node **u** not in **S** then **u** will be closer to **s** than **v**. Implies **v** is not the **i**th closest node to **s** - recall that **S** already has the **i − 1** closest nodes. □

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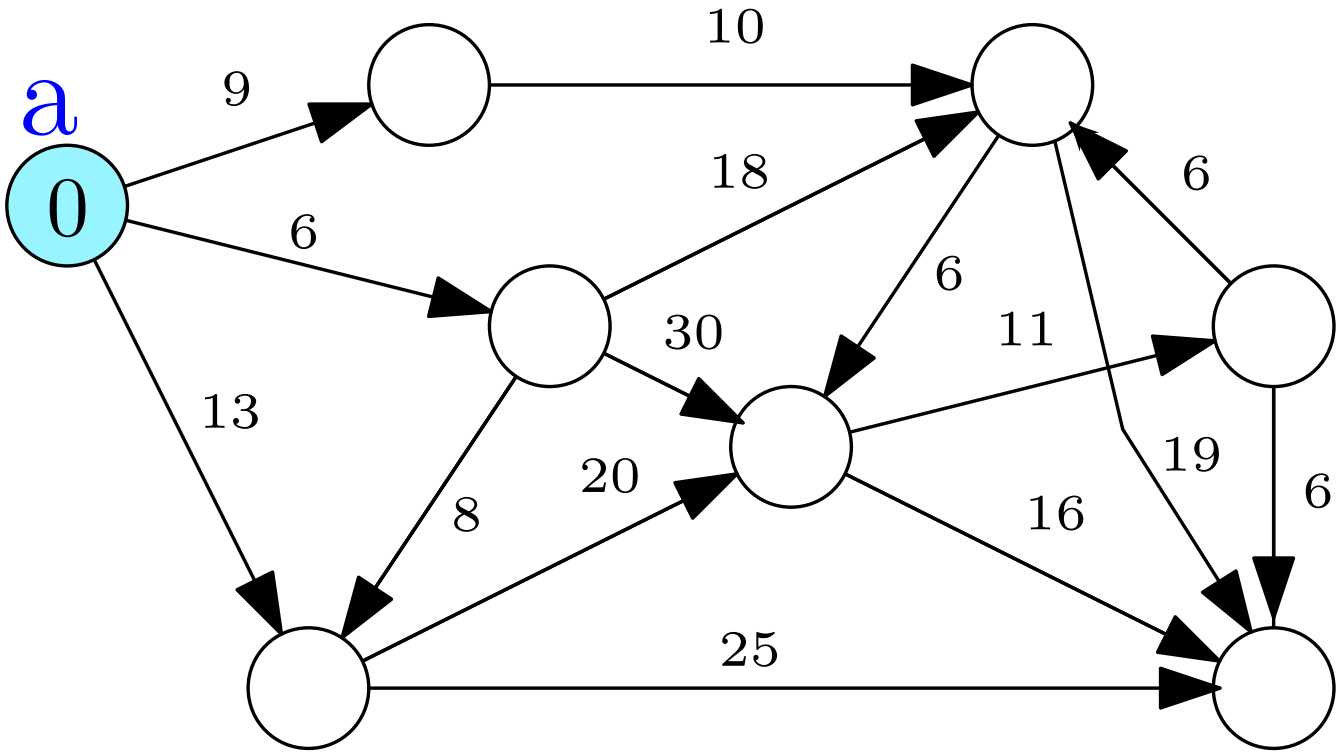
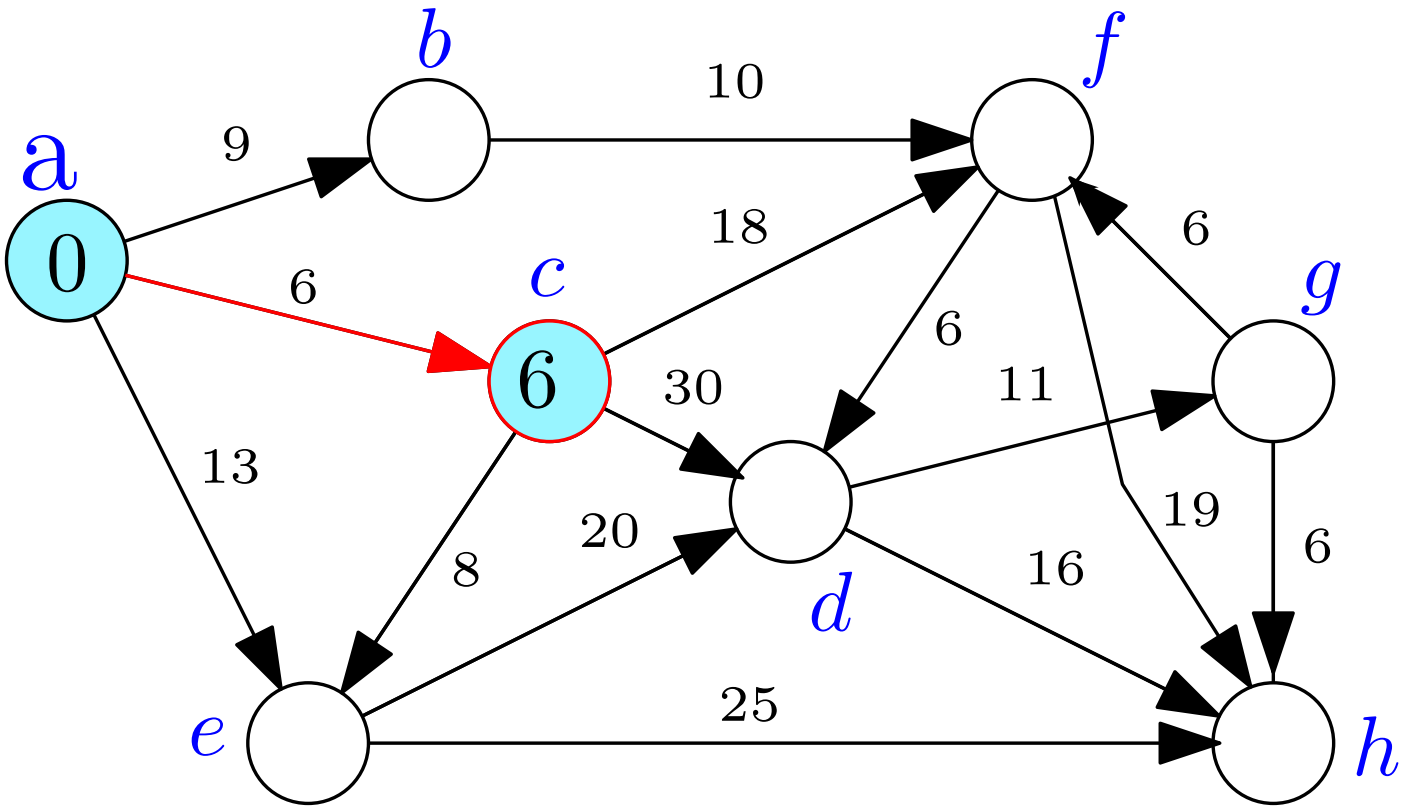
An example

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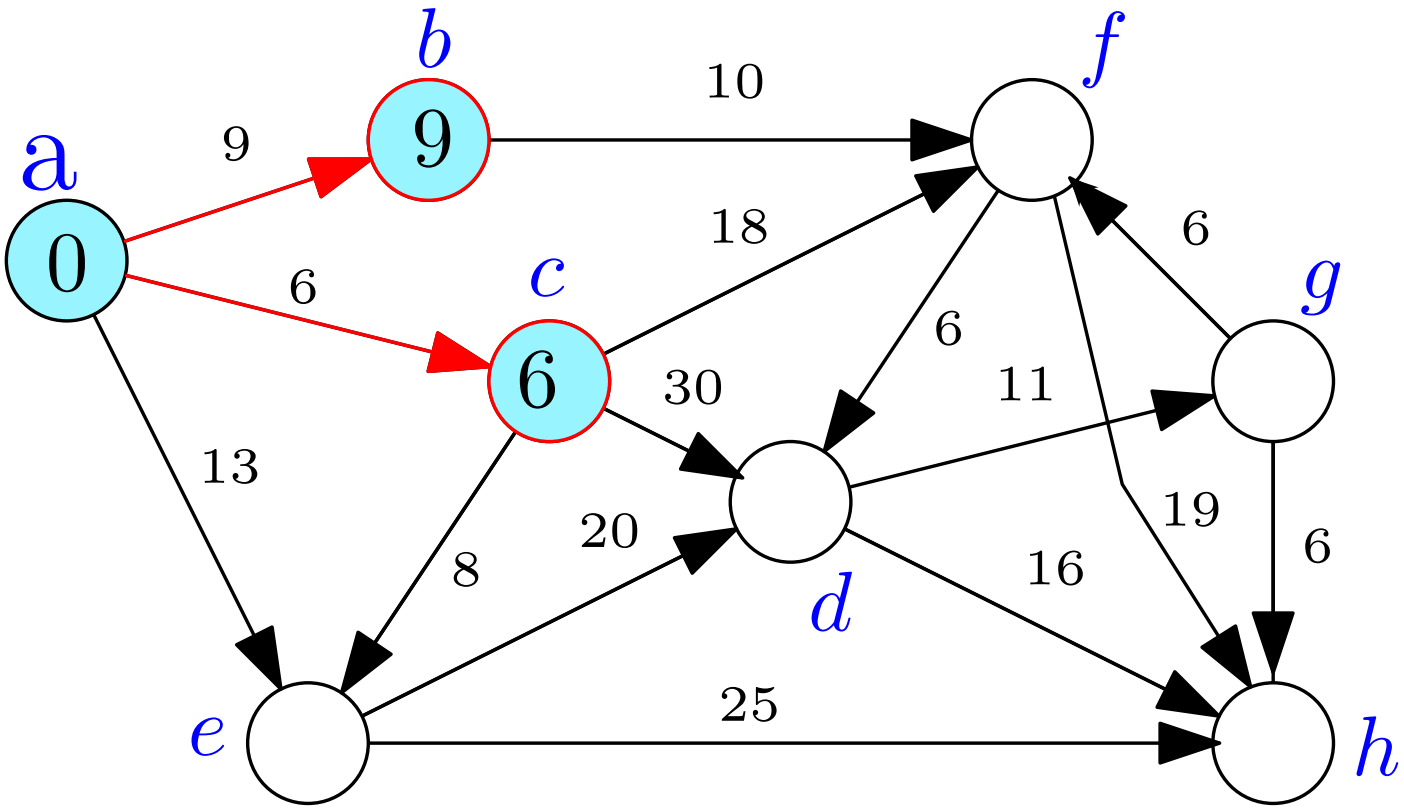
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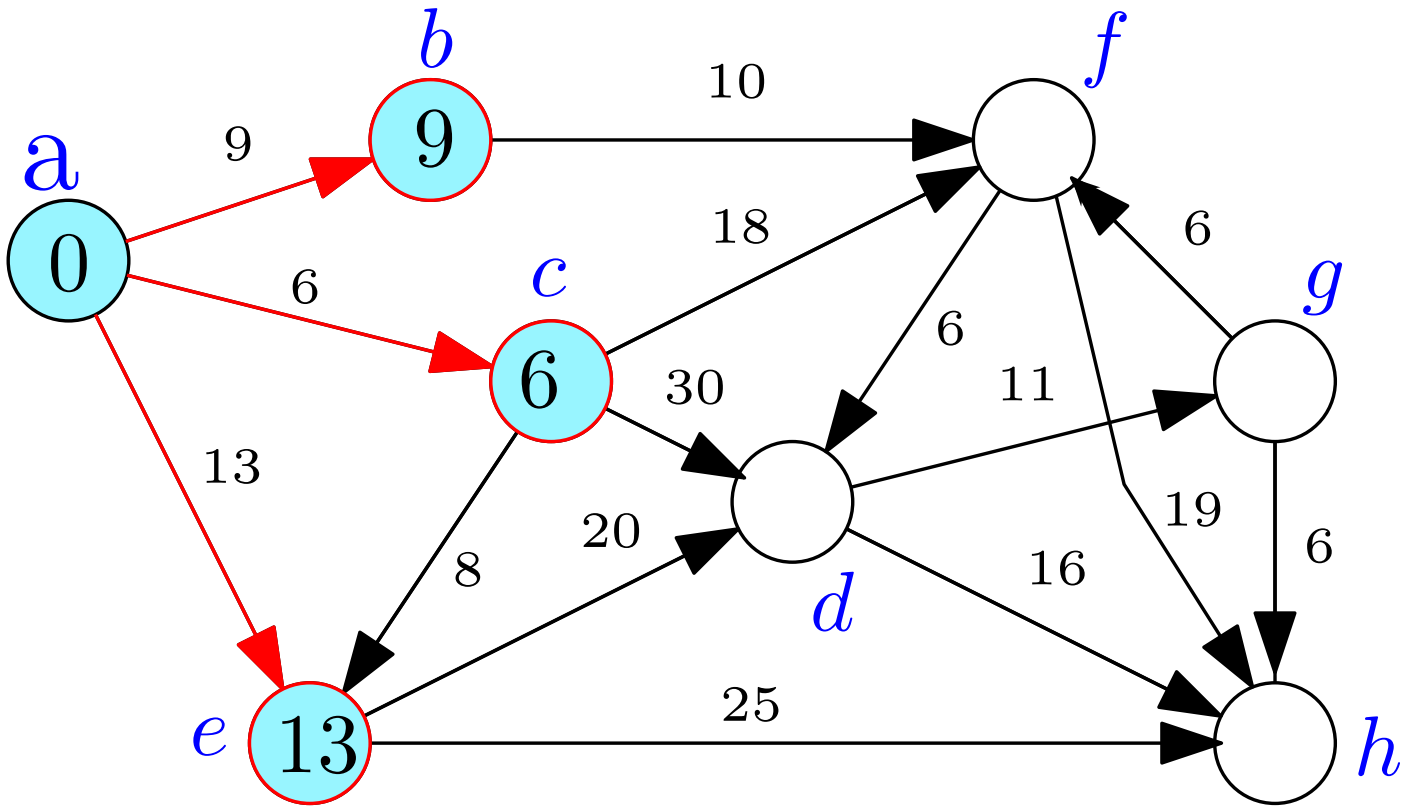
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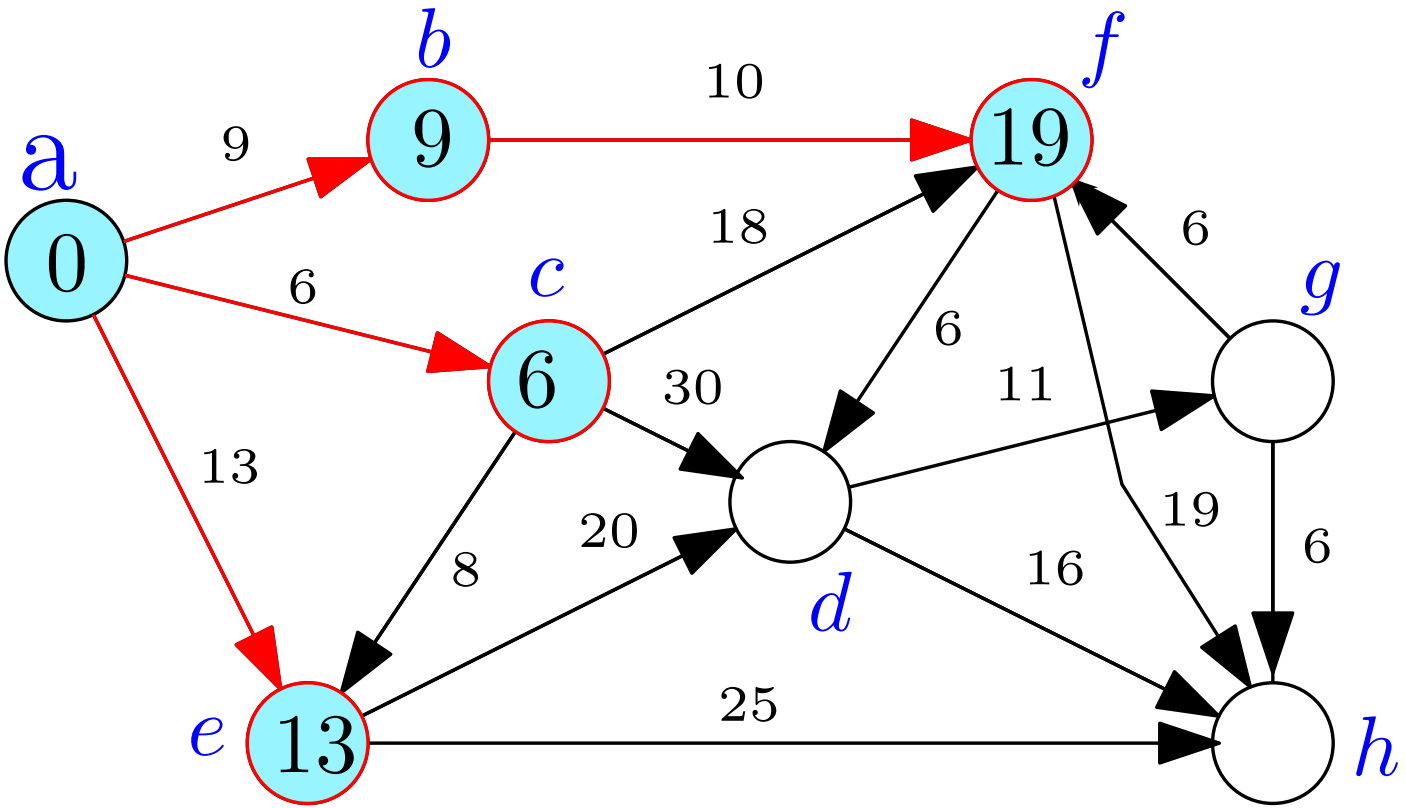
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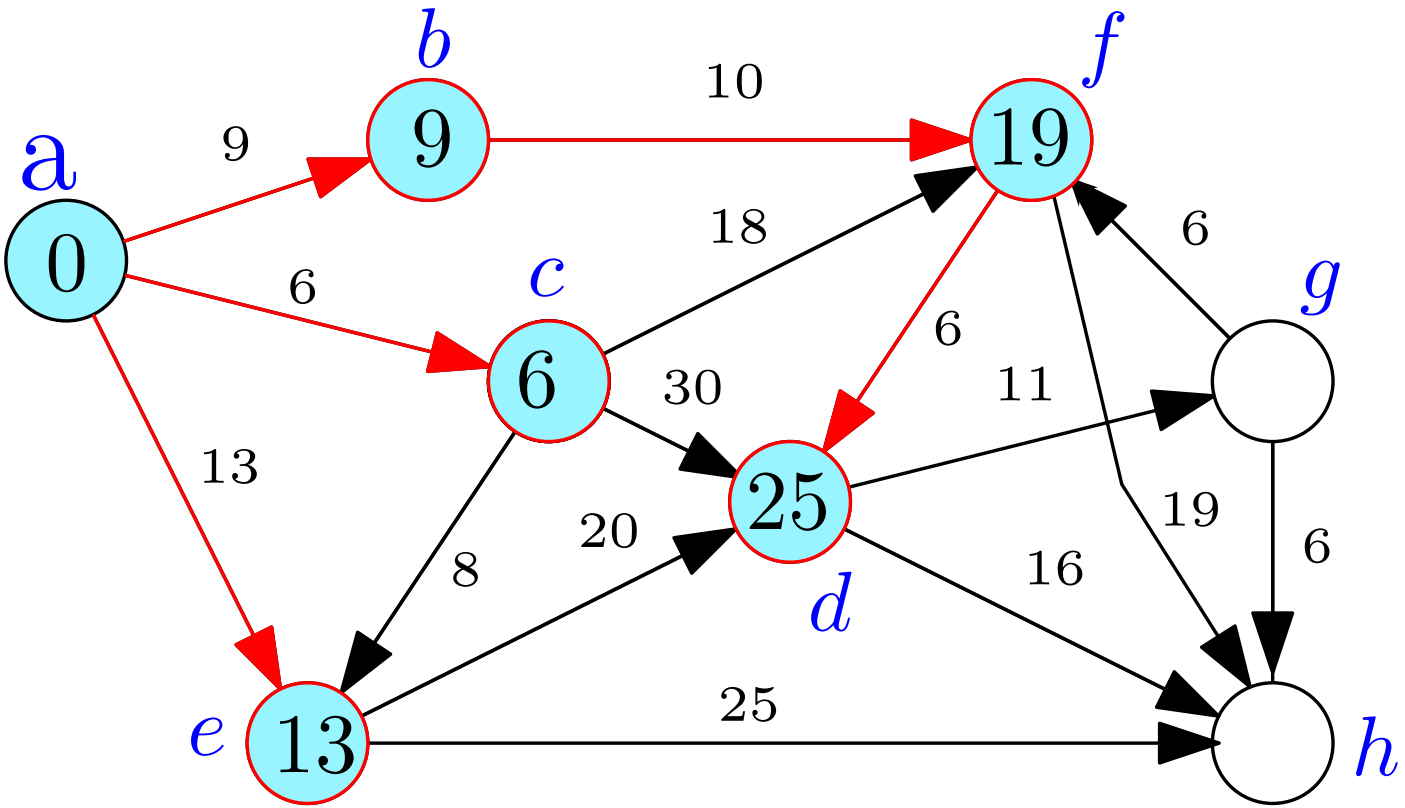
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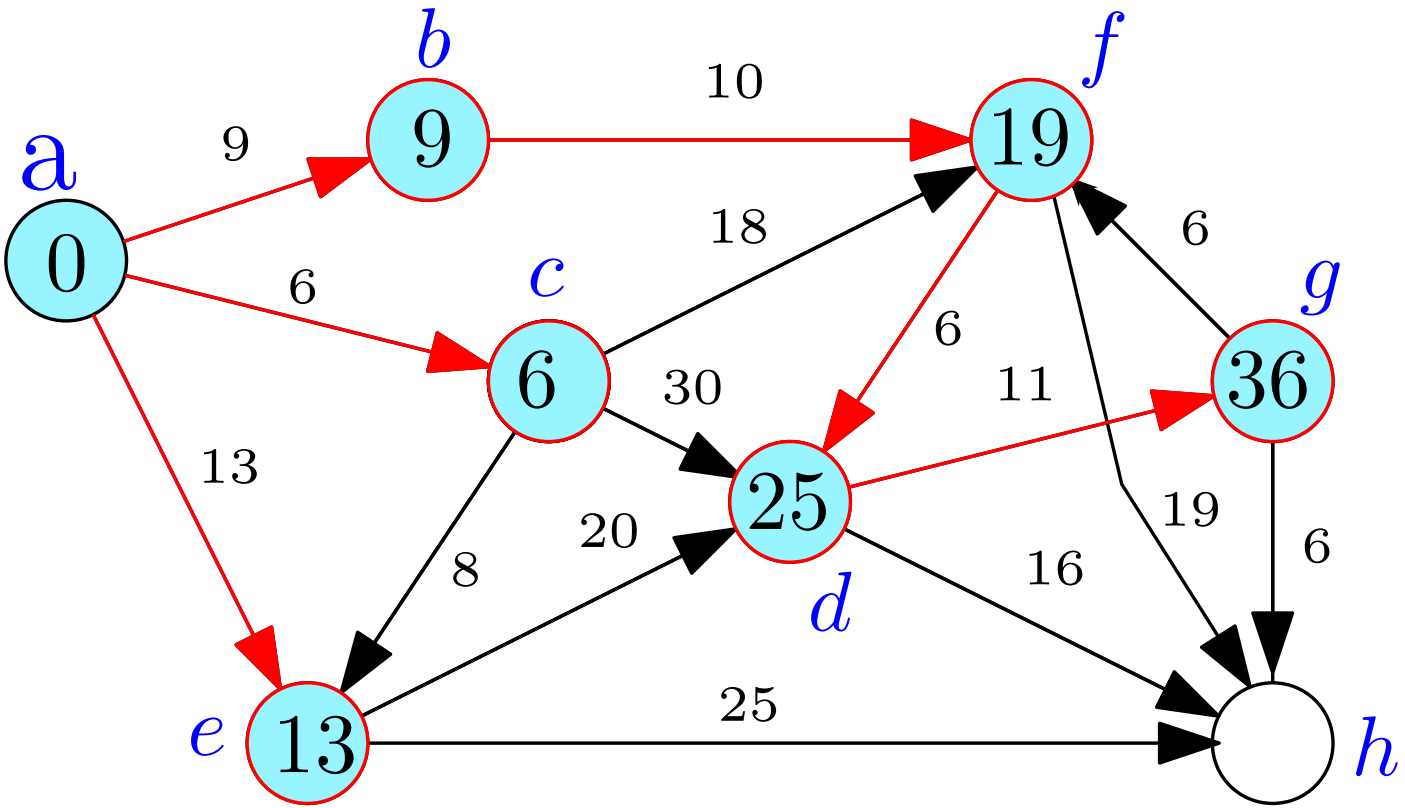
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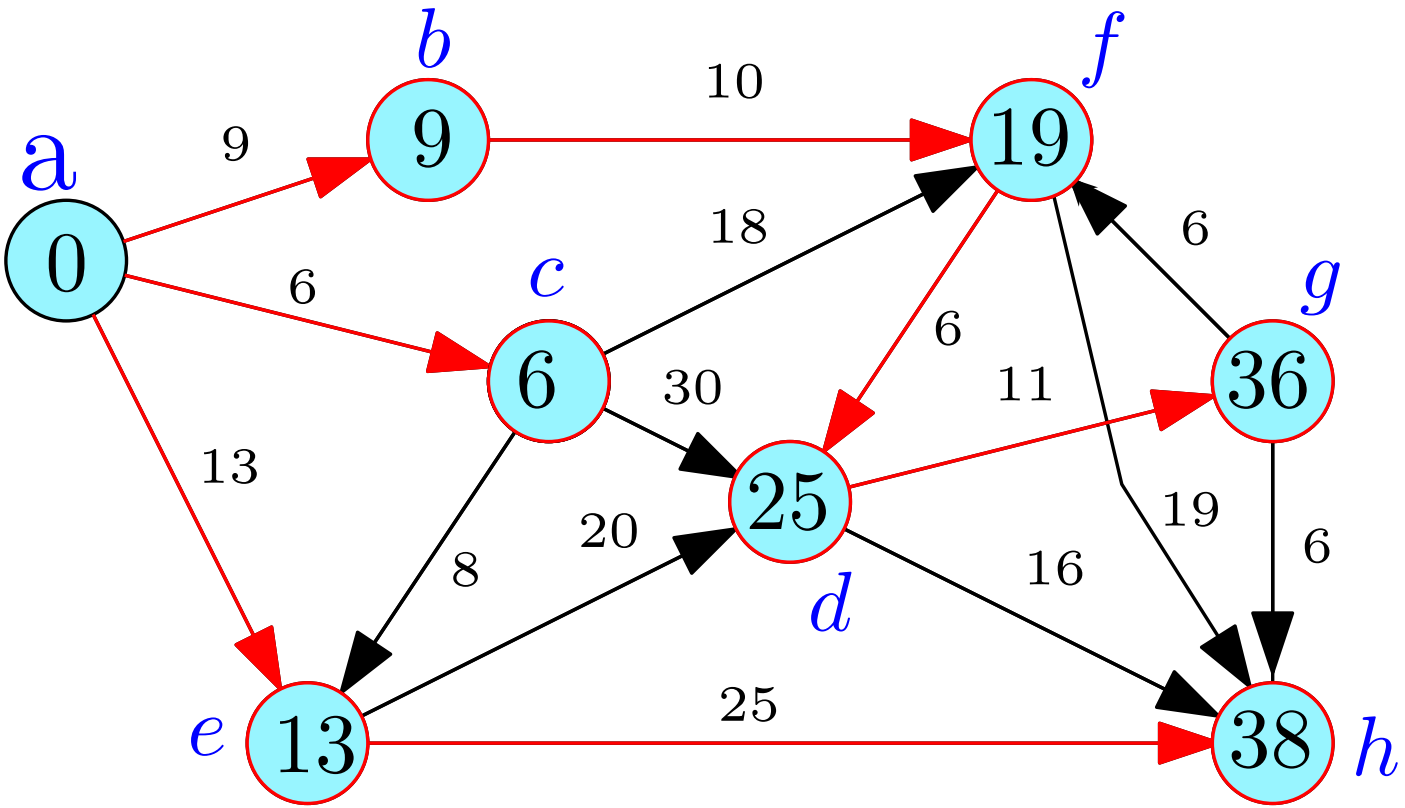
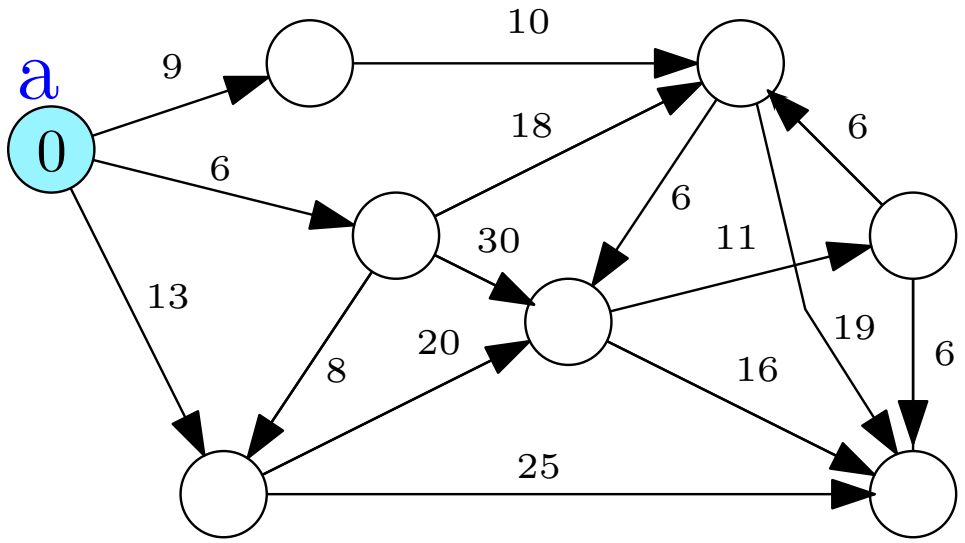
An example





Corollary

The **i***th closest node is adjacent to* **S***.*

- S contains the $i-1$ closest nodes to s
- Want to find the ith closest node from $V-S$.

- For each $u \in V-S$ let $P(s,u,S)$ be a shortest path from s to u using only nodes in S as intermediate vertices.
- Let $d'(s,u)$ be the length of $P(s,u,S)$

Observations: for each $u \in V-S$,

- $dist(s,u) \leq d'(s,u)$ since we are constraining the paths
- $d'(s,u) = \min_{a \in S}(dist(s,a) + \ell(a,u))$ - Why?

Lemma

If v is the ith closest node to s, then $d'(s,v) = dist(s,v)$.

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Proof.

Let $\mathbf{v}$ be the $\mathbf{i}$th closest node to $\mathbf{s}$. Then there is a shortest path $\mathbf{P}$ from $\mathbf{s}$ to $\mathbf{v}$ that contains only nodes in $\mathbf{S}$ as intermediate nodes (see previous claim). Therefore $\mathbf{d'(s, v) = \mathrm{dist}(s, v)}$. □

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The $\mathbf{i}$*th closest node to* $\mathbf{s}$ *is the node* $\mathbf{v \in V - S}$ *such that* $\mathbf{d'(s, v) = \min_{u \in V - S} d'(s, u)}$.

Proof.

For every node $\mathbf{u \in V - S}$, $\mathbf{dist(s, u) \leq d'(s, u)}$ and for the $\mathbf{i}$th closest node $\mathbf{v}$, $\mathbf{dist(s, v) = d'(s, v)}$. Moreover, $\mathbf{dist(s, u) \geq dist(s, v)}$ for each $\mathbf{u \in V - S}$. □

Initialize for each node v: $\mathbf{dist(s,v) = \infty}$
Initialize S = $\emptyset$, $\mathbf{d'(s,s) = 0}$
for $\mathbf{i = 1}$ to $\mathbf{|V|}$ **do**
    (* Invariant: S contains the i-1 closest nodes to s *)
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    Let v be such that d'(s,v) = $\mathbf{\min_{u \in V-S}}$ d'(s,u)
    $\mathbf{dist(s,v) = d'(s,v)}$
    $\mathbf{S = S \cup \{v\}}$
    for each node u in $\mathbf{V \setminus S}$
        compute d'(s,u) = $\mathbf{\min_{a \in S}}$ $(\mathbf{dist(s,a) + \ell(a,u)})$

Correctness: By induction on i using previous lemmas.
Running time: $\mathbf{O(n \cdot (n+m))}$ time.

- n outer iterations. In each iteration, $\mathbf{d'(s,u)}$ for each u by scanning all edges out of nodes in S; $\mathbf{O(m+n)}$ time/iteration.

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    S = S ∪ {v}
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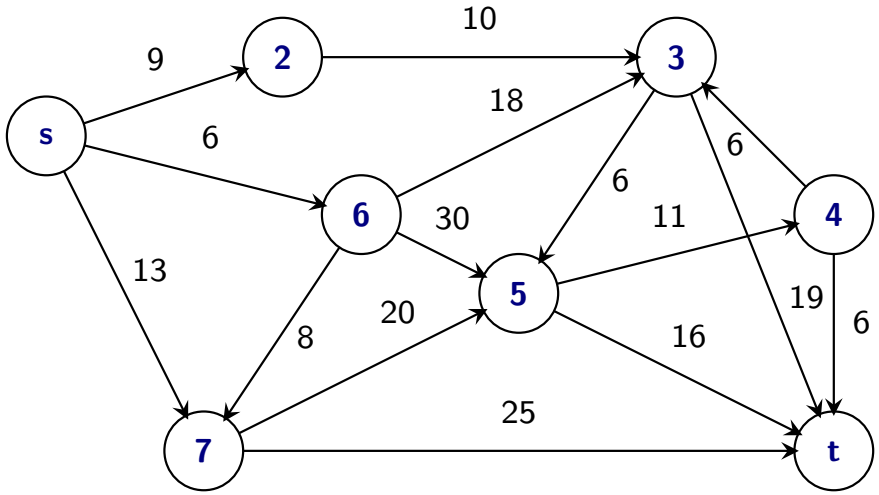
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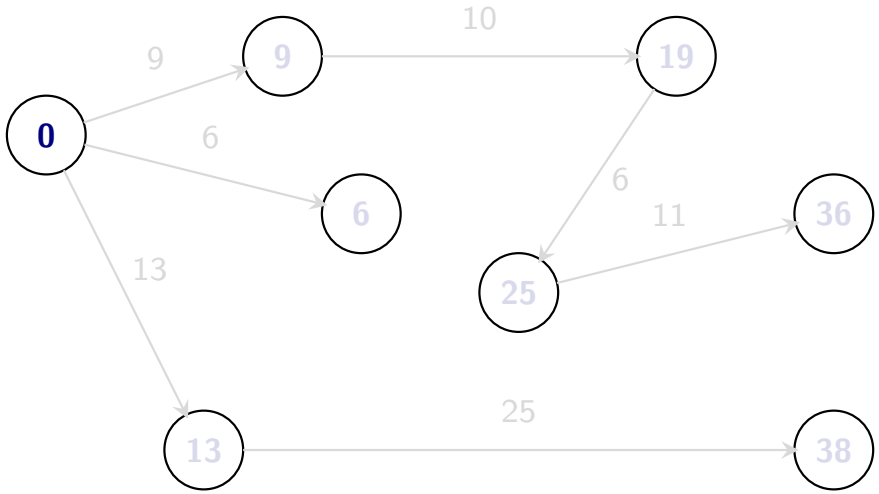
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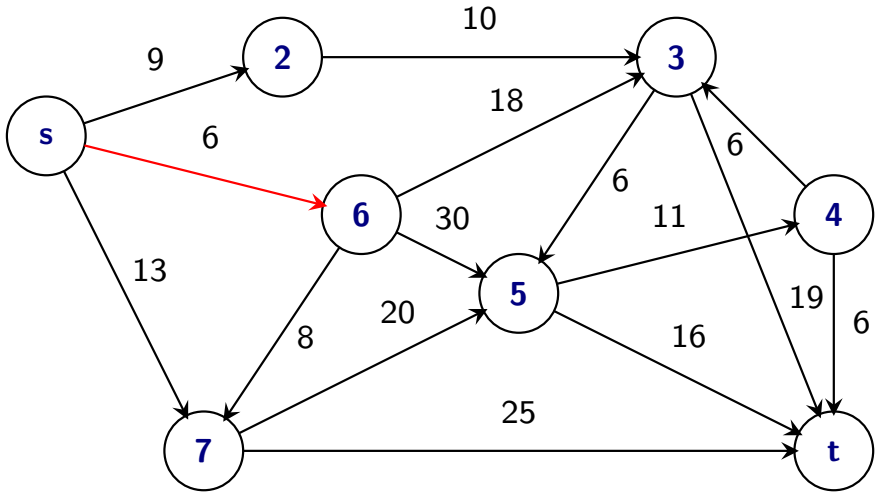
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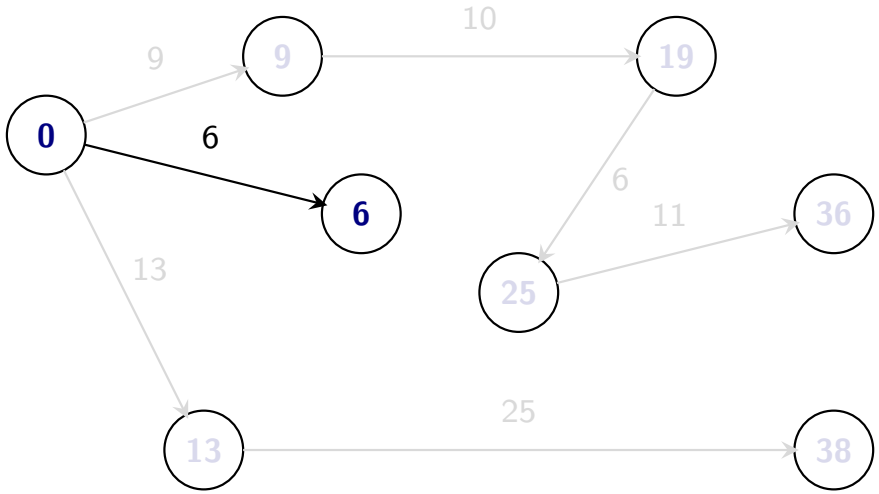
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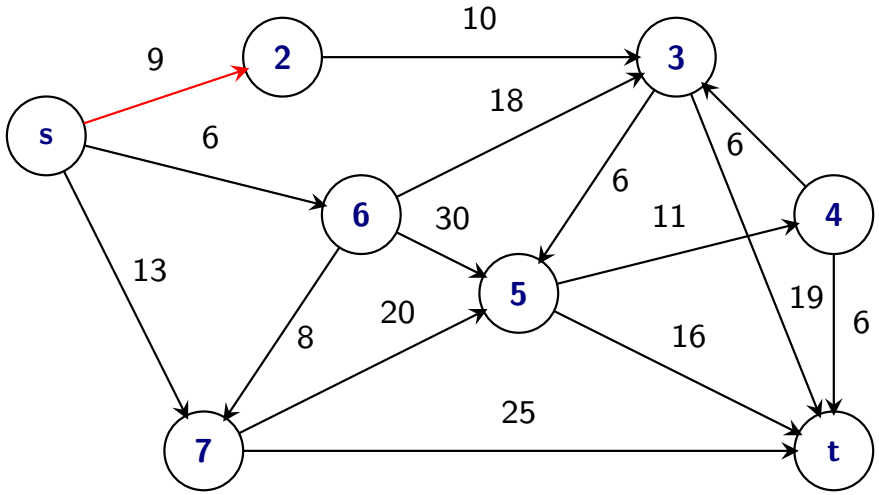
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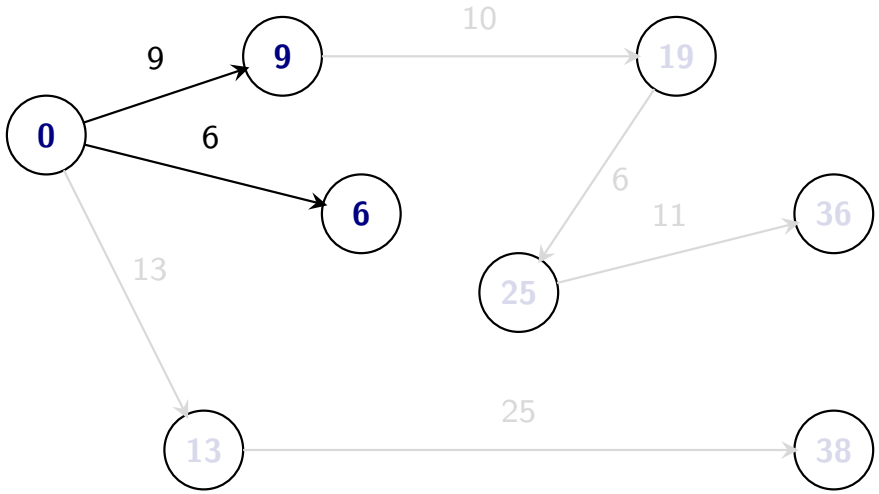
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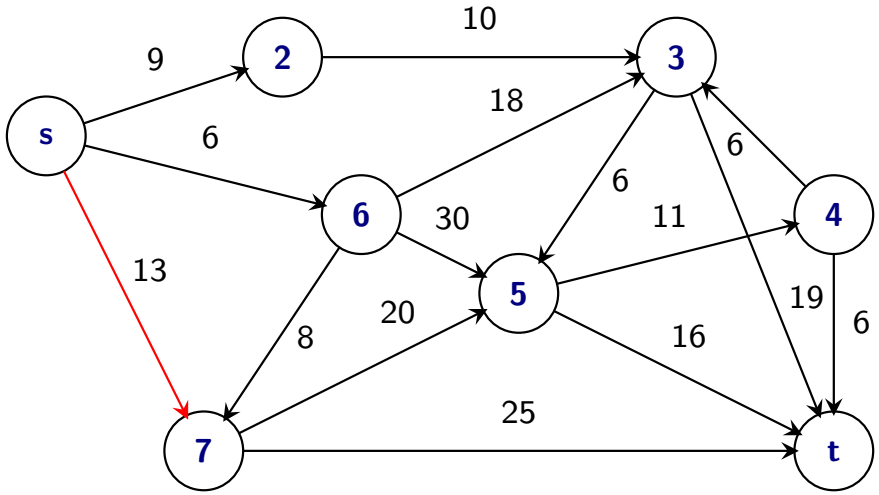
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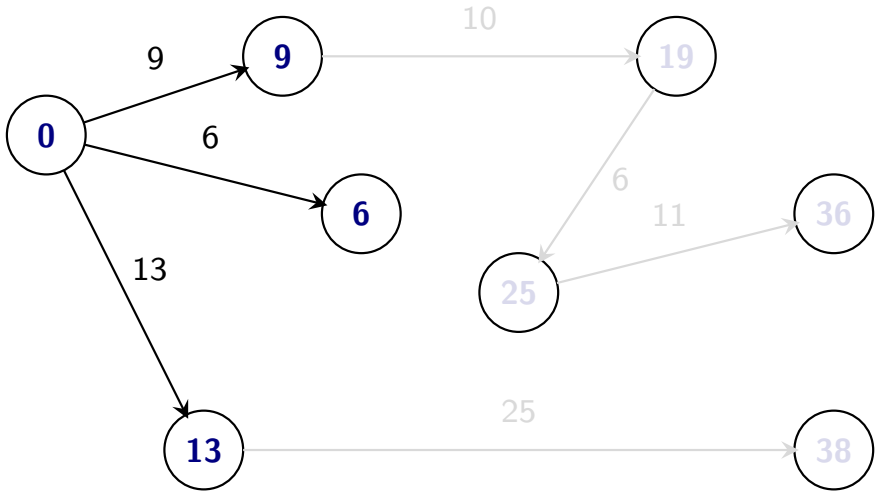
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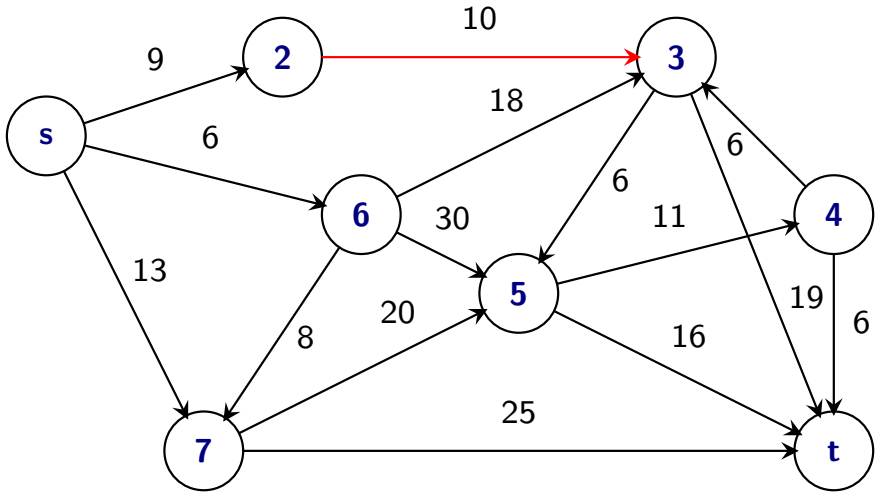
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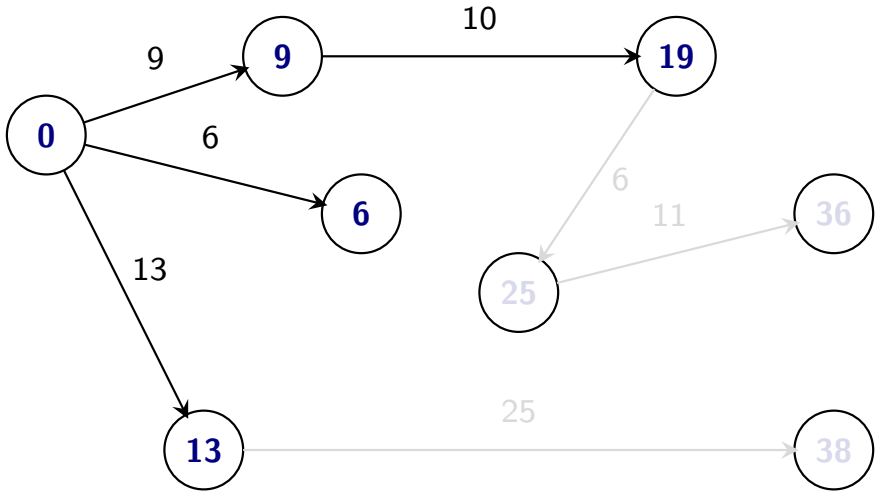
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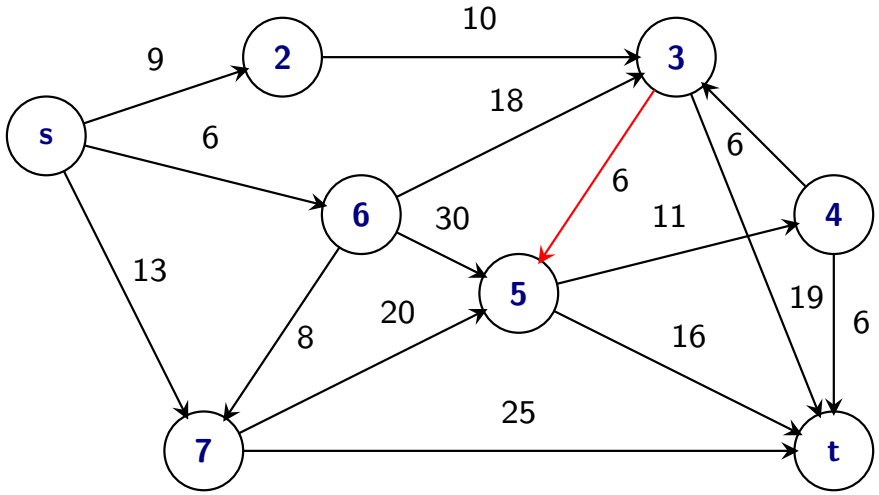
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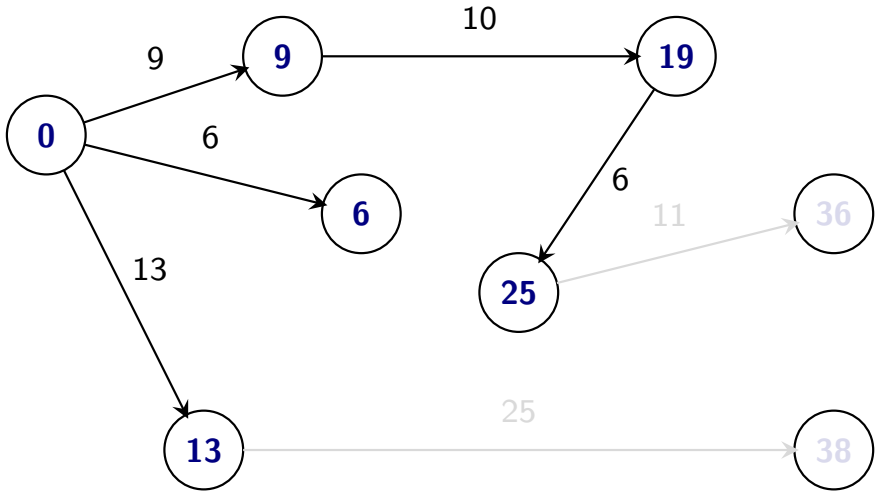
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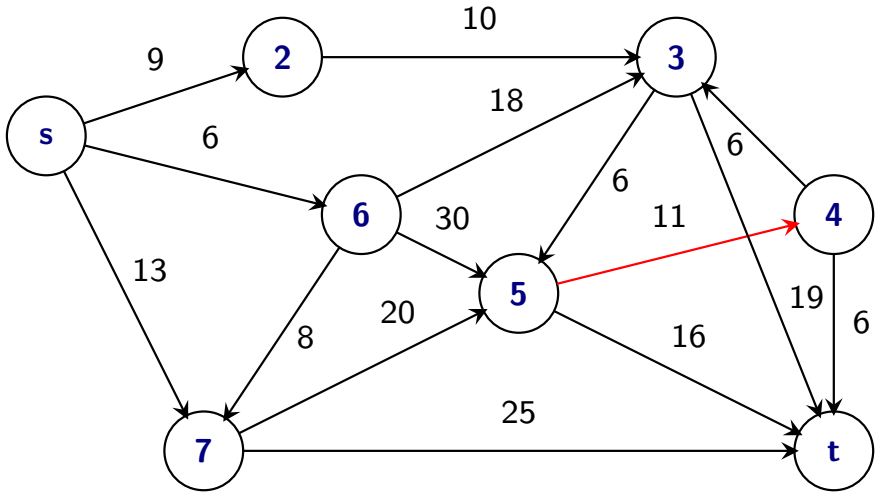
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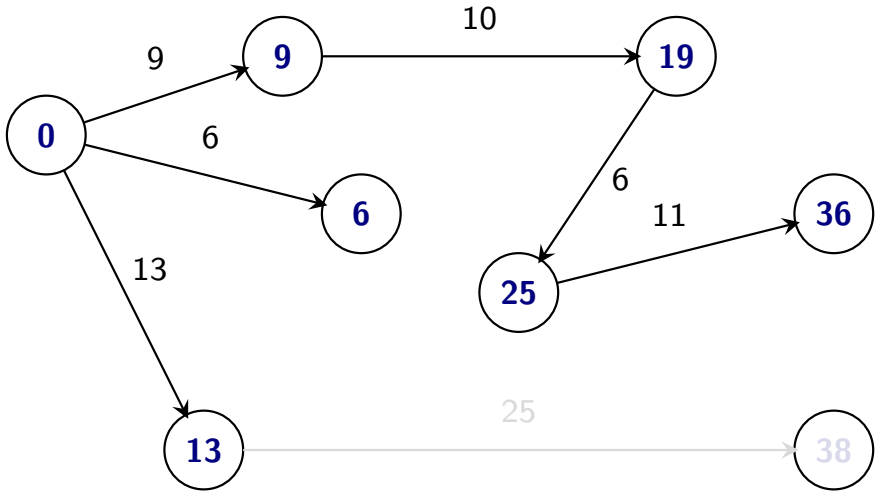
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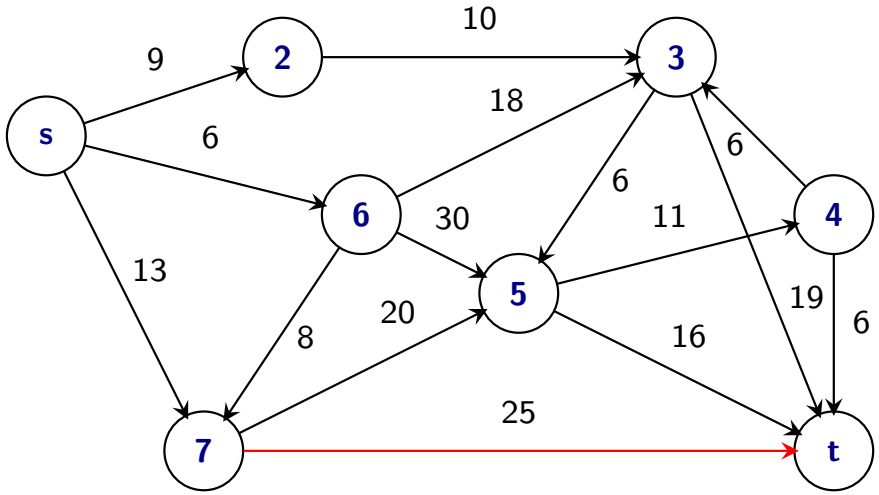
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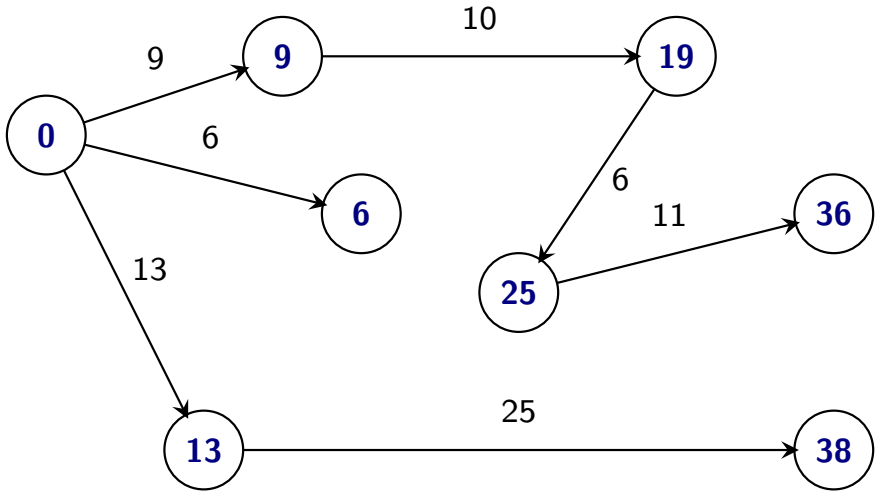
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- Main work is to compute the $d'(s,u)$ values in each iteration
- $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to S in iteration i.

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Initialize for each node v, dist(s,v) = d'(s,v) = ∞
Initialize S = ∅, d'(s,s) = 0
for i = 1 to |V| do
    // S contains the i − 1 closest nodes to s,
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    Update d'(s,u) for each u in V-S as follows:
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))
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Running time: $O(m+n^2)$ time.

- n outer iterations and in each iteration following steps
- updating $d'(s,u)$ after v added takes $O(\deg(v))$ time so total

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- updating $d'(s,u)$ after v added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters S only once
- Finding v from $d'(s,u)$ values is $O(n)$ time

- eliminate $d'(s,u)$ and let $dist(s,u)$ maintain it
- update **dist** values after adding **v** by scanning edges out of **v**

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Priority Queues to maintain **dist** values for faster running time

- Using heaps and standard priority queues: $O((m+n)\log n)$
- Using Fibonacci heaps: $O(m + n\log n)$.

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Data structure to store a set **S** of **n** elements where each element **v** $\in$ **S** has an associated real/integer key **k(v)** such that the following operations

- `makeQ`: create an empty queue
- `findMin`: find the minimum key in **S**
- `extractMin`: Remove **v** $\in$ **S** with smallest key and return it
- `add(v, k(v))`: Add new element **v** with key **k(v)** to **S**
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- `decreaseKey(v, k'(v))`: *decrease* key of **v** from **k(v)** (current key) to **k′(v)** (new key). Assumption: **k′(v)** $\leq$ **k(v)**
- `meld`: merge two separate priority queues into one

can be performed in **O(log n)** time each.

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Priority Queue operations:

- **O(n)** `insert` operations
- **O(n)** `extractMin` operations
- **O(m)** `decreaseKey` operations

Using Heaps

Store elements in a heap based on the key value

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- `extractMin, add, delete, meld` in $\mathbf{O(\log n)}$ time
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— Dijkstra's algorithm can be implemented in $\mathbf{O(n \log n + m)}$ time. If $\mathbf{m = \Omega(n \log n)}$, running time is linear in input size.

— Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

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Dijkstra's algorithm finds the shortest path distances from s to V.

Question: How do we find the paths themselves?

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for each node u ≠ s do
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    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u) ) then
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Lemma

The edge set $(\mathbf{u}, \mathbf{prev(u)})$ *is the* reverse *of a shortest path tree rooted at* **s**. *For each* **u**, *the reverse of the path from* **u** *to* **s** *in the tree is a shortest path from* **s** *to* **u**.

Proof Sketch.

- The edgeset $\{(\mathbf{u}, \mathbf{prev(u)}) \mid \mathbf{u} \in \mathbf{V}\}$ induces a directed in-tree rooted at **s** (Why?)
- Use induction on $|\mathbf{S}|$ to argue that the tree is a shortest path tree for nodes in **V**.

Dijkstra's algorithm gives shortest paths from **s** to all nodes in **V**.

How do we find shortest paths from all of **V** to **s**?

- In undirected graphs shortest path from **s** to **u** is a shortest path from **u** to **s** so there is no need to distinguish.
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