Groups in Cryptography

Groups in Cryptography

- A set S and a binary operation $\oplus$
- A group $G = (S, \oplus)$ if S and $\oplus$ satisfy:
  - Closure: If $a, b \in S$ then $a \oplus b \in S$
  - Associativity: For $a, b, c \in S$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
  - A neutral element: $e \in S$ such that $a \oplus e = e \oplus a = a$
  - Every element $a \in S$ has an inverse $\text{inv}(a) \in S$:

$$a \oplus \text{inv}(a) = \text{inv}(a) \oplus a = e$$

  - Commutativity: If $a \oplus b = b \oplus a$, then the group G is called an a commutative group or an Abelian group
  - In cryptography we deal with Abelian groups

Multiplicative Groups

- The operation $\oplus$ is a multiplication
- The neutral element is generally called the **unit element** $e = 1$
- The inverse of an element a is denoted as a^{-1}
- Multiplication of an element k times by itself is denoted as

$$a^k = \overbrace{a \cdot a \cdots a}^{k \text{ copies}}$$

- Example: $G = (\mathcal{Z}_p, * \bmod p)$ where p is prime
- The set $\mathcal{Z}_p = \{1, 2, \ldots, p-1\}$
- The operation $*$ is multiplication mod p

Multiplicative Group Example

- $G = (\mathcal{Z}_5, * \bmod 5)$
- The set $\mathcal{Z}_5 = \{1, 2, 3, 4\}$
- The operation multiplication mod 5 over $\mathcal{Z}_5$
- The unit element $e = 1$
- The multiplication table, powers and inverses

* mod 5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$2^1 = 2$
$2^2 = 2 \cdot 2 = 4$
$2^3 = 2 \cdot 2 \cdot 2 = 3$
$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 1$

$1^{-1} = 1$
$2^{-1} = 3$
$3^{-1} = 2$
$4^{-1} = 4$

Multiplicative Groups

- Example: $(\mathcal{Z}_n^*, * \bmod n)$
- The operation $*$ is multiplication mod n
- If n is prime, $\mathcal{Z}_n^* = \{1, 2, \ldots, n-1\}$
- If n is not a prime, $\mathcal{Z}_n^*$ consists of elements a with $\gcd(a, n) = 1$
- In other words, $\mathcal{Z}_n^*$ is the set of invertible elements mod n

Multiplicative Group Examples

- Consider the multiplication tables for mod 5 and mod 6

* mod 5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

* mod 6	**1**	2	3	4	**5**
1	**1**	2	3	4	**5**
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	**5**	4	3	2	**1**

- mod 5 multiplication on the set $\mathcal{Z}_5 = \{1, 2, 3, 4\}$ forms the group $\mathcal{Z}_5^*$
- mod 6 multiplication on the set $\mathcal{Z}_6 = \{1, 2, 3, 4, 5\}$ does not form a group since 2, 3 and 4 are not invertible
- However, mod 6 multiplication on the set of invertible elements forms a group: $(\mathcal{Z}_6^*, * \text{ mod } 6) = (\{1, 5\}, * \text{ mod } 6)$

Additive Groups

- The operation $\oplus$ is an addition
- The neutral element is generally called the zero element $e = 0$
- Addition of an element a k times by itself, denoted as

$$[k]\, a = \overbrace{a + \cdots + a}^{k \text{ copies}}$$

- The inverse of an element a is denoted as $-a$
- Example: $(\mathcal{Z}_n, + \bmod n)$ is a group; the set is $\mathcal{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and the operation is addition mod n

Additive Group Examples

- Consider the addition tables mod 4 and mod 5

+ mod 4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

+ mod 5	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

- mod 4 addition on $\mathcal{Z}_4 = \{0, 1, 2, 3\}$ forms the group $(\mathcal{Z}_4, + \text{ mod } 4)$
- mod 5 addition on $\mathcal{Z}_5 = \{0, 1, 2, 3, 4\}$ forms the group $(\mathcal{Z}_5, + \text{ mod } 5)$

Order of a Group

- **The order of a group** is the number of elements in the set
- The order of $(\mathcal{Z}_{11}^*, * \bmod 11)$ is 10, since the set $\mathcal{Z}_{11}^*$ has 10 elements: $\{1, 2, \ldots, 10\}$
- The order of group $(\mathcal{Z}_p^*, * \bmod p)$ is equal to $p - 1$
- Note that, since p is prime, the group order $p - 1$ is not prime
- The order of $(\mathcal{Z}_{11}, + \bmod 11)$ is 11, since the set $\mathcal{Z}_{11}$ has 11 elements: $\{0, 1, 2, \ldots, 10\}$
- The order of $(\mathcal{Z}_n, + \bmod n)$ is n, since the set $\mathcal{Z}_n$ has n elements: $\{0, 1, 2, \ldots, n - 1\}$; here n could be prime or composite

Order of an Element

- **The order of an element** a in a multiplicative group is the smallest integer k such that $a^k = 1$, where 1 is the unit element of the group
- order(3) $= 5$ in $(\mathcal{Z}_{11}^*, * \bmod 11)$ since

$$\{ \, 3^i \bmod 11 \mid 1 \le i \le 10\} = \{3, 9, 5, 4, 1\}$$

- order(2) $= 10$ in $(\mathcal{Z}_{11}^*, * \bmod 11)$ since

$$\{ \, 2^i \bmod 11 \mid 1 \le i \le 10\} = \{2, 4, 8, 5, 10, 9, 7, 3, 6, 1\}$$

- Note that order(1) $= 1$

Order of an Element

- **The order of an element** a in an additive group is the smallest integer k such that $[k]\,a = 0$, where 0 is the zero element
- order(3) in $(\mathcal{Z}_{11}, + \bmod 11)$ is computed by finding the smallest k such that $[k]\,3 = 0$
- This is obtained by successively computing

$$3 = 3, \quad 3+3 = 6, \quad 3+3+3 = 9, \quad 3+3+3+3 = 1, \quad \cdots$$

  until we obtain the zero element
- We find order(3) $= 11$ in $(\mathcal{Z}_{11}, + \bmod 11)$

$$\{\ [i]\,3 \bmod 11 \mid 1 \leq i \leq 11\} = \{3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 0\}$$

- Note that order(0) $= 1$

Lagrange's Theorem

Theorem

The order of an element divides the order of the group.

- The order of the group $(\mathcal{Z}_{11}^*, * \bmod 11)$ is equal to 10, while order(3) $= 5$ in $(\mathcal{Z}_{11}^*, * \bmod 11)$, and 5 divides 10
- order(2) $= 10$ in $(\mathcal{Z}_{11}^*, * \bmod 11)$, and 10 divides 10
- Similarly, order(1) $= 1$ in $(\mathcal{Z}_{11}^*, * \bmod 11)$, and 1 divides 10
- Since the divisors of 10 are 1, 2, 5, and 10, the element orders can only be 1, 2, 5, or 10

Lagrange Theorem

- On the other hand, order(3) $= 11$ in $(\mathcal{Z}_{11}, +$ mod $11)$, and $11|11$
- Similarly, order(2) $= 11$ in $(\mathcal{Z}_{11}, +$ mod $11)$
- We also found order(0)=1
- The order of the group $(\mathcal{Z}_{11}, +$ mod $11)$ is 11
- Since 11 is a prime number, the order of any element in this group can be either 1 or 11
- 0 is the only element in $(\mathcal{Z}_{11}, +$ mod $11)$ whose order is 1
- All other elements have the same order 11 which is the group order

Primitive Elements

- An element whose order is equal to the group order is called **primitive**
- The order of the group $(\mathcal{Z}_{11}^*, * \bmod 11)$ is 10 and order$(2) = 10$, therefore, 2 is a primitive element of the group
- order$(2) = 11$ and order$(3) = 11$ in $(\mathcal{Z}_{11}, + \bmod 11)$, which is the order of the group, therefore 2 and 3 are both primitive elements — in fact all elements of $(\mathcal{Z}_{11}, + \bmod 11)$ are primitive except 0

Theorem

The number of primitive elements in $(\mathcal{Z}_p^*, * \bmod p)$ *is* $\phi(p-1)$.

- There are $\phi(10) = 4$ primitive elements in $(\mathcal{Z}_{11}^*, * \bmod 11)$,
- The primitive elements are: 2, 6, 7, 8
- All of these elements are of order 10

Cyclic Groups and Generators

- We call a group **cyclic** if all elements of the group can be generated by repeated application of the group operation on **a single element**
- This element is called a **generator**
- Any primitive element is a generator
- For example, 2 is a generator of $(\mathcal{Z}_{11}^*, * \bmod 11)$ since

$$\{2^i \mid 1 \leq i \leq 10\} = \{2, 4, 8, 5, 10, 9, 7, 3, 6, 1\} = \mathcal{Z}_{11}^*$$

- Also, 2 is a generator of $(\mathcal{Z}_{11}, + \bmod 11)$ since

$$\{\ [i]\, 2 \bmod 11 \mid 1 \leq i \leq 11\} = \{2, 4, 6, 8, 10, 1, 3, 5, 7, 9, 0\} = \mathcal{Z}_{11}$$