Computing π

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1 Leonhard Euler (1748)

$$\frac{\pi^2}{6} = \sum_{i=1} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

2 Gottfried Wilhelm Leibniz (1674)

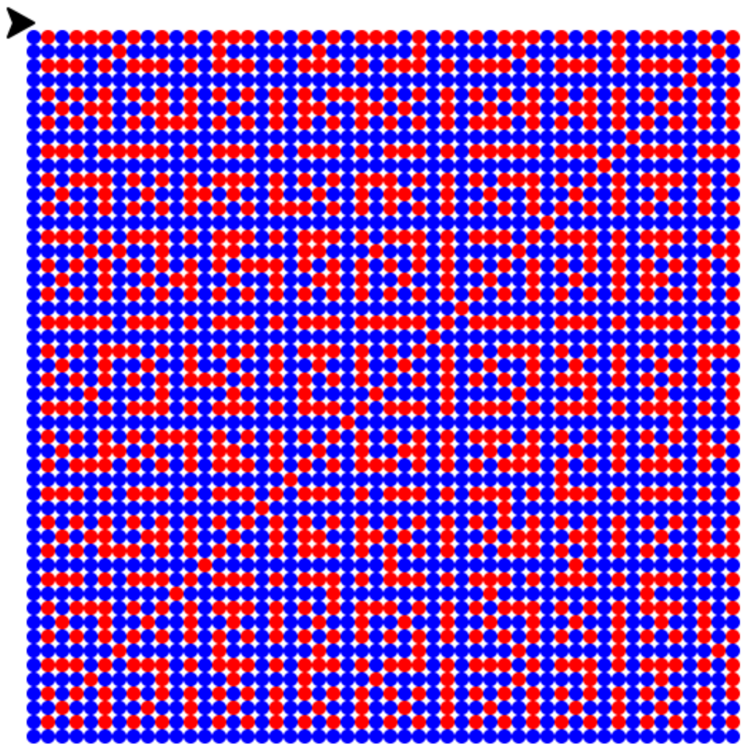
$$\frac{\pi}{4} = \sum_{i=0} \frac{(-1)^i}{2i+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

3 John Wallis (1655)

$$\frac{\pi}{2} = \prod_{i=1} \left(\frac{2i}{2i-1} \cdot \frac{2i}{2i+1} \right) = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdots$$

4 Counting Relatively Prime Pairs of Integers

Two integers are relatively prime if they share no common positive factors (divisors) except 1. Using the notation $\gcd(m,n)$ to denote the greatest common divisor, two integers m and n are relatively prime if $\gcd(m,n) = 1$. The plot below plots m and n along the two axes and colors a square blue if $\gcd(m,n) = 1$ and read otherwise.



The probability that two integers i and n picked at random are relatively prime is

$$P(\gcd(m,n) = 1) \;=\; \frac{1}{\zeta(2)} \;=\frac{6}{\pi^2} \;=\; 0.6079271018540267\cdots$$

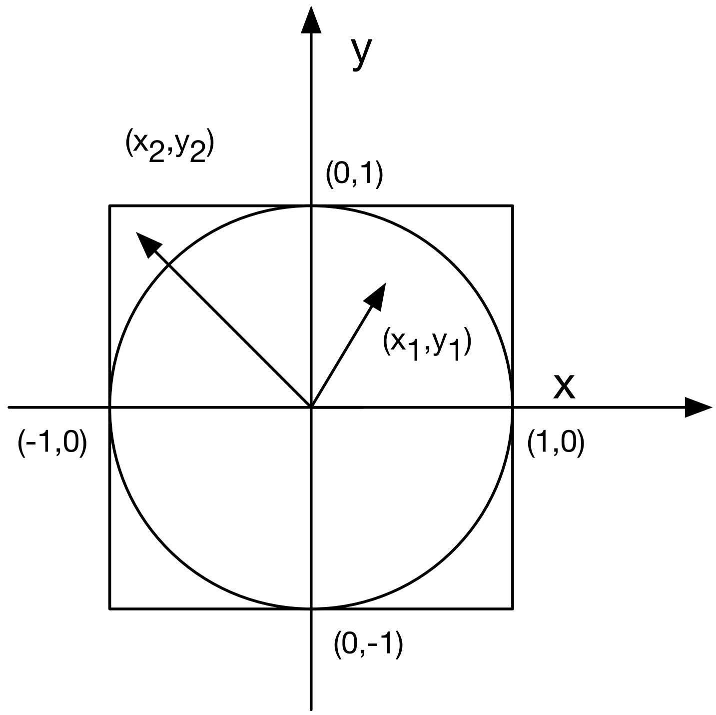
where $\zeta(z)$ is the Riemann zeta function.[1] Therefore, we can compute $\frac{6}{\pi^2}$ by scanning pairs of integers (m,n) from 1 to K, and counting the number times they are relatively prime, and dividing that by K^2.

$$\frac{6}{\pi^2} \;=\; \frac{N[\gcd(m,n) = 1;\ \text{for } m, = 1,2,\ldots,K \text{ and } n = 1,2,\ldots,K]}{K^2}$$

[1] For references, see: `http://oeis.org/A059956`; Cesáro and Sylvester 1883; Lehmer 1900; Sylvester 1909; Nymann 1972; Wells 1986; Borwein and Bailey 2003; Havil 2003; Moree 2005.

5 Monte Carlo Method

Consider the unit circle drawn inside a square of side 2:



The area of the circle: $\pi r^2 = \pi \cdot 1^2 = \pi$
The area of the square: $2^2 = 4$
The ratio of circle area to the square area: $\frac{\pi}{4}$
Consider that this figure depicts a square dart board; a dart thrown to it will either land it inside the circle (such as (x_1, y_1)) or outside the circle (such as (x_2, y_2)). Therefore, we can calculate the ratio, by throwing darts randomly to the board, and counting the number of darts falling into the circle, and then dividing this number by the total number of darts. This ratio would be approximately equal to $\pi/4$:

$$\frac{\pi}{4} = \frac{\text{Number of Darts Falling into the Circle}}{\text{Total Number of Darts}}$$

To determine whether a dart that falls on the point (x_i, y_i) is inside the circle, we check if

$$x_i^2 + y_i^2 \leq 1$$

since the radius of the circle is 1.