MINIMUM BISECTION IS FIXED PARAMETER TRACTABLE*

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Abstract. In the classic Minimum Bisection problem we are given as input an undirected graph G and an integer k. The task is to determine whether there is a partition of $V(G)$ into two parts A and B such that $||A| - |B|| \leq 1$ and there are at most k edges with one endpoint in A and the other in B. In this paper we give an algorithm for Minimum Bisection with running time $2^{\mathcal{O}(k^3)} n^3 \log^3 n$. This is the first fixed parameter tractable algorithm for Minimum Bisection parameterized by k. At the core of our algorithm lies a new decomposition theorem that states that every graph G can be decomposed by small separators into parts where each part is "highly connected" in the following sense: any separator of bounded size can separate only a limited number of vertices from each part of the decomposition. Our techniques generalize to the weighted setting, where we seek for a bisection of minimum weight among solutions that contain at most k edges.

Key words. minimum bisection, fixed-parameter tractability, graph decomposition

AMS subject classifications. 68Q25, 68R10, 68W05

1. Introduction. In the Minimum Bisection problem the input is a graph G on n vertices together with an integer k, and the objective is to find a partition of the vertex set into two parts A and B such that $|A| = \lfloor \frac{n}{2} \rfloor$, $|B| = \lceil \frac{n}{2} \rceil$, and there are at most k edges with one endpoint in A and the other endpoint in B. The problem can be seen as a variant of Minimum Cut, and is one of the classic NP-complete problems [17]. Minimum Bisection has been studied extensively from the perspective of approximation algorithms [15, 14, 22, 28], heuristics [6, 8] and average case complexity [5].

In this paper we consider the complexity of Minimum Bisection when the solution size k is small relative to the input size n. A naïve brute-force algorithm solves the problem in time $n^{\mathcal{O}(k)}$. Until this work, it was unknown whether there exists a *fixed parameter tractable* algorithm, that is an algorithm with running time $f(k)n^{\mathcal{O}(1)}$, for the Minimum Bisection problem. In fact Minimum Bisection was one of very few remaining classic NP-hard graph problems whose parameterized complexity status was unresolved. Our main result is the first fixed parameter tractable algorithm for Minimum Bisection.

Theorem 1. Minimum Bisection *admits an* $2^{\mathcal{O}(k^3)} n^3 \log^3 n$ *time algorithm.*

Theorem 1 implies that Minimum Bisection can be solved in polynomial time for $k = \mathcal{O}(\sqrt[3]{\log n})$. In fact, our techniques can be generalized to solve the more general

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problem where the target $|A|$ is given as input, the edges have non-negative weights, and the objective is to find, among all partitions of $V(G)$ into A and B such that A has the prescribed size and there are at most k edges between A and B, such a partition where the total weight of the edges between A and B is minimized.

Our methods. The crucial technical component of our result is a new graph decomposition theorem. Roughly speaking, the theorem states that for any k, every graph G may be decomposed in a tree-like fashion by separators of size $2^{\mathcal{O}(k)}$ such that each part of the decomposition is "highly connected". To properly define what we mean by "highly connected" we need a few definitions. A *separation* of a graph G is a pair $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and there are no edges between $A \setminus B$ and $B \setminus A$. The *order* of the separation (A, B) is $|A \cap B|$. A vertex set $X \subseteq V(G)$ is called (q, k)-*unbreakable* if every separation (A, B) of order at most k satisfies $|(A \setminus B) \cap X| \leq q$ or $|(B \setminus A) \cap X| \leq q$. The parts of our decomposition will be "highly connected" in the sense that they are $(2^{\mathcal{O}(k)}, k)$-unbreakable. We can now state the decomposition theorem as follows.

THEOREM 2. *There is an algorithm that given G and k runs in time $2^{\mathcal{O}(k^2)} n^2 m$ and outputs a tree decomposition (T, β) of G such that (i) for each $a \in V(T)$, $\beta(a)$ is $(2^{\mathcal{O}(k)}, k)$-unbreakable in G, (ii) for each $ab \in E(T)$ we have that $|\beta(a) \cap \beta(b)| \leq 2^{\mathcal{O}(k)}$, and $\beta(a) \cap \beta(b)$ is $(2k, k)$-unbreakable in G.*

Here $\beta(a)$ denotes the bag at node $a \in V(T)$; the completely formal definition of tree decompositions may be found in the preliminaries. It is not immediately obvious that a set X which is (q, k)-unbreakable is "highly connected". To get some intuition it is helpful to observe that if a set X of size at least $3q$ is (q, k)-unbreakable then removing any k vertices from G leaves almost all of X, except for at most q vertices, in the same connected component. In other words, one cannot separate two large chunks of X with a small separator. From this perspective Theorem 2 can be seen as an approximate way to "decompose a graph by k vertex-cuts into it's $k+1$-connected components" [9], which is considered an important quest in structural graph theory. The proof strategy of Theorem 2 is inspired by the recent decomposition theorem of Marx and Grohe [19] for graphs excluding a topological subgraph. Contrary to the approach of Marx and Grohe [19], however, the crucial technical tool we use to decompose the graph are the important separators of Marx [23].

Our algorithm for MINIMUM BISECTION applies Theorem 2 and then proceeds by performing bottom up dynamic programming on the tree decomposition. The states in the dynamic program are similar to the states in the dynamic programming algorithm for MINIMUM BISECTION on graphs of bounded treewidth [20]. Property (ii) of Theorem 2 ensures that the size of the dynamic programming table is upper bounded by $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$. For graphs of bounded treewidth all bags have small size, making it easy to compute the dynamic programming table at a node b of the decomposition tree, if the tables for the children of b have already been computed. In our setting we do not have any control over the size of the bags, we only know that they are $(2^{\mathcal{O}(k)}, k)$-unbreakable. We show that the sole assumption that the bag at b is $(2^{\mathcal{O}(k)}, k)$-unbreakable is already sufficient to efficiently compute the table at b from the tables of its children, despite no a priori guarantee on the bag's size. The essence of this step is an application of the "randomized contractions" technique [11].

We remark here that the last property of the decomposition of Theorem 2—the one that asserts that adhesions $\beta(a) \cap \beta(b)$ are $(2k, k)$-unbreakable in G—is not essential to establish the fixed-parameter tractability of MINIMUM BISECTION. This high unbreakability of adhesions is used to further limit the number of states of the dynamic programming, decreasing the dependency on k in the algorithm of Theorem 1 from double- to single-exponential.

Related work on balanced separations. There are several interesting results concerning the parameterized complexity of finding balanced separators in graphs. Marx [23] showed that the the vertex-deletion variant of the bisection problem is W[1]-hard. In Minimum Vertex Bisection the task is to partition the vertex set into three parts A, S and B such that $|S| \leq k$ and $|A| = |B|$, and there are no edges between A and B. It is worth mentioning that the hardness result of Marx [23] applies to the more general problem where $|A|$ is given as input, however the hardness of Minimum Vertex Bisection easily follows from the results presented in [23].

As the vertex-deletion variant of the bisection problem is W[1]-hard, we should not expect that our approach would work also in this case. Observe that one can compute the decomposition of Theorem 2 and define the states of the dynamic programming over the tree decomposition, as it is done for graphs of bounded treewidth. However, we are unable to perform the computations needed for one bag of the decomposition in fixed-parameter tractable time. Moreover, it is not only the artifact of the "randomized contractions" technique, but the hard instances obtained from the reduction of [23] are in fact highly unbreakable by our definition, and Theorem 2 would return a trivial decomposition.

Feige and Mahdian [16] studied cut problems that may be considered as approximation variants of Minimum Bisection and Minimum Vertex Bisection. We say that a vertex (edge) set S is an α-(edge)-separator if every connected component of $G \setminus S$ has at most αn vertices. The main result of Feige and Mahdian [16] is a randomized algorithm that given an integer k, $\frac{2}{3} \leq \alpha < 1$ and $\epsilon > 0$ together with a graph G which has an α-separator of size at most k, outputs in expected time $2^{f(\epsilon)k} n^{\mathcal{O}(1)}$ either an α-separator of size at most k or an $(\alpha + \epsilon)$-separator of size strictly less than k. They also give a deterministic algorithm with similar running time for the edge variant of this problem. To complement this result they show that, at least for the vertex variant, the exponential running time dependence on $1/\epsilon$ is unavoidable. Specifically, they prove that for any $\alpha > \frac{1}{2}$ finding an α-separator of size k is W[1]-hard, and therefore unlikely to admit an algorithm with running time $f(k)n^{\mathcal{O}(1)}$, for any function f. On the other hand, our methods imply a $2^{\mathcal{O}(k^3)} n^{\mathcal{O}(1/\alpha)}$ time algorithm for finding an α-edge-separator of size at most k, for any $\alpha > 0$.

Minimum Bisection on planar graphs was shown to be fixed parameter tractable by Bui and Peck [7]. It is interesting to note that Minimum Bisection is not known to be NP-hard on planar graphs, and the complexity of Minimum Bisection on planar graphs remains a challenging open problem. More recently, van Bevern et al. [3, 4] used the treewidth reduction technique of Marx et al. [24] to give a fixed parameter tractable algorithm for Minimum Bisection for the special case when removing the cut edges leaves a constant number of connected components. Their algorithm also works for the vertex-deletion variant under the same restrictions. Since Minimum Vertex Bisection is known to be W[1]-hard, it looks difficult to extend their methods to give a fixed parameter tractable algorithm for Minimum Bisection without any restrictions. Thus, Theorem 1 resolves an open problem stated in the conference version of the work of van Bevern et al. [3] on the existence of such an algorithm.

Related work on graph decompositions. The starting point of our decomposition theorem is the "recursive understanding" technique pioneered by Grohe et al. [18], and later used by Kawarabayashi and Thorup [21] and by Chitnis et al. [11] to design a number of interesting parameterized algorithms for cut problems. Recursive understanding can be seen as a reduction from a parameterized problem on general graphs to the same problem on graphs with a particular structure. Grohe et al. [18] essentially use recursive understanding to reduce the problem of deciding whether G contains H as a topological subgraph to the case where G either excludes a clique on $f(|H|)$ vertices as a *minor* or contains at most $f(|H|)$ vertices of degree more than $f(|H|)$, for some function f. Marx and Grohe [19] subsequently showed that any graph which excludes H as a topological subgraph can be decomposed by small separators, in a tree-like fashion, into parts such that each part either excludes a clique on $f(|H|)$ vertices as a minor or contains at most $f(|H|)$ vertices of degree more than $f(|H|)$, for some function f. Thus, the decomposition theorem of Marx and Grohe [19] can be seen as a "structural" analogue of the recursive understanding technique for topological subgraph containment.

Both Kawarabayashi and Thorup [21] and Chitnis et al. [11] apply recursive understanding to reduce certain parameterized cut problems on general graphs to essentially the same problem on a graph G where $V(G)$ is $(f(k), k)$-unbreakable for some function f. Then they proceed to show that the considered problem becomes fixed parameter tractable on $(f(k), k)$-unbreakable graphs. Observe that MINIMUM BISECTION on $(f(k), k)$-unbreakable graphs is trivially fixed parameter tractable - if the number of vertices is more than $2f(k)$ we can immediately say no, while if the number of vertices is at most $2f(k)$, then a brute force algorithm is already fixed parameter tractable. More importantly, it turns out that even the more general problem where $|A|$ is given on the input can be solved in fixed parameter tractable time on $(f(k), k)$-unbreakable graphs via an application of the "randomized contractions" technique of Chitnis et al [11]. It is therefore very natural to try to use recursive understanding in order to reduce MINIMUM BISECTION on general graphs to MINIMUM BISECTION on $(f(k), k)$-unbreakable graphs.

Unfortunately, it seems very difficult to pursue this route. In particular, recursive understanding works by cutting the graph into two parts by a small separator, "understanding" the easier of the two parts recursively, and then replacing the "understood" part by a constant size gadget. For MINIMUM BISECTION it seems unlikely that the understood part can be emulated by any constant size gadget because of the balance constraint in the problem definition. Intuitively, we would need to encode the behaviour of the understood part for every possible cardinality of A, which gives us amount of information that is not bounded by a function of k. The issue has strong connections to the fact that the best known algorithm for MINIMUM BISECTION on graphs of bounded treewidth is at least quadratic [20] rather than linear.

At this point our decomposition theorem comes into play. It precisely allows us to *structurally* decompose the graph in a tree-like fashion into $(f(k), k)$-unbreakable parts, which provides much more robust foundations for further algorithmic applications. Thus, essentially our decomposition theorem does the same for cut problems as the decomposition theorem of Marx and Grohe [19] does for topological subgraph containment. Notably, the "recursive understanding" step used by Kawarabayashi and Thorup [21] and Chitnis et al. [11] for their problems could be replaced by dynamic programming over the tree decomposition given by Theorem 2.

We remark here that it has been essentially known, and observed earlier by Chitnis, Cygan and Hajiaghayi (private communication), that MINIMUM BISECTION can be solved in FPT time on sufficiently unbreakable graphs via the "randomized contractions" technique. Furthermore, although our application of this framework to handle one bag of the decomposition is more technical than in [11], due to the presence of the information for children bags, it uses no novel tools compared to [11]. Hence, we emphasize that our main technical contribution is the decomposition theorem (Theorem 2), with the fixed-parameter algorithm for MINIMUM BISECTION being its corollary via an involved application of known techniques.

Organisation of the paper. After setting up notation and recalling useful results on (important) separators in Section 2, we turn our attention to the decomposition theorem and prove Theorem 2 in Section 3. The algorithm for MINIMUM BISECTION in the unweighted setting, promised by Theorem 1, is presented in Section 4. We discuss the weighted extension in Section 5 and how to find an α-edge-separator of size at most k in Section 6. Section 7 concludes the paper.

2. Preliminaries.

We use standard graph notation, see e.g. [13]. We use n and m to denote cardinalities of the vertex and edge sets, respectively, of a given graph provided it is clear from the context. We begin with some definitions and known results on separators and separations in graphs.

DEFINITION 3 (**separator**). *For two sets $X, Y \subseteq V(G)$ a set $W \subseteq V(G)$ is called an $X - Y$ separator if in $G \setminus W$ no connected component contains a vertex of X and a vertex of Y at the same time.*

DEFINITION 4 (**separation**). *A pair (A, B) where $A \cup B = V(G)$ is called a* separation *if $E(A \setminus B, B \setminus A) = \emptyset$. The* order *of a separation (A, B) is defined as $|A \cap B|$.*

DEFINITION 5 (**important separator**). *An inclusion-wise minimal $X - Y$ separator W is called an* important *$X - Y$ separator if there is no $X - Y$ separator W' with $|W'| \leq |W|$ and $R_{G \setminus W}(X \setminus W) \subsetneq R_{G \setminus W'}(X \setminus W')$, where $R_H(A)$ is the set of vertices reachable from A in the graph H.*

LEMMA 6 ([10, 25]). *For any two sets $S, T \subseteq V(G)$ there are at most 4^k important $S - T$ separators of size at most k and one can list all of them in $\mathcal{O}(4^k k(n + m))$ time.*

We proceed to define tree decompositions. For a rooted tree T and a non-root node $t \in V(T)$, by parent(t) we denote the parent of t in the tree T. For two nodes $u, t \in T$, we say that u is a *descendant* of t, denoted $u \preceq t$, if t lies on the unique path connecting u to the root. Note that every node is thus its own descendant.

DEFINITION 7 (**tree decomposition**). *A* tree decomposition *of a graph G is a pair (T, β), where T is a rooted tree and $\beta : V(T) \to 2^{V(G)}$ is a mapping such that:*

- *for each node $v \in V(G)$ the set $\{t \in V(G) \mid v \in \beta(t)\}$ induces a nonempty and connected subtree of T,*
- *for each edge $e \in E(G)$ there exists $t \in V(T)$ such that $e \subseteq \beta(t)$.*

The set $\beta(t)$ is called the *bag at t*, while sets $\beta(u) \cap \beta(v)$ for $uv \in E(T)$ are called *adhesions*. Following the notation from [19], for a tree decomposition (T, β) of a graph G we define auxiliary mappings $\sigma, \gamma : V(T) \to 2^{V(G)}$ as

$$\sigma(t) = \begin{cases} \emptyset & \text{if t is the root of } T \\ \beta(t) \cap \beta(\text{parent}(t)) & \text{otherwise} \end{cases}$$

$$\gamma(t) = \bigcup_{u \preceq t} \beta(u)$$

Finally, we proceed to the definition of unbreakability.

Definition 8 (**(q,k)-unbreakable set).** *Let G be an undirected graph. We say that a subset of vertices A is* (q,k)-unbreakable *in G, if for any separation (X,Y) of order at most k we have $|(X \setminus Y) \cap A| \leq q$ or $|(Y \setminus X) \cap A| \leq q$. Otherwise A is* (q,k)-breakable*, and any separation (X,Y) certifying this is called a* witnessing separation*.*

Let us repeat the intuition on unbreakable sets from the introduction. If a set X of size at least $3q$ is (q,k)-unbreakable then removing any k vertices from G leaves almost all of X, except for at most q vertices, in the same connected component. In other words, one cannot separate two large chunks of X with a small separator.

Observe that if a set A is (q,k)-unbreakable in G, then any of its subset $A' \subseteq A$ is also (q,k)-unbreakable in G. Moreover, if A is (q,k)-unbreakable in G, then A is also (q,k)-unbreakable in any supergraph of G. For a small set A it is easy to efficiently verify whether A is (q,k)-unbreakable in G, or to find a witnessing separation.

Lemma 9. *Given a graph G, a set $A \subseteq V(G)$ and an integer q one can check in $\mathcal{O}(|A|^{2q+2}k(n+m))$ time whether A is (q,k)-unbreakable in G, and if not, then find a separation (X,Y) of order at most k such that $|(X \setminus Y) \cap A| > q$ and $|(Y \setminus X) \cap A| > q$.*

Proof. Our algorithm guesses, by trying all possibilities, two disjoint subsets $X_0, Y_0 \subseteq A$ of $q+1$ vertices each. Having fixed X_0 and Y_0 we may, in $\mathcal{O}(k(n+m))$ time by applying $(k+1)$ rounds of the Ford-Fulkerson algorithm, find a minimum $X_0 - Y_0$ separator in G, or conclude that its size is larger than k. If a separator Z of size at most k exists, then obtain a separation (X', Y') as follows: set $X' \cap Y' = Z$, add connected components of $G \setminus Z$ intersecting X_0 to $X' \setminus Y'$, add connected components intersecting Y_0 to $Y' \setminus X'$, and distribute all the other connected component arbitrarily between $X' \setminus Y'$ and $Y' \setminus X'$. Observe that, since $|X' \cap Y'| \leq k$, $|X' \setminus Y'| \geq |X_0| = q+1$, and $|Y' \setminus X'| \geq |Y_0| = q+1$, the separation (X', Y') witnesses that A is (q,k)-breakable, and thus can be output by the algorithm. If none of the pairs (X_0, Y_0) admits a $X_0 - Y_0$ separator of size at most k, then we conclude that A is (q,k)-unbreakable.

It remains to argue that if A is (q,k)-breakable, then for some pair (X_0, Y_0) the minimum $X_0 - Y_0$ separator has size at most k. Indeed, let (X,Y) be any separation of order at most k witnessing that A is (q,k)-breakable, and let $X_0 \subseteq X \setminus Y$ and $Y_0 \subseteq Y \setminus X$ be any subsets of size $q+1$. Then $X \cap Y$ is an $X_0 - Y_0$ separator of size at most k. ∎

3. Decomposition. We now restate our decomposition theorem in a slightly stronger form that will emerge from the proof.

Theorem 10. *There is an $2^{\mathcal{O}(k^2)}n^2m$ time algorithm that, given a connected graph G together with an integer k, computes a tree decomposition (T,β) of G with at most n nodes such that the following conditions hold:*

(i) for each $t \in V(T)$, the graph $G[\gamma(t)] \setminus \sigma(t)$ is connected and $N(\gamma(t) \setminus \sigma(t)) = \sigma(t)$,
(ii) for each $t \in V(T)$, the set $\beta(t)$ is $(2^{\mathcal{O}(k)}, k)$-unbreakable in $G[\gamma(t)]$,
(iii) for each non-root $t \in V(T)$, we have that $|\sigma(t)| \leq 2^{\mathcal{O}(k)}$ and $\sigma(t)$ is $(2k,k)$-unbreakable in $G[\gamma(\mathrm{parent}(t))]$.

3.1. Proof overview. We first give an overview of the proof of Theorem 10, ignoring the requirement that each adhesion is supposed to be $(2k,k)$-unbreakable. As discussed in the introduction, this property is only used to improve the running time of the algorithm, and is not essential to establish fixed-parameter tractability.

We prove the decomposition theorem using a recursive approach, similar to the standard framework used for instance by Robertson and Seymour [29] or by Marx and Grohe [19]. That is, in the recursive step we are given a graph G together with a relatively small set $S \subseteq V(G)$ (i.e., of size bounded by $2^{\mathcal{O}(k)}$), and our goal is to construct a decomposition of G satisfying the requirements of Theorem 10 with an additional property that S is contained in the root bag of the decomposition. The intention is that the recursive step is invoked on some subgraph of the input graph, and the set S is the adhesion towards the decomposition of the rest of the graph.

Henceforth we focus on one recursive step, and consider three cases. In the base case, if $|S| \leq 3k$, we add an arbitrary vertex to S and repeat. In what follows, we assume $|S| > 3k$.

First, assume that S is $(2k, k)$-breakable in G, and let (X, Y) be a witnessing separation. We proceed in a standard manner (cf. [29]): we create a root bag $A := S \cup (X \cap Y)$, for each connected component C of $G \setminus A$ recurse on the subgraph induced by $N_G[C]$, where $N_G(C)$ is set to be the set S in the recursive call, and glue the obtained trees as children of the root bag. It is straightforward from the definition of the witnessing separation that in every recursive call we have $|N_G(C)| \leq |S|$. Moreover, clearly $|A| \leq |S| + k$ and hence A is appropriately unbreakable (recall that $|S| \leq 2^{\mathcal{O}(k)}$). Note that so far we have not ensured that the adhesions $N_G(C)$ are $(2k, k)$-unbreakable; this is achieved by running an additional augmentation procedure.

In the last, much more interesting case the adhesion S turns out to be $(2k, k)$-unbreakable. Hence, any separation (X, Y) in G partitions S very unevenly: almost the entire set S, up to $\mathcal{O}(k)$ elements, lies on only one side of the separation. Let us call this side the "big" side, and the second side the "small" one.

The main idea now is as follows. We would like to partition the graph into a part that is "well-attached" to S, and a part consisting of vertices that can be separated from S using a separation of small order (say, *easily separable*). The unbreakability of S ensures us that every separation of small order has a well-defined "big" side that includes almost the whole S, and a well-defined "small" side consisting of vertices that are separated from almost whole S by this separation. In order to separate all easily separable vertices at once, we do the following: for each vertex $v \in V(G)$, we mark all important separators of size $\mathcal{O}(k)$ between v and S. Then the marked vertices will separate all "small" sides of separations from the set S, thus effectively separating all easily separable vertices.

Let B be the set of marked vertices and let A be the set of all vertices of G that are either in $B \cup S$, or are not separated from S by any of the considered important separators. We observe that the strong structure of important separators—in particular, the single-exponential bound on the number of important separators for one vertex v—allows us to argue that each connected component C of $G \setminus A$ that is separated by some important separator from S has only bounded number of neighbours in A. Moreover, the fact that we cut all "small" sides of separations implies that A is appropriately unbreakable in G. Hence, we may recurse, for each connected component C of $G \setminus A$ that is separated by some important separator from S, on $(N_G[C], N_G(C))$, and take A as the root bag.

The section is organised as follows. In Section 3.2 we define the notion of *chips*, which are parts of the graph cut out by important separators, and provide all the properties needed later on. In Section 3.3 we also show how to proceed with the case of S being unbreakable, that is, how to extract the root bag containing S by cutting away all the chips. In Section 3.4 we perform some technical augmentation to ensure that the adhesions are $(2k, k)$-unbreakable. Finally in Section 3.5 we combine the obtained results and construct the main decomposition of Theorem 10.

3.2. Chips. In this subsection we define fragments of the graph which are easy to chip (i.e. cut out of the graph) from some given set of vertices S, and show their basic properties.

Definition 11 (**chips**). *For a fixed set of vertices $S \subseteq V$, a subset $C \subseteq V$ is called a* chip*, if*
(a) $G[C]$ is connected,
(b) $|N(C)| \leq 3k$,
(c) $N(C)$ is an important $C - S$ separator.
Let $\mathcal{C}$ be the set of all inclusion-wise maximal chips.

The next lemma is straightforward from the definition of important separators.

Lemma 12. *For any nonempty set $C \subseteq V(G)$ such that $G[C]$ is connected, the following conditions are equivalent.*
(i) $N(C)$ is an important $C - S$ separator;
(ii) for any $v \in C$, $N(C)$ is an important $v - S$ separator;
(iii) there exists $v \in C$ such that $N(C)$ is an important $v - S$ separator.

Note also that for a connected set of vertices D and any important $D-S$ separator Z of size at most $3k$ that is disjoint with D, the set of vertices reachable from D in $G \setminus Z$ forms a chip. We now show how to enumerate inclusion-wise maximal chips.

Lemma 13. *Given a set $S \subseteq V(G)$ one can compute the set $\mathcal{C}$ of all inclusion-wise maximal chips in time $2^{\mathcal{O}(k)}n(n+m)$. In particular, $|\mathcal{C}| \leq 4^{3k}n$.*

Proof. For any $v \in V$, we use Lemma 6 to enumerate the set $\mathcal{Z}_v$ of all important $v - S$ separators of size at most $3k$. Recall that for any $Z \in \mathcal{Z}_v$, the set $R_{G\setminus Z}(v)$ is the vertex set of the connected component of $G \setminus Z$ containing v. Define $\mathcal{A}_v = \{R_{G\setminus Z}(v) \mid Z \in \mathcal{Z}_v\}$ and let $\mathcal{C}_v$ be the set of inclusion-wise maximal elements of $\mathcal{A}_v$. By Lemma 12 we infer that if some chip $C \in \mathcal{A}_v$ is not inclusion-wise maximal, then there exists $C' \in \mathcal{A}_v$ such that $C \subsetneq C'$. Therefore, we have that $\mathcal{C} = \bigcup_{v\in V(G)} \mathcal{C}_v$.

As $|\mathcal{Z}_v| \leq 4^{3k}$ for any $v \in V(G)$, the bound on $|\mathcal{C}|$ follows. For each $v \in V(G)$, the sets $\mathcal{Z}_v$, $\mathcal{A}_v$ and $\mathcal{C}_v$ can be computed in $2^{\mathcal{O}(k)}(n+m)$ time in a straightforward manner. The computation of $\mathcal{C} = \bigcup_{v\in V(G)} \mathcal{C}_v$ in $2^{\mathcal{O}(k)}n(n+m)$ time can be done by inserting all the elements of $\bigcup_{v\in V(G)} \mathcal{C}_v$ into a prefix tree (trie), each in $\mathcal{O}(n)$ time, and ignoring encountered duplicates. ∎

Definition 14 (**chips touching**). *We say that two chips $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$,* touch *each other, denoted $C_1 \sim C_2$, if $C_1 \cap C_2 \neq \emptyset$ or $E(C_1, C_2) \neq \emptyset$.*

The following lemma provides an alternative definition of touching that we will find useful.

Lemma 15. *$C_1 \in \mathcal{C}$ touches $C_2 \in \mathcal{C}$ if and only if $N(C_1) \cap C_2 \neq \emptyset$.*

Proof. From right to left, if $v \in N(C_1) \cap C_2$ then there exists a neighbour u of v that belongs to C_1, and consequently $uv \in E(C_1, C_2)$.

From left to right, first assume $C_1 \cap C_2 \neq \emptyset$. Since $\mathcal{C}$ contains only inclusion-wise maximal chips, we have that $C_2 \setminus C_1 \neq \emptyset$. By property (a) of Definition 11 the graph $G[C_2]$ is connected, hence there is an edge between $C_2 \setminus C_1$ and $C_1 \cap C_2$ inside $G[C_2]$. This proves $N(C_1) \cap C_2 \neq \emptyset$.

In the other case, assume that $C_1 \cap C_2 = \emptyset$ but there exists $uv \in E(C_1, C_2)$ such that $u \in C_1$ and $v \in C_2$. Since $C_1 \cap C_2 = \emptyset$, it follows that $v \notin C_1$, and hence $v \in N(C_1) \cap C_2$. ∎

The next result provides the most important tool for bounding the size of adhesions in the constructed decomposition.

Lemma 16. *Any chip $C \in \mathcal{C}$ touches at most $3k \cdot 4^{3k}$ other chips of $\mathcal{C}$.*

Proof. Assume that C touches some $C' \in \mathcal{C}$. By Lemma 15 there exists a vertex $v \in N(C) \cap C'$. Observe that since $N(C')$ is an important $C' - S$ separator, then $N(C')$ is also an important $v - S$ separator. By Lemma 6 there are at most 4^{3k} important $v - S$ separators of size at most $3k$. Since $|N(C)| \leq 3k$ (by property (b) of Definition 11), we infer that C touches at most $3k \cdot 4^{3k}$ chips from $\mathcal{C}$. ∎

3.3. Local decomposition. Equipped with basic properties of chips we are ready to prove the main step of the decomposition part of the paper. In what follows we show that given a $(2k, k)$-unbreakable set S of size bounded in k one can find a (potentially large) unbreakable part $A \subseteq V$ of the graph, such that $S \subseteq A$ and each connected component of $G \setminus A$ is adjacent to a small number of vertices of A. In what follows, let us define

$$\eta = 3k \cdot (3k \cdot 4^{3k} + 1),$$
$$\tau = (3k)^2 \cdot 8^{3k} + 2k.$$

Theorem 17. *There is an $2^{\mathcal{O}(k)} nm$ time algorithm that, given a connected graph G together with an integer k and a $(2k, k)$-unbreakable set $S \subseteq V(G)$, computes a set $A \subseteq V(G)$ such that:*
(a) $S \subseteq A$,
(b) for each connected component D of $G \setminus A$ we have $|N_G(D)| \leq \eta$,
(c) A is (τ, k)-unbreakable in G, and
(d) if $|S| > 3k$, $G \setminus S$ is connected and $N(V(G) \setminus S) = S$, then $S \neq A$.

The remainder of this subsection is devoted to the proof of Theorem 17, which is broken into a few lemmas.

Let $\mathcal{C}$ be the set of inclusion-wise maximal chips, enumerated by Lemma 13. Define the set A as follows:

$$A = \left(V(G) \setminus \bigcup_{C \in \mathcal{C}} C \right) \cup \bigcup_{C \in \mathcal{C}} N(C).$$

In the definition we assume that when $\mathcal{C}$ is empty, then $A = V(G)$. See Fig. 1a for an illustration. The claimed running time of the algorithm follows directly from Lemma 13.

For property (a), note that no vertex of S is contained in a chip of $\mathcal{C}$, hence $S \subseteq A$. We now show property (d). Note that $N(V(G) \setminus S) = S$ implies $S \neq V(G)$. Consequently, if $\mathcal{C} = \emptyset$, property (d) is straightforward. Otherwise, let $C \in \mathcal{C}$. Note that $|S| > 3k$ implies that $S \setminus N(C) \neq \emptyset$ and the connectivity of $G \setminus S$ together with $N(V(G) \setminus S) = S$ further implies that $N(C) \setminus S \neq \emptyset$. Consequently, $A \setminus S \neq \emptyset$ and property (d) is proven.

We now move to the remaining two properties.

Lemma 18. *For any connected component D of $G \setminus A$ there exists a chip $C_1 \in \mathcal{C}$ such that $D \subseteq C_1$.*

Proof. Observe that a vertex which is not contained in any chip belongs to the set A, as it is either contained in $N(C)$ for some $C \in \mathcal{C}$ or it belongs to $V(G) \setminus N[C]$ for every $C \in \mathcal{C}$. Let D be an arbitrary connected component of $G \setminus A$ and let $v \in D$ be its arbitrary vertex. As $v \notin A$, there is a chip $C_v \in \mathcal{C}$ such that $v \in C_v$. Recall that by its definition the set A contains all the neighbours of all the chips in $\mathcal{C}$, hence $N(C_v) \cap D = \emptyset$ and by the connectivity of $G[D]$ we have $D \subseteq C_v$. ☐

In the following lemma we show that the set A satisfies property (b) of Theorem 17.

LEMMA 19. *For any connected component D of $G \setminus A$ it holds that $|N(D)| \leq \eta$.*

Proof. Let D be an arbitrary connected component of $G \setminus A$. By Lemma 18 there exists $C \in \mathcal{C}$ such that $D \subseteq C$. Intuitively each vertex of $N(D)$ belongs to the set A for one of two reasons: (i) it belongs to $N(C)$, or (ii) it is adjacent to a vertex of some other chip, which touches C. In both cases we show that there is only a bounded number of such vertices, which is formalized as follows.

Let v be any vertex of $N(D)$. Clearly $v \in N[C]$, hence we either have $v \in N(C)$ or $v \in C$. Suppose $v \in C$. Then, since $v \in A$, by the definition of the set A we have $v \in N(C')$ for some $C' \in \mathcal{C}$, $C' \neq C$. Consequently $v \in N(C') \cap C$, so C' touches C by Lemma 15. We infer that $N(D) \subseteq N(C) \cup \bigcup_{C' \in \mathcal{C}, C \sim C'} N(C')$. The claimed upper bound on $|N(D)|$ follows from Lemma 16. ☐

Next, we show that the set A is unbreakable. A short and informal rationale behind this property is that everything what could be easily cut out of the graph was already excluded in the definition of A.

LEMMA 20. *The set A is (τ, k)-unbreakable.*

Proof. Assume the contrary, and let (X, Y) be a witnessing separation, i.e. we have that $|X \cap Y| \leq k$, $|(X \setminus Y) \cap A| > \tau$ and $|(Y \setminus X) \cap A| > \tau$. Since S is $(2k, k)$-unbreakable, then either $|(X \setminus Y) \cap S| \leq 2k$ or $|(Y \setminus X) \cap S| \leq 2k$. Without loss of generality we assume that $|(X \setminus Y) \cap S| \leq 2k$. Let us define a set $Q = (X \cap Y) \cup (X \cap S)$ and observe that $|Q| \leq 3k$.

Note that each connected component of $G \setminus Q$ is either entirely contained in $X \setminus Y$ or in $Y \setminus X$ (see Fig. 1b). Consider connected components of the graph $G \setminus Q$ that are contained in $X \setminus Y$ and observe that they contain at least $|((X \setminus Y) \cap A) \setminus S| > \tau - 2k$ vertices of A in total. Therefore, by grouping the connected components of $G \setminus Q$ contained in $X \setminus Y$ by their neighbourhoods in Q, we infer that there exists a set of connected components $\mathcal{D} = \{D_1, \ldots, D_r\}$, such that $\forall_{1 \leq i,j \leq r} N_G(D_i) = N_G(D_j)$ and

$$(1) \qquad \left| \bigcup_{i=1}^{r} D_i \cap A \right| > \frac{\tau - 2k}{2^{3k}} = (3k)^2 \cdot 4^{3k}\,.$$

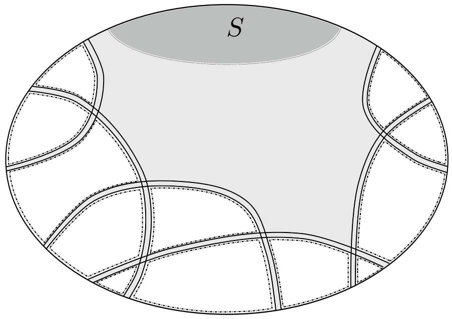
We now need the following claim.

CLAIM 21. *There is a subset $\mathcal{C}_0 \subseteq \mathcal{C}$, such that each $v \in \bigcup_{i=1}^{r} D_i \cap A$ belongs to some chip of $\mathcal{C}_0$, and there are at most $3k \cdot 4^{3k}$ chips in $\mathcal{C}$ that touch some chip of $\mathcal{C}_0$.*

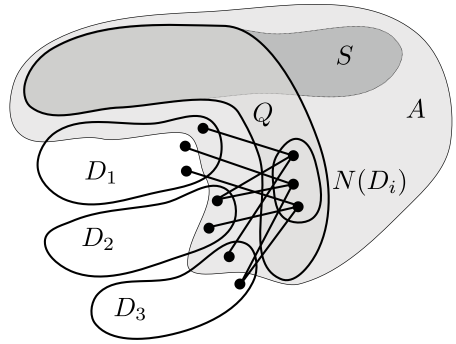
Proof. Observe that, for each $1 \leq i \leq r$, Q is a $D_i - S$ separator (see Fig. 1b) of size at most $3k$. Therefore, for each D_i there is an important $D_i - S$ separator of size at most $3k$ disjoint with D_i, hence each D_i is contained in some chip of $\mathcal{C}$. Consider two cases.

First, assume that for each $1 \leq i \leq r$ we have $D_i \in \mathcal{C}$. Define $\mathcal{C}_0$ as $\{D_1, \ldots, D_r\}$. Observe that since components D_i have the same neighbourhoods in G, by Lemma 15 each chip of $\mathcal{C}$ that touches some chip D_i touches also D_1. Therefore, by Lemma 16 there are at most $3k \cdot 4^{3k}$ chips in $\mathcal{C}$ that touch some chip of $\mathcal{C}_0$.

In the second case assume that there exist $1 \leq i_0 \leq r$ and a chip $C \in \mathcal{C}$ such that $D_{i_0} \subsetneq C$. We shall prove that for each $1 \leq i \leq r$ we have $D_i \subseteq C$. Since C is connected and $C \setminus D_{i_0}$ is non-empty, we have that $C \cap N(D_{i_0}) \neq \emptyset$. Let $C' = C \cup \bigcup_{1 \leq i \leq r} D_i$.



(a) Construction of the bag A



(b) Situation in the proof of Claim 21.

Fig. 1: Illustrations of the proof of Theorem 17

Clearly $C' \cap S = \emptyset$, and C' is connected since each component D_i is adjacent to every vertex of $C \cap N(D_{i_0})$. Moreover, as each D_i has the same neighbourhood in Q we have $|N(C')| \leq |N(C)| \leq 3k$ (see Fig. 1b). As $\mathcal{C}$ contains only maximal chips we have $C' = C$ and hence $\bigcup_{1 \leq i \leq r} D_i \subseteq C$. Define $\mathcal{C}_0$ as $\{C\}$. By Lemma 16 a single chip touches at most $3k \cdot 4^{3k}$ other chips, which finishes the proof of Claim 21. ⌟

Let $v \in A \cap D_i$ for some $1 \leq i \leq r$. Since v is contained in some $C' \in \mathcal{C}_0$, we have $v \notin V(G) \setminus N[C']$. Consequently, by the definition of the set A there exists a chip $C_v \in \mathcal{C}$ such that $v \in N(C_v)$. Note that $C' \neq C_v$ and $N(C_v) \cap C' \neq \emptyset$. Hence, by Lemma 15, C' touches C_v. By Claim 21, there are at most $3k \cdot 4^{3k}$ chips touching a chip of $\mathcal{C}_0$. As each C_v satisfies $|N(C_v)| \leq 3k$, we infer that the number of vertices of A in $\bigcup_{1 \leq i \leq r} D_i$ is at most $(3k)^2 4^{3k}$. This contradicts (1) and finishes the proof of Lemma 20. ◻

Lemma 19 and Lemma 20 ensure properties (b) and (c) of the set A, respectively. This concludes the proof of Theorem 17.

3.4. Strengthening unbreakability of adhesions. So far Theorem 17 provides us with a construction of the bag that meets almost all the requirements, apart from $(2k, k)$-unbreakability of adhesions. For this reason, in this section we want to show that the set A from Theorem 17 can be extended to a set A' in such a way that for each connected component D of $G \setminus A'$ the set $N_G(D)$ is even $(2k, k)$-unbreakable. During this extension we may weaken unbreakability of A', but if we are careful enough, then this loss will be limited to a single-exponential function of k. We start with the following recursive procedure.

Lemma 22. *Let G be a graph, and $L \subseteq V(G)$ be a subset of vertices of size at least $2k+1$. Then one can in $\mathcal{O}(|L|^{4k+3} kn(n+m))$ time find a set L', $L \subseteq L'$, such that $|L' \setminus L| \leq (|L| - 2k - 1) \cdot k$ and for each connected component D of $G \setminus L'$, we have that $|N_G(D)| \leq |L|$ and $N_G(D)$ is $(2k, k)$-unbreakable in G.*

Proof. We give a recursive procedure with the following two base cases. If $L = V(G)$, clearly we may return $L' = L$. In the second base case we assume that L is $(2k, k)$-unbreakable in G, which can be checked in $\mathcal{O}(|L|^{4k+2} k(n+m))$ time using Lemma 9. Then for each connected component D of $G \setminus L$ we have that $N_G(D) \subseteq L$, and thus $|N_G(D)| \leq |L|$ and $N_G(D)$ is also $(2k, k)$-unbreakable in G. Hence we can set $L' = L$, and since $|L| \geq 2k + 1$, we have that $|L' \setminus L| \leq (|L| - 2k - 1) \cdot k$.

Now let us assume that L is $(2k, k)$-breakable in G, and hence there exists a separation (X, Y) of G such that $|X \cap Y| \leq k$, $|(X \setminus Y) \cap L| > 2k$ and $|(Y \setminus X) \cap L| > 2k$, found by the algorithm of Lemma 9. We apply the procedure recursively for the pair $(G_1 = G[X], L_1 = (X \cap L) \cup (X \cap Y))$ and for the pair $(G_2 = G[Y], L_2 = (Y \cap L) \cup (X \cap Y))$, to obtain sets L'_1 and L'_2, respectively. Note here that $|L_1|, |L_2| \geq 2k + 1$ and $|L_1|, |L_2| < |L|$, so the sizes of sets L are smaller in the recursive subcalls. Define $L' = L'_1 \cup L'_2$. Each connected component D of $G \setminus L'$ is either a connected component of $G_1 \setminus L'_1$ and is adjacent only to L'_1, or is a connected component of $G_2 \setminus L'_2$ and is adjacent only to L'_2. Assume without loss of generality the first case. From the recursion we infer that $|N_{G_1}(D)| \leq |L_1|$ and $N_{G_1}(D)$ is $(2k, k)$-unbreakable in G_1, and since $N_{G_1}(D) = N_G(D)$, $|L_1| < |L|$, and G_1 is a subgraph of G, it follows that $|N_G(D)| \leq |L|$ and $N_G(D)$ is $(2k, k)$-unbreakable in G. It remains to argue that the cardinality of $L' \setminus L$ is not too large. Observe that

$$L' \setminus L \subseteq (L'_1 \setminus L_1) \cup (L'_2 \setminus L_2) \cup (X \cap Y)\,;$$

therefore, by the invariants given by the recursion we have

$$\begin{aligned}
|L' \setminus L| &\leq (|L_1| - 2k - 1) \cdot k + (|L_2| - 2k - 1) \cdot k + k \\
&= (|L_1| + |L_2| - 4k - 1) \cdot k \\
&\leq (|L| + 2|X \cap Y| - 4k - 1) \cdot k \\
&\leq (|L| - 2k - 1) \cdot k\,.
\end{aligned}$$

Let us now bound the running time of the presented recursive procedure. Clearly, as the size of the set L decreases in the recursive calls, the depth of the recursion is at most $|L|$. Moreover, note that any vertex may appear in $V(G) \setminus L$ in at most one recursive call (G, L) at any fixed level of the recursion tree. Hence, there are at most $|L|n$ recursive calls that do not correspond to the first base case, and, consequently, at most $2|L|n + 1$ recursive calls in total. As each recursive call takes $\mathcal{O}(|L|^{4k+2}k(n + m))$ time, the promised running time bound follows. ∎

We can now proceed to strengthen Theorem 17 by including also the procedure of Lemma 22. In the following, let

$$\tau' = \tau + \left(\binom{\tau + k}{2} \cdot k + k\right) \cdot k\eta\,.$$

Theorem 23. *There is an $2^{\mathcal{O}(k^2)}nm$ time algorithm that, given a connected graph G together with an integer k and a $(2k, k)$-unbreakable set S, computes a set $A' \subseteq V(G)$ such that the following conditions are satisfied:*

(a) $S \subseteq A'$;

(b) for each connected component D of $G \setminus A'$ the set $N_G(D)$ is $(2k, k)$-unbreakable, and $|N_G(D)| \leq \eta$;

(c) A' is (τ', k)-unbreakable in G; and

(d) if $|S| > 3k$, $G \setminus S$ is connected, and $N(V(G) \setminus S) = S$, then $S \neq A'$.

Proof. We start by finding the set A by running the algorithm Theorem 17. Next, for each connected component D of $G \setminus A$ using Lemma 9 we check whether $N(D)$ is $(2k, k)$-breakable in G. By Theorem 17, the cardinality of $N(D)$ is bounded by η, hence all tests require total running time $\mathcal{O}(\eta^{4k+2}knm) = 2^{\mathcal{O}(k^2)}nm$ time. Note that if $N(D)$ is $(2k,k)$-breakable in G, then in particular $|N(D)| > 2k$, hence we can use Lemma 22 for the pair $(G[N[D]], L_D = N(D))$; let L'_D be the obtained set. As $|L_D| \le \eta$, the algorithm of Lemma 22 runs in $\mathcal{O}(\eta^{4k+3} k |N[D]| m)$ time for a fixed component D, and total time used by calls to Lemma 22 is:

$$\sum_D \mathcal{O}(\eta^{4k+3} k(|D| + |N(D)|)m) \le \mathcal{O}(\eta^{4k+3} km) \cdot \left(\sum_D |D| + \sum_D \eta \right)$$
$$= \mathcal{O}(\eta^{4k+4} knm) = 2^{\mathcal{O}(k^2)} nm.$$

In the case when $N(D)$ is $(2k,k)$-unbreakable, we simply define $L'_D = L_D = N(D)$. Define $A' = A \cup (\bigcup_D L'_D)$, where the union is taken over all the connected components D of $G \setminus A$.

Since $S \subseteq A \subseteq A'$, we have that $S \subseteq A'$, and, moreover, the property (d) follows directly from property (d) of Theorem 17. Moreover, as $|L_D| \le \eta$ for each connected component D of $G \setminus A$, by Lemma 22 for each connected component D' of $G \setminus A'$ we also have $|N_G(D')| \le \eta$. The fact that $N_G(D')$ is $(2k,k)$-unbreakable in G follows directly from Lemma 22. It remains to show that A' is (τ', k)-unbreakable in G.

Consider any separation (X,Y) of G of order at most k. By Theorem 17 the set A is (τ,k)-unbreakable, hence either $|(X \setminus Y) \cap A| \le \tau$ or $|(Y \setminus X) \cap A| \le \tau$, and without loss of generality assume the former. As (X,Y) is an arbitrary separation of order at most k, to show that A' is (τ',k)-unbreakable it suffices to prove that $|(X \setminus Y) \cap (A' \setminus A)| \le (\binom{\tau+k}{2} \cdot k + k) \cdot k\eta$.

Note that $A' \setminus A \subseteq \bigcup_D L'_D \setminus L_D$. As for each D we have $|L'_D \setminus L_D| \le k\eta$ by Lemma 22, to finish the proof of Theorem 23 we are going to show that there are at most $\binom{\tau+k}{2} \cdot k + k$ connected components D of $G \setminus A$ such that $D \cap (X \setminus Y) \neq \emptyset$ and $L'_D \neq L_D$. As (X,Y) is of order at most k, there are at most k connected components D of $G \setminus A$ intersecting $X \cap Y$. Hence we restrict our attention to connected components D of $G \setminus A$, such that $D \subseteq X \setminus Y$, which in turn implies $N(D) \subseteq A \cap X$. Recall that if $L'_D \neq L_D$ for such a connected component D, then $N(D)$ is $(2k,k)$-breakable in G, and hence there exist two vertices $v_a, v_b \in N(D) \subseteq A \cap X$, such that the minimum vertex cut separating v_a and v_b in G is at most k. However, such a pair of vertices v_a, v_b may be simultaneously contained in neighbourhoods of at most k connected components D, since within each component D adjacent both to v_a and to v_b, we can select a path connecting them so that all the selected paths are pairwise vertex-disjoint. The theorem follows since $|A \cap X| \le \tau + k$. □

3.5. Constructing a decomposition. In this subsection we prove our main decomposition theorem, i.e., Theorem 10. However, for the inductive approach to work we need a bit stronger statement, where additionally we have a set $S \subseteq V(G)$ that has to be contained in the root bag of the tree decomposition. Note that Theorem 10 follows from the following by setting $S = \emptyset$.

THEOREM 24. *There is an $2^{\mathcal{O}(k^2)} n^2 m$ time algorithm that, given a connected graph G together with an integer k and a set $S \subseteq V(G)$ of size at most η such that $G \setminus S$ is connected and $N(V(G) \setminus S) = S$, computes a tree decomposition (T, β) such that S is contained in the root bag of the tree decomposition, and the following conditions are satisfied:*

(i) for each $t \in V(T)$, the graph $G[\gamma(t)] \setminus \sigma(t)$ is connected and $N(\gamma(t) \setminus \sigma(t)) = \sigma(t)$,
(ii) for each $t \in V(T)$, the set $\beta(t)$ is (τ', k)-unbreakable in $G[\gamma(t)]$,
(iii) for each non-root $t \in V(T)$, we have that $|\sigma(t)| \le \eta$ and $\sigma(t)$ is $(2k,k)$-unbreakable in $G[\gamma(\mathrm{parent}(t))]$.

(iv) $|V(T)| \le |V(G) \setminus S|$.

Proof. We provide a recursive procedure that computes the decomposition. The halting condition of the procedure will be clear from the running time analysis that is provided at the end of the proof.

If $|V(G)| \le \tau'$, the algorithm creates a single bag containing the entire $V(G)$. It is straightforward to verify that such a decomposition satisfies all the required properties. Thus, in the rest of the proof we assume that $|V(G)| > \tau'$, in particular, $|V(G)| > 3k$.

Define $S' = S$ and, if $|S| \le 3k$, add $3k + 1 - |S|$ arbitrary vertices of $V(G) \setminus S$ to S'. Note that, as $\eta > 3k$, we have $3k < |S'| \le \eta$.

We now define a set A' as follows. First, we verify, using Lemma 9, whether S' is $(2k, k)$-breakable in G or not. If it turns out to be $(2k, k)$-breakable in G, we apply Lemma 22 to the pair (G, S'), obtaining a set which we denote by A'. Otherwise, we can use Theorem 23 on the pair (G, S') to obtain a set A'. Note that in both cases $S \subseteq S' \subseteq A'$ and all computations so far require $2^{\mathcal{O}(k^2)}nm$ time in total.

Regardless of the way the set A' was obtained, we proceed with it as follows. For each connected component D of $G \setminus A'$, we apply the algorithm recursively to the graph $G[N[D]]$ and $S_D = N(D)$. Let us now verify (a) that each S_D is $(2k, k)$-unbreakable in G, (b) that the assumptions of the theorem are satisfied, and (c) that the recursive call is applied to a strictly smaller instance in the sense defined below.

For the first two claims, if S' is $(2k, k)$-breakable, then Lemma 22 asserts that $|S_D| \le |S'| \le \eta$ and S_D is $(2k, k)$-unbreakable in G. Otherwise, property (b) of Theorem 23 ensures that $|S_D| \le \eta$ and S_D is $(2k, k)$-unbreakable in G. The other assumptions on the set S_D required in the recursive calls, namely that $G[N[D]] \setminus S_D$ is connected and $N_{G[N[D]]}(N[D] \setminus S_D) = S_D$, follow directly from the definitions of these calls.

For the last claim, we show that either $N[D] \subsetneq V(G)$, or $N[D] = V(G)$ and $D \subsetneq V(G) \setminus S$. Assume the contrary, which implies that $D = V(G) \setminus S$ and $N(D) = S_D = S = S' = A'$. In particular, as S_D is $(2k, k)$-unbreakable in G, the set A' was obtained using Theorem 23. However, as $|S'| > 3k$, property (d) of Theorem 23 ensures that $S' \subsetneq A'$, a contradiction.

Let (T_D, β_D) be the tree decomposition obtained in the recursive call for the pair $(G[N[D]], S_D)$. Construct a tree decomposition (T, β), by creating an auxiliary node r, which will be the root of T, and attach T_D to r, by making the root r_D of T_D a child of r in T. Finally, define $\beta = \bigcup_D \beta_D$ and set $\beta(r) = A'$. A straightforward check shows that (T, β) is indeed a valid tree decomposition. We now proceed to verify its promised properties.

Since $S \subseteq S' \subseteq A'$, we have that S is indeed contained in the root bag of (T, β). For any connected component D of $G \setminus A'$, note that $\gamma(r_D) = N[D]$ and $\sigma(r_D) = N(D) = S_D$. This, together with the assumption about the correctness of the recursive calls and the fact that each S_D has size at most η and is $(2k, k)$-unbreakable in G, proves properties (i) and (iii).

If A' is obtained using Lemma 22, then $|A'| \le k|S'| \le k\eta < \tau'$, hence clearly $A' = \beta(r)$ is (τ', k)-unbreakable. In the other case, property (c) of Theorem 23 ensures the unbreakability promised in property (ii).

It remains to bound the number of bags of (T, β); as each bag is processed in $2^{\mathcal{O}(k^2)}nm$ time this would also prove the promised running time bound. Note that by property (iv) for the recursive calls we have that $|V(T_D)| \le |D|$ and, consequently, $|V(T)| \le |V(G) \setminus A'| + 1 = |V(G) \setminus S| + 1 - |A' \setminus S|$. To finish the proof of property (iv) it suffices to show that $S \subsetneq A'$. If $S \subsetneq S'$, the claim is straightforward. Otherwise, if $S =$

S' is $(2k,k)$-breakable, then the algorithm Lemma 22 cannot return $A' = S'$ as $G \setminus S'$ is connected and $N(V(G) \setminus S') = S'$ is not $(2k,k)$-unbreakable. Consequently, $S' \subsetneq A'$ in this case. In the remaining case, when $S = S'$ is $(2k,k)$-unbreakable, property (d) of Theorem 23 ensures that $S' \subsetneq A'$. This finishes the proof of Theorem 24. ☐

4. Bisection. In this section we present a dynamic programming algorithm working over the decomposition given by Theorem 10. When handling one bag of the decomposition, we essentially follow the approach of the high connectivity phase of "randomized contractions" [11]. That is, we apply colour-coding in a quite involved fashion to highlight the solution in a bag, relying heavily on its unbreakability. Then we analyse the outcome by a technical, but natural knapsack-style dynamic programming.

The section is organized as follows. First, in Section 4.1 we define an abstract problem which encapsulates the computational task one needs to perform in a single step of the dynamic programming procedure. Then the overall dynamic procedure is presented in Section 4.2.

Through this section we mostly ignore the study of factors polynomial in the graph size in the running time of the algorithm, and we use the $\mathcal{O}^\star(\cdot)$ notation. We do not optimize the exponent of the polynomial in this dependency for the following reason: this would add an unnecessary level of technicalities to the description, distracting the reader from the main points of the reasoning. Perhaps more importantly, it is in fact less relevant to the main result of this paper—the fixed-parameter tractability of Minimum Bisection. In Section 4.3 we briefly argue how to obtain the running time promised in Theorem 1.

4.1. Hypergraph painting. Hypergraph Painting is defined as follows.

Hypergraph Painting[a] (HP)
Input: Positive integers k, b, d, a multihypergraph H with hyperedges of size at most d, a partial function $\mathrm{col}_0 : V(H) \nrightarrow \{\mathbf{B}, \mathbf{W}\}$, and a function $f_F : \{\mathbf{B}, \mathbf{W}\}^F \times \{0, \ldots, b\} \to \{0, 1, \ldots, k, \infty\}$ for each $F \in E(H)$.
Goal: For each $0 \le \mu \le b$, compute the value w_μ,

$$w_\mu = \min_{\mathrm{col} \supseteq \mathrm{col}_0, (a_F)_{F \in E(H)}} \sum_{F \in E(H)} f_F(\mathrm{col}|_F, a_F),$$

where the minimum is taken over colourings $\mathrm{col} : V(H) \to \{\mathbf{B}, \mathbf{W}\}$ extending col_0 and partitions of μ into non-negative integers $\mu = \sum_{F \in E(H)} a_F$, and the sum attains value ∞ whenever its value exceeds k.

[a] We are intentionally avoiding the name Hypergraph Colouring, as it has an established, and different, meaning.

As mentioned before, Hypergraph Painting is an abstraction for one step of the final dynamic programming for Minimum Bisection, namely processing one bag of the decomposition. The vertex set $V(H)$ represents the bag itself. The hyperedges $E(H)$ essentially come from two sources. Firstly, each edge contained in the bag gives raise to one hyperedge of H of arity 2, with a simple function f_F that checks whether the edge is monochromatic or not. Second, for each subtree attached below the bag, we add one hyperedge being the corresponding adhesion. The function f_F associated with this hyperedge measures, for each colouring of the adhesion into black and white, the optimum cost (that is, the number of edges with endpoints receiving different colors) of extending this colouring into the part of the graph contained in the subtree;

the second argument is the requested number of white vertices. Thus, functions f_F for such hyperedges represent precomputed optimum values for subtrees. For the considered bag, we need to compute optimum values for all possible colourings of the top adhesion (between the bag and its parent), and hence we need the possibility of pre-coloring some vertices using the partial function col_0. Finally, the variable μ represents the total number of white vertices that we expect in a sought solution for the subtree of the decomposition rooted at the currently considered bag. As we need to be prepared for every possible balance between white and black vertices, this variable ranges over integers between 0 and some upper bound b, which will usually be the total number of vertices contained in this subtree of the decomposition.

We denote $n = |V(H)|$ and $m = |E(H)|$ throughout the analysis of the Hypergraph Painting problem. We call an instance $(k, b, d, H, \mathrm{col}_0, (f_F)_{F \in E(H)})$ a *q-proper* instance of Hypergraph Painting if the following conditions hold:

- (**local unbreakability**) for each $F \in E(H)$, each colouring $\mathrm{col} : F \to \{\mathbf{B}, \mathbf{W}\}$ marking more than $3k$ vertices of each colour, i.e. $|\mathrm{col}^{-1}(\mathbf{B})|, |\mathrm{col}^{-1}(\mathbf{W})| > 3k$, and each $0 \leq \mu \leq b$ the value $f_F(\mathrm{col}, \mu)$ equals ∞,
- (**connectivity**) for each $F \in E(H)$, each colouring $\mathrm{col} : F \to \{\mathbf{B}, \mathbf{W}\}$ that is *bichromatic*, i.e. $|\mathrm{col}^{-1}(\mathbf{B})| > 0$ and $|\mathrm{col}^{-1}(\mathbf{W})| > 0$, and each $0 \leq \mu \leq b$ the value $f_F(\mathrm{col}, \mu)$ is non-zero,
- (**global unbreakability**) for each $0 \leq \mu \leq b$ such that $w_\mu < \infty$ there is a witnessing colouring $\mathrm{col} : V(H) \to \{\mathbf{B}, \mathbf{W}\}$, which colours at most q vertices with one of the colours, i.e. $\min(|\mathrm{col}^{-1}(\mathbf{B})|, |\mathrm{col}^{-1}(\mathbf{W})|) \leq q$.

Note that, by local unbreakability, for proper instances each function f_F can be represented by at most $(2\sum_{i=0}^{3k} d^i) \cdot (b+1) \leq 4(b+1)d^{3k}$ values which are smaller than ∞. Observe, moreover, that these values can be stored in a dictionary data structure, say a trie, where for a given colouring, the array consisting of values for all $\mu \in \{0, 1, \ldots, b\}$ can be accessed in time polynomial in d and k. Then the value for a particular choice of μ can be accessed in constant time, because we work in the RAM model. This access time, being polynomial in d and k, will be always dominated by other factors in the running times of our algorithms, and hence from now on we ignore it in the running time analysis and treat the access to values of functions f_F as a constant-time operation.

We are going to use the well-established tool of fixed parameter tractability, namely the colour-coding technique of Alon, Yuster and Zwick [1]. A standard method of derandomizing the colour-coding technique is to use *splitters* of Naor et al. [27]. We present our algorithm already in its derandomized form, and for this reason we use the following abstraction of splitters.

Lemma 25 (Lemma 1 of [11]). *Given a set U of size n and integers $0 \leq a, b \leq n$, one can in time $2^{\mathcal{O}(\min(a,b)\log(a+b))} n \log n$ construct a family $\mathcal{F}$ of at most $2^{\mathcal{O}(\min(a,b)\log(a+b))} \log n$ subsets of U such that for any sets $A, B \subseteq U$ with $A \cap B = \emptyset$, $|A| \leq a$, and $|B| \leq b$, there exists a set $S \in \mathcal{F}$ with $A \subseteq S$ and $B \cap S = \emptyset$.*

We will use the following operators in order to make the description of the algorithm more concise. The first one is often called the *min-plus convolution*.

Definition 26. *For two functions $g, h : \{0, \ldots, b\} \to \{0, 1, \ldots, k, \infty\}$ we define functions $g \oplus h, \min(g, h)$ as follows:*

$$(g \oplus h)(\mu) = \min_{\mu_1 + \mu_2 = \mu} g(\mu_1) + h(\mu_2)\,,$$
$$\min(g, h)(\mu) = \min(g(\mu), h(\mu)),$$

where each integer larger than k is treated as ∞.

Note that these operators are commutative and associative, and that given two functions g, h one can compute $g \oplus h$ in $\mathcal{O}(b^2)$ time, and $\min(g, h)$ in $\mathcal{O}(b)$ time.

LEMMA 27. *There is an $q^{\mathcal{O}(k)} \cdot d^{\mathcal{O}(k^2)} \cdot |I|^{\mathcal{O}(1)}$-time algorithm solving the* HYPERGRAPH PAINTING *problem for q-proper instances I where $q \geq k$. Here, $|I|$ denotes the total size of the encoding of a q-proper instance I, which is at most $(|V(H)| + |E(H)| \cdot (b+1)(k+2)d^{\mathcal{O}(k)})^{\mathcal{O}(1)}$.*

Proof. First, let us fix the value of μ, $0 \leq \mu \leq b$. Our goal is to compute a single value w_μ[1]. Throughout the proof we work under the assumption that

$$w_\mu < \infty \,. \tag{2}$$

More precisely, the algorithm shall produce a number of finite candidates for w_μ, each not smaller than the actual value of w_μ. We will argue that under the assumption $w_\mu < \infty$, some candidates will be produced and the smallest among them will actually be equal to w_μ. Note that this implies that $w_\mu = \infty$ if and only if no candidate is produced, so if the algorithm failed to find any candidates, then it can safely conclude that $w_\mu = \infty$.

By the global unbreakability property, assumption (2) implies that there is a colouring $\mathrm{col}_{opt} \colon V(H) \to \{\mathbf{B}, \mathbf{W}\}$ that witnesses the value w_μ and colours at most q vertices with one of the colours. Without loss of generality let us assume that

$$|\mathrm{col}_{opt}^{-1}(\mathbf{W})| \leq q \,, \tag{3}$$

as the other case $|\mathrm{col}^{-1}(\mathbf{B})| \leq q$ is symmetric.

Observe that by the local unbreakability property for each $F \in E(H)$ there are at most $\ell = 4d^{3k} = d^{\mathcal{O}(k)}$ colourings of F that lead to a value of f_F which is different than ∞; call such colourings *admissible*. For each $F \in E(H)$ let us order the admissible bichromatic colourings of F arbitrarily, and for $1 \leq i \leq \ell$ let $\mathrm{col}_{F,i}$ be the i-th of the admissible colourings which is bichromatic on F (if the number of such colourings is smaller than ℓ we append the sequence with arbitrary bichromatic colourings).

We want to assign each $F \in E(H)$ to be in one of the following states:

- F is definitely monochromatic,
- F is either monochromatic, or should be coloured as in $\mathrm{col}_{F,i}$ for a fixed $1 \leq i \leq \ell$.

Formally, for an assignment $p \colon E(H) \to \{0, \ldots, \ell\}$ by $p(F) = 0$ we express the "definitely monochromatic" state, and by $p(F) = i > 0$ we express the "either monochromatic or i-th bichromatic colouring" state.

Let $E_{\mathbf{W}} = \{F \in E(H) \mid \mathrm{col}_{opt}(F) = \{\mathbf{W}\}\}$ be the multiset of monochromatic hyperedges of $E(H)$ coloured all white with respect to col_{opt}. Moreover let $E_0 \subseteq E_{\mathbf{W}}$ be any spanning forest of the hypergraph $(V(H), E_{\mathbf{W}})$. Here and later, by a spanning forest of a hypergraph we mean a subhypergraph on the same vertex set which has the same connected components (treated as subsets of vertices), while the set of hyperedges is inclusion-wise minimal subject to the first condition. By (3) we have $|E_0| \leq q$. Let $E_1 \subseteq E(H)$ be the set of hyperedges which are bichromatic with respect to col_{opt}. Note that by the connectivity property together with (2) we have $|E_1| \leq k$.

We call an assignment $p \colon E(H) \to \{0, \ldots, \ell\}$ *good*, with respect to col_{opt}, if:

[1] Actually our algorithm after a minor modification computes all the values w_μ at once, however for the sake of simplicity we focus on a single value of μ, at the cost of higher polynomial factor.

- for each $F \in E_1$ we have $p(F) > 0$ and $\mathrm{col}_{opt}|_F = \mathrm{col}_{F,p(F)}$,
- for each $F \in E_0$ we have $p(F) = 0$.

Let $\mathcal{F}$ be a family constructed by the algorithm of Lemma 25 for the universe $E(H)$ and integers k, q in time $2^{\mathcal{O}(\min(q,k)\cdot\log(q+k))} \cdot |E(H)|^{\mathcal{O}(1)} = q^{\mathcal{O}(k)} \cdot |E(H)|^{\mathcal{O}(1)}$; the last equality holds due to $q \geq k$. By the properties of $\mathcal{F}$ there exists $S_0 \in \mathcal{F}$ such that $E_1 \subseteq S_0$ and $E_0 \cap S_0 = \emptyset$. We iterate through all possible $S \in \mathcal{F}$; in one of the cases we have $S = S_0$.

In the second level of derandomization we use the standard notion of perfect families. An (N, r)-*perfect family* is a family of functions from $\{1, 2, \ldots, N\}$ to $\{1, 2, \ldots, r\}$, such that for any subset $X \subseteq \{1, 2, \ldots, N\}$ of size r, one of the functions in the family is injective on X. Naor et al. [27] gave an explicit construction of an (N, r)-perfect family of size $e^r r^{\mathcal{O}(\log r)} \log N$ using $e^r r^{\mathcal{O}(\log r)} N \log N$ time. We construct a $(|S|, k)$-perfect family $\mathcal{D}$ of size $2^{\mathcal{O}(k)} \log |S|$. Assuming that we consider the case when $S = S_0$, there exists a function $\delta_0 \in \mathcal{D}$, $\delta_0 : S_0 \to \{1, \ldots, k\}$, such that δ_0 is injective on E_1. We iterate through all possible functions $\delta \in \mathcal{D}$; provided that $S = S_0$, in one case we have $\delta = \delta_0$.

Finally, we guess, by trying all $\ell^{\mathcal{O}(k)} = d^{\mathcal{O}(k^2)}$ possibilities, a function $\delta' \colon \{1, \ldots, k\} \to \{1, \ldots, \ell\}$. In the case where $S = S_0$ and $\delta = \delta_0$, for at least one such δ' we have that $\delta'(\delta_0(F)) = i$ holds for each $F \in E_1$, where $\mathrm{col}_{opt}|_F = \mathrm{col}_{F,i}$. Consequently, if we define $p \colon E(H) \to \{0, \ldots, \ell\}$ by

$$p(F) = \begin{cases} 0 & \text{if } F \notin S; \\ \delta'(\delta(F)) & \text{otherwise,} \end{cases}$$

then in one of the $q^{\mathcal{O}(k)} \cdot d^{\mathcal{O}(k^2)} \cdot |E(H)|^{\mathcal{O}(1)}$ cases we will end up having a good assignment p in hand. The algorithm runs through all possible guesses of the assignment p and performs the same computations, yielding a candidate for w_μ per each guess of p. The smallest among the candidates is then reported as the final value of w_μ. As usual, at the end of the proof we will argue that not good guesses lead to candidate values not smaller than w_μ, whereas in the good guess we obtain the exact value w_μ. Thus, the returned value will be equal to w_μ.

For an assignment $p : E(H) \to \{0, \ldots, \ell\}$ define an auxiliary undirected simple graph L_p, with a vertex set $V(L_p) = V(H)$. For each hyperedge $F \in E(H)$ such that $p(F) = 0$ make F a clique in L_p. For each hyperedge $F \in E(H)$ such that $p(F) = i > 0$ make the sets $\mathrm{col}^{-1}_{F,i}(\mathbf{B})$, $\mathrm{col}^{-1}_{F,i}(\mathbf{W})$ cliques in L_p.

Claim 28. *If p is a good assignment, then all the vertices contained in the same connected component of L_p are coloured with the same colour in col_{opt}.*

Proof. Follows directly from the assumption that p is a good assignment and from the definition of the graph L_p. ⌟

Claim 29. *Let p be a good assignment. If D is a connected component of L_p coloured white by col_{opt}, then each hyperedge $F \in E(H)$ such that $F \cap D \neq \emptyset$ and $F \setminus D \neq \emptyset$, belongs to E_1.*

Proof. Assume that $F \notin E_1$, that is, F is monochromatic in col_{opt}. As $F \cap D \neq \emptyset$, col_{opt} needs to colour all elements of F white, and $F \in E_{\mathbf{W}}$. However, since E_0 is a spanning forest of the hypergraph $(V(H), E_{\mathbf{W}})$ and p is a good assignment, all the elements of F are contained in the same connected component of L_p, and consequently $F \subseteq D$. ⌟

Let us fix an assignment p; the reader may think of it as an assignment that is intended to be good, but we would also like to reason about it without such supposition. We exhaustively apply to p the following two cleaning operations; note that we modify also the graph L_p along with p. Here, we always prefer application of the first operation to the second; that is, the second is applied only if the first is inapplicable.

(1) If there exists a hyperedge $F \in E(H)$ such that $F \subseteq D$ for some connected component D of L_p and $p(F) > 0$, then set $p(F) = 0$.

(2) If there exist a connected component D of L_p and hyperedges F_1, F_2 (possibly $F_1 = F_2$) such that $F_1 \cap D \neq \emptyset$, $F_2 \cap D \neq \emptyset$, $p(F_1) > 0$, $p(F_2) > 0$, and $\mathrm{col}_{F_1,p(F_1)}(v_1) = \mathbf{W}$ while $\mathrm{col}_{F_2,p(F_2)}(v_2) = \mathbf{B}$ for some $v_1, v_2 \in D$ (possibly $v_1 = v_2$), then put $p(F_1) = 0$. This is because the guess on $\mathrm{col}_{F_1,p(F_1)}$ is clearly incorrect.

As in each round the number of hyperedges of $E(H)$ assigned zeros is strictly increasing, the process finishes in polynomial time. Let us check that the cleaning operations do not spoil a good assignment.

Claim 30. *Provided p is a good assignment, after the application of each cleaning operation p remains a good assignment.*

Proof. Suppose first the first cleaning operation is applied, say to a hyperedge F and a component D. By Claim 28, all the vertices in D are coloured with the same colour by col_{opt}, hence in particular all the vertices of F are coloured with the same colour in col_{opt} and definitely $F \notin E_1$. Hence it is safe to set $p(F) = 0$.

Suppose now the second cleaning operation is applied, say to a component D and hyperedge F_1 and this is witnessed by a hyperedge F_2 and vertices $v_1, v_2 \in D$ as in the formulation of the operation. Since the first cleaning operation could not be applied to F_1 and F_2, we have $F_1 \setminus D \neq \emptyset$ and $F_2 \setminus D \neq \emptyset$. Recall that $\mathrm{col}_{F_1,p(F_1)}(v_1) = \mathbf{W}$ and $\mathrm{col}_{F_2,p(F_1)}(v_2) = \mathbf{B}$. We want to show that F_1 is monochromatic in col_{opt}. Assume the contrary, i.e., $F_1 \in E_1$. Since p was a good assignment (before the operation) we have $\mathrm{col}_{opt}|_{F_1} = \mathrm{col}_{F_1,p(F_1)}$, which together with Claim 28 and $\mathrm{col}_{F_1,p(F_1)}(v_1) = \mathbf{W}$ implies that all the vertices of D are coloured white by col_{opt}. However by Claim 29 this means that $F_2 \in E_1$, and as p is a good assignment this means that col_{opt} colours all the vertices of D black, a contradiction. ⌟

Having applied the cleaning steps exhaustively, we proceed with further analysis. We call a connected component D of L_p a *black component* if there exists a hyperedge $F \in E(H)$, such that $p(F) = i > 0$ and $\mathrm{col}_{F,i}^{-1}(\mathbf{B}) \cap D \neq \emptyset$. Otherwise we call D a *potentially white* component. The next two claims show that these names are in fact meaningful.

Claim 31. *If p is a good assignment, then for any black connected component D of L_p, col_{opt} colours all vertices of D black.*

Proof. Let F be a hyperedge witnessing that D is a black component. Note that $F \nsubseteq D$, as otherwise the first cleaning operation would be applicable to F. By Claim 28, col_{opt} colours all vertices of D in the same colour. If this colour is white, then, by Claim 29, $F \in E_1$, and, as p is a good assignment, $\mathrm{col}_{F,p(F)} = \mathrm{col}_{opt}|_F$. However, this contradicts the assumption that $\mathrm{col}_{F,p(F)}$ colours some vertex of D black, and, consequently, col_{opt} colours D black. ⌟

Claim 32. *Suppose p is a good assignment. Let D be a potentially white component of L_p and let $E_D \subseteq E(H)$ be the subset of hyperedges with non-empty intersection with D. Exactly one of the following conditions holds:*

- col_{opt} *colours all vertices of D white, and for each hyperedge $F \in E_D$ either*
  - $F \nsubseteq D$, $p(F) > 0$, $F \in E_1$, $\mathrm{col}_{opt}|_F = \mathrm{col}_{F,p(F)}$, *or*

– $F \subseteq D$, $p(F) = 0$ *and* col_{opt} *colours all vertices of* F *white;*

- col_{opt} *colours* D *and each each hyperedge of* E_D *entirely black.*

Proof. Let $E' \subseteq E_D$ be the subset of hyperedges of E_D which are not fully contained in D. By Claim 28, col_{opt} colours D monochromatically.

Assume first that col_{opt} colours all vertices of D white, and consider $F \in E_D$. If $F \subseteq D$ then $p(F) = 0$ by the application of the first cleaning operation, and $F \in E_{\mathbf{W}}$. If $F \not\subseteq D$ then, by Claim 29, $F \in E_1$. Since p is good, $p(F) > 0$ and $\mathrm{col}_{opt}|_F = \mathrm{col}_{F,p(F)}$.

We are left with the case when col_{opt} colours all vertices of D black. Consider $F \in E_D$. If $p(F) = 0$ then the assumption that p is good implies that col_{opt} colours F monochromatically; as $F \cap D \neq \emptyset$, the hyperedge F is coloured black by col_{opt}. If $p(F) = i > 0$ then, since D is potentially white, $\mathrm{col}_{F,i}(v) = \mathbf{W} \neq \mathrm{col}_{opt}(v)$ for any $v \in F \cap D$. Consequently, $\mathrm{col}_{opt}|_F \neq \mathrm{col}_{F,i}$ and, since p is good, $F \notin E_1$. Therefore col_{opt} colours F monochromatically, and, since $F \cap D \neq \emptyset$, it colours F black. ⌟

Let E_{black} be the set of all hyperedges of $E(H)$ contained in black components of L_p. The following claim states that we may consider sets E_{black} and E_D for different potentially white components D independently.

Claim 33. *Every hyperedge* $F \in E(H)$ *belongs to exactly one of the sets: to* E_{black} *or to one of the sets* E_D *for potentially white connected components* D *of* L_p.

Proof. Assume first that $F \subseteq D$ for some connected component D of L_p. Then either D is black and $F \in E_{black}$, or D is potentially white and $F \in E_D$.

Assume now that F is not entirely contained in any connected component of L_p. By the construction of L_p, we have that $p(F) = i > 0$ and F intersects exactly two different components D_1, D_2 of L_p, such that w.l.o.g. $\mathrm{col}_{F,i}^{-1}(\mathbf{W}) = D_1 \cap F$ and $\mathrm{col}_{F,i}^{-1}(\mathbf{B}) = D_2 \cap F$. To prove the claim it suffices to show that (a) D_1 is potentially white, and (b) D_2 is black, as then F will belong only to E_{D_1} among the sets present in the statement of the claim. For (a), observe that otherwise the second cleaning operation would set $p(F) = 0$, and (b) follows from the definition of being black. ⌟

Armed with Claims 31, 32 and 33, we proceed to presenting the algorithm. First, we need to include the constraints imposed by the colouring col_0. To this end, for any hyperedge F, $0 \le \mu \le b$, and any colouring $\mathrm{col} : F \to \{\mathbf{B}, \mathbf{W}\}$ we set

$$\hat{f}_F(\mathrm{col}, \mu) = \begin{cases} \infty & \text{if } \exists_{v \in F}\ \mathrm{col}(v) \neq \mathrm{col}_0(v) \\ f_F(\mathrm{col}, \mu) & \text{otherwise.} \end{cases}$$

That is, we set the cost of colouring F with col as ∞ whenever col conflicts with col_0 on some vertex.

Now we handle hyperedges contained entirely in black components. For a hyperedge $F \in E_{black}$ let $\hat{f}_F^{black} : \{0, \ldots, b\} \to \{0, 1, \ldots, k, \infty\}$ be the function

$$\hat{f}_F^{black}(\mu) = \hat{f}_F(\{\mathbf{B}\}^F, \mu)\,.$$

Let $t : \{0, \ldots, b\} \to \{0, 1, \ldots, k, \infty\}$ be a function such that $t(0) = 0$ and $t(\mu) = \infty$ for $\mu > 0$. For each hyperedge $F \in E_{black}$ we update the function t by setting $t := t \oplus \hat{f}_F^{black}$. It remains to process all the hyperedges of $E(H) \setminus E_{black}$.

Consider all the potentially white components D of L_p one by one. Let $t_1, t_2 : \{0, \ldots, b\} \to \{0, 1, \ldots, k, \infty\}$ be functions such that $t_1(0) = t_2(0) = 0$ and $t_1(\mu) = t_2(\mu) = \infty$ for $\mu > 0$. We want to make t_1 represent the case when all the hyperedges of E_D are black, while t_2 is to represent the other case of Claim 32. First, for each hyperedge $F \in E_D$ set $t_1 := t_1 \oplus \hat{f}_F(\{\mathbf{B}\}^F, \cdot)$. Moreover, for each hyperedge $F \in E_D$ such that $p(F) = 0$ do $t_2 := t_2 \oplus \hat{f}_F(\{\mathbf{W}\}^F, \cdot)$, while for each hyperedge $F \in E_D$ such that $p(F) > 0$ set $t_2 := t_2 \oplus \hat{f}_F(\mathrm{col}_{F,p(F)}, \cdot)$. Finally make the update

$$t := t \oplus \min(t_1, t_2)\,.$$

This concludes the procedure computing t.

From the way we compute t it easily follows that for each $0 \leq \mu \leq b$, the value $t(\mu)$ is equal to one of the candidate costs considered in the minimum in the definition of Hypergraph Painting, unless it is equal to ∞. Precisely, we have the following claim; note that it holds also when p is not necessarily good.

Claim 34. *For each $0 \leq \mu \leq b$, there is a colouring $\mathrm{col}\colon V(H) \to \{\mathbf{B}, \mathbf{W}\}$ and nonnegative integers $(a_F)_{F \in E(H)}$ such that*

$$\sum_{F \in E(H)} a_F = \mu \qquad \textit{and} \qquad t(\mu) = \sum_{F \in E(H)} \hat{f}_F(\mathrm{col}|_F, a_F).$$

Proof. By the way we construct t via min-plus convolutions, it follows that for each hyperedge F we can choose a colouring $\mathrm{col}^F\colon F \to \{\mathbf{B}, \mathbf{W}\}$ and a nonnegative integer a_F such that

$$\sum_{F \in E(H)} a_F = \mu \qquad \text{and} \qquad t(\mu) = \sum_{F \in E(H)} \hat{f}_F(\mathrm{col}^F, a_F).$$

Also, the definition of functions taken into the convolution imply that each colouring col^F must satisfy the following:

- If $F \in E_{black}$ then $\mathrm{col}^F = \{\mathbf{B}\}^F$.
- If $F \in E_D$ for some potentially white component D of L_p, then either (a) $\mathrm{col}^F = \{\mathbf{B}\}^F$, or (b) if $p(F) = 0$ then $\mathrm{col}^F = \{W\}^F$ and if $p(F) > 0$ then $\mathrm{col}^F = \mathrm{col}_{F,p(F)}$.

Further, for every potentially white component D either option (a) holds simultaneously for all hyperedges of E_D, or option (b) holds simultaneously for all of them. Note that, by Claim 33, the cases described above are exclusive and cover all the hyperedges.

We now claim that the colourings col^F for $F \in E(H)$ are pairwise compatible, i.e., no vertex of $V(H)$ is colored white in the coloring of one hyperedge and black in the coloring of another. Note that this will conclude the proof, as then taking the union of colourings $(\mathrm{col}^F)_{F \in E(H)}$ and extending it in any way to vertices not contained in any hyperedge yields a colouring $\mathrm{col}\colon V(H) \to \{\mathbf{B}, \mathbf{W}\}$ such that $\mathrm{col}^F = \mathrm{col}|_F$ for each $F \in E(H)$. It follows that this colouring satisfies the statement of the claim.

For the sake of contradiction, suppose there is a vertex $u \in V(H)$ and two hyperedges F_1, F_2, both containing u, such that $\mathrm{col}^{F_1}(u) = \mathbf{W}$ and $\mathrm{col}^{F_2}(u) = \mathbf{B}$. Then F_1 belongs to E_D for some potentially white component D of L_p.

Suppose first that $p(F_1) = 0$. Then F_1 is a clique in L_p, hence all the vertices of F_1, in particular u, belong to D. This implies that $F_2 \in E_D$. Since col^{F_1} colors u white, the colourings for hyperedges of E_D are all chosen according to the option (b) above. This in particular applies to F_2, so since col^{F_2} colors u black, we have that $p(F_2) > 0$ and $\mathrm{col}^{F_2} = \mathrm{col}_{F_2,p(F_2)}$. As $\mathrm{col}_{F_2,p(F_2)}(u) = \mathbf{B}$, the hyperedge F_2 witnesses that D is a black component, contrary to the assumption that it is potentially white.

Suppose now that $p(F_1) > 0$. Since col^{F_1} colors u white, it follows that the colourings for hyperedges of E_D are all chosen according to the option (b) above, and in particular $\mathrm{col}^{F_1} = \mathrm{col}_{F_1,p(F_1)}$. In the definition of L_p we have made $\mathrm{col}^{-1}_{F_1,p(F_1)}(\mathbf{W})$ and $\mathrm{col}^{-1}_{F_1,p(F_1)}(\mathbf{B})$ into two cliques that form a partition of F_1. Each of these cliques is either entirely contained in D or entirely disjoint from D, and at least one of them must be contained in D, because $F_1 \in E_D$. However, if $\mathrm{col}^{-1}_{F_1,p(F_1)}(\mathbf{B})$ was nonempty and contained in D, then F_1 would witness that D should be a black component, contradicting the assumption that it is potentially white. Therefore, $\mathrm{col}_{F_1,p(F_1)}$ colours $F_1 \cap D$ entirely white and $F_1 \setminus D$ entirely black, and in particular $u \in \mathrm{col}^{-1}_{F_1,p(F_1)}(\mathbf{W}) \subseteq D$. Since $u \in F_2$, we infer that $F_2 \in E_D$, so in particular F_2 is coloured according to the option (b) above. If we had $p(F_2) > 0$, then F_2 would witness that D should be a black component, contradicting the assumption that it is potentially white. Otherwise $p(F_2) = 0$ and, according to (b) above, col^{F_2} colours F_2 entirely white, a contradiction with $\mathrm{col}^{F_2}(u) = \mathbf{B}$. ⌟

The procedure for computing t that we described above is applied for all the considered guesses of p, and the value of $t(\mu)$ is recorded as a candidate for w_μ. Recall that at the beginning we have assumed that $|\mathrm{col}^{-1}_{opt}(\mathbf{W})| \leq q$, while there is also the symmetric case $|\mathrm{col}^{-1}_{opt}(\mathbf{B})| \leq q$. Therefore, in addition we execute the same algorithm with the roles of colours swapped, yielding a second family of candidates for w_μ. Finally, we claim that the smallest among the obtained candidates is equal to w_μ. This easily follows from Claims 31, 32, 33, and 34, as explained next.

On one hand, by Claim 34, $t(\mu)$ is equal to $\sum_{F\in E(H)} \hat{f}_F(\mathrm{col}|_F, a_F)$ for some colouring col and nonnegative integers $(a_F)_{F\in E(H)}$ that sum up to μ. This cost is ∞ if col does not extend col_0, and otherwise it is equal to $\sum_{F\in E(H)} f_F(\mathrm{col}|_F, a_F)$, which matches the definition of the Hypergraph Painting problem. Since w_μ is defined to be the minimum among all such costs, we infer that each value $t(\mu)$ taken as the candidate for w_μ is actually not smaller than w_μ, and hence we never report a value smaller than w_μ.

On the other hand, by symmetry suppose that indeed $|\mathrm{col}^{-1}_{opt}(\mathbf{W})| \leq q$, and consider the guess when p is good. Then, from Claims 31, 32, and 33 it follows that some computation path of the presented dynamic programming, that is, a sequence of choices whether to take the colouring corresponding to t_1 or t_2 in consecutive steps, leads to the discovery of col_{opt}. Hence the candidate value found for a good p cannot be larger than w_μ. We conclude that the value reported by the algorithm is neither larger nor smaller than w_μ, so it is equal to w_μ. □

4.2. Dynamic programming. In this section we show that by constructing a tree decomposition from Theorem 10 and invoking the algorithm of Lemma 27 one can solve the Minimum Bisection problem in $\mathcal{O}^\star(2^{\mathcal{O}(k^3)})$ time, proving Theorem 1.

Proof of Theorem 1. First note that, without loss of generality, we may focus on the following variant: the input graph G is required to be connected, and our goal is to partition $V(G)$ into parts A and B of prescribed size minimizing $|E(A,B)|$. The algorithm for the classic Minimum Bisection problem follows from a standard knapsack-type dynamic programming on connected components of the input graph.

As the input graph is connected, we may use Theorem 10. Let (T,β) be a tree decomposition constructed by the algorithm of Theorem 10 in $\mathcal{O}^\star(2^{\mathcal{O}(k^2)})$ time.

As is usual for tree decompositions, we will use a dynamic programming approach. For a node $t \in V(T)$ of the tree decomposition, an integer μ, $0 \leq \mu \leq n$, and a colouring $\mathrm{col}_0 : \sigma(t) \to \{\mathbf{B}, \mathbf{W}\}$ satisfying

$$\min(|\mathrm{col}_0^{-1}(\mathbf{B})|, |\mathrm{col}_0^{-1}(\mathbf{W})|) \leq 3k\,, \tag{4}$$

we consider a variable $x_{t,\mathrm{col}_0,\mu}$.

The variable $x_{t,\mathrm{col}_0,\mu}$ equals the minimum cardinality of a set $Z \subseteq E(G[\gamma(t)])$, such that there exists a colouring $\mathrm{col} : \gamma(t) \to \{\mathbf{B}, \mathbf{W}\}$, where $\mathrm{col}|_{\sigma(t)} = \mathrm{col}_0$, no edge of $E(G[\gamma(t)]) \setminus Z$ is incident to two vertices of different colours in col, and the total number of white vertices equals μ, i.e. $|\mathrm{col}^{-1}(\mathbf{W})| = \mu$. Additionally if it is impossible to find such a colouring col, or the number of edges one needs to include in Z is greater than k, then we define $x_{t,\mathrm{col}_0,\mu} = \infty$. The fact that we require col_0 to satisfy inequality (4), and we do not browse through all possible colourings col_0 of $\sigma(t)$, will be used to optimize the running time.

As Theorem 10 upper bounds the cardinality of $\sigma(t)$ by $2^{\mathcal{O}(k)}$ and of $V(T)$ by n, the total number of values $x_{t,\mathrm{col}_0,\mu}$ we want to compute is $2^{\mathcal{O}(k^2)}n^2$. Note that having all those values is enough to solve the considered variant of the Minimum Bisection problem as the minimum possible size of the cut $E(A,B)$ equals $x_{r,\emptyset,a}$, where r is the root of (T,β), $\emptyset$ plays the role of the single colouring of $\sigma(r) = \emptyset$ and a is the prescribed size of one part of the partition we are looking for. The value $x_{r,\emptyset,a}$ attains ∞ if any feasible cut $E(A,B)$ is of size larger than k. We will compute the values $x_{t,\cdot,\cdot}$ in a bottom-up manner, that is our computation is performed for a node $t \in V(T)$ only after all the values $x_{t',\cdot,\cdot}$ for $t' \prec t$ have been already computed.

Consider a fixed $t \in V(T)$ and a colouring $\mathrm{col}_0 : \sigma(t) \to \{\mathbf{B}, \mathbf{W}\}$ satisfying (4). In what follows we show how to find all the values $x_{t,\mathrm{col}_0,\cdot}$ by solving a single q-proper instance of the Hypergraph Painting problem, for an appropriate q. Create an auxiliary hypergraph H, with a vertex set $V(H) = \beta(t)$ and the following set of hyperedges; in the following we use Iverson notation, i.e., $[\varphi]$ is equal to 1 if the condition φ is true and 0 otherwise.

(a) For each vertex $v \in \beta(t)$ add to H a hyperedge $F = \{v\}$, and define a function $f_F : \{\mathbf{B}, \mathbf{W}\}^F \times \{0, \ldots, n\} \to \{0, 1, \ldots, k, \infty\}$

$$f_F(\mathrm{col}_F, \mu) = \begin{cases} 0 & \text{if } \mu = [\mathrm{col}_F(v) = \mathbf{W}], \\ \infty & \text{otherwise.} \end{cases}$$

We introduce those hyperedges in order to keep track of the number of white vertices in $\beta(t)$.

(b) For each edge $uv \in E(G[\beta(t)])$ add to H a hyperedge $F = \{u, v\}$, and define a function $f_F : \{\mathbf{B}, \mathbf{W}\}^F \times \{0, \ldots, n\} \to \{0, 1, \ldots, k, \infty\}$

$$f_F(\mathrm{col}_F, \mu) = \begin{cases} [\mathrm{col}_F(u) \neq \mathrm{col}_F(v)] & \text{if } \mu = 0, \\ \infty & \text{otherwise.} \end{cases}$$

We introduce those hyperedges in order to keep track of the number of edges with endpoints of different colours in $G[\beta(t)]$.

(c) For each $t' \in V(T)$ which is a child of t in the tree decomposition add to H a hyperedge $F = \sigma(t')$, and define a function $f_F : \{\mathbf{B}, \mathbf{W}\}^F \times \{0, \ldots, n\} \to \{0, 1, \ldots, k, \infty\}$ as follows

$$f_F(\mathrm{col}_F, \mu) = \begin{cases} \infty & \text{if } \min(|\mathrm{col}_F^{-1}(\mathbf{B})|, |\mathrm{col}_F^{-1}(\mathbf{W})|) > 3k, \\ \infty & \text{if } x_{t',\mathrm{col}_F,\mu+\mu_0} = \infty, \\ x_{t',\mathrm{col}_F,\mu+\mu_0} - x_0 & \text{otherwise,} \end{cases}$$

where $\mu_0 = |\mathrm{col}_F^{-1}(\mathbf{W})|$, and $x_0 = |\{uv \in E(G[\sigma(t')]) : \mathrm{col}_F(u) \neq \mathrm{col}_F(v)\}|$. Less formally, we are shifting the values by x_0 and μ_0 in order not to overcount white vertices of $\sigma(t')$ and edges of $G[\sigma(t')]$ having endpoints of different colours in col_F, as a vertex of $\sigma(t')$ might appear in several bags, and similarly an edge of $G[\sigma(t')]$ may have both endpoints in several bags being children of t'.

Note that each of the hyperedges of H is of size at most η (by Theorem 10, or more precisely the technical statement of Theorem 24), hence

$$I = (k, n, \eta, H, \mathrm{col}_0, (f_F)_{F\in E(H)})$$

is an instance of the Hypergraph Painting problem. In the following, we will refer to the hyperedges of H introduced in consecutive points above as to hyperedges of type (a), (b), and (c), respectively.

Claim 35. *Let $(w_\mu)_{0\leq\mu\leq n}$ be the solution for the instance I of the* Hypergraph Painting *problem. Then for any $0 \leq \mu \leq n$ we have $x_{t,\mathrm{col}_0,\mu} = w_\mu$.*

Moreover for any colouring $\mathrm{col} : \beta(t) \to \{\mathbf{B}, \mathbf{W}\}$ witnessing $w_\mu \leq k$ there is an extension $\mathrm{col}' : \gamma(t) \to \{\mathbf{B}, \mathbf{W}\}$, such that $\mathrm{col}'|_{\beta(t)} = \mathrm{col}$, and the number of bichromatic edges of $G[\gamma(t)]$ with respect to col' equals w_μ.

Proof. Fix an arbitrary $0 \leq \mu \leq n$. First, we show that $x_{t,\mathrm{col}_0,\mu} \geq w_\mu$. Note that the inequality holds trivially for $x_{t,\mathrm{col}_0,\mu} = \infty$, hence let us assume $x_{t,\mathrm{col}_0,\mu} \leq k$ and let $\mathrm{col} : \gamma(t) \to \{\mathbf{B}, \mathbf{W}\}$ be a colouring such that

- $\mathrm{col}|_{\sigma(t)} = \mathrm{col}_0$,
- $|Z| \leq k$, where $Z = \{uv \in E(G[\gamma(t)]) \mid \mathrm{col}(u) \neq \mathrm{col}(v)\}$,
- $|\mathrm{col}^{-1}(\mathbf{W})| = \mu$.

Recall that Theorem 10 ensures that for any child t' of t in the tree decomposition the adhesion $\sigma(t')$ is $(2k, k)$-unbreakable in $G[\gamma(t)]$. Therefore,

$$\min(|\mathrm{col}^{-1}(\mathbf{B}) \cap \sigma(t')|, |\mathrm{col}^{-1}(\mathbf{W}) \cap \sigma(t')|) \leq 3k\,,$$

as otherwise $(X = N_{G[\gamma(t)]}[\mathrm{col}^{-1}(\mathbf{B})], Y = \mathrm{col}^{-1}(\mathbf{W}))$ would be a separation of $G[\gamma(t)]$ of order at most k with $|(X \setminus Y) \cap \sigma(t')|, |(Y \setminus X) \cap \sigma(t')| > 2k$, contradicting the fact that $\sigma(t')$ is $(2k, k)$-unbreakable in $G[\gamma(t)]$. Consequently, the values $x_{t',\mathrm{col}|_{\sigma(t')},\cdot}$ are well-defined, i.e., $\mathrm{col}|_{\sigma(t')}$ satisfies (4). Furthermore, observe that

$$x_{t',\mathrm{col}|_{\sigma(t')},|\mathrm{col}^{-1}(\mathbf{W})\cap\gamma(t')|} \leq |Z \cap E(G[\gamma(t')])|, \tag{5}$$

which is witnessed by the colouring $\mathrm{col}|_{\gamma(t')}$. For the hyperedge $F \in E(H)$ of type (c) that corresponds to t', we define $a_F = |\mathrm{col}^{-1}(\mathbf{W}) \cap (\gamma(t') \setminus \sigma(t'))|$. For hyperedges $F \in E(H)$ of type (b), we simply set $a_F = 0$. Finally, for hyperedges $F \in E(H)$ of type (a), say $F = \{v\}$ for some vertex v, we put $a_F = [\mathrm{col}(v) = \mathbf{W}]$. In other words, a_F is equal to 0 if v is black under col, and 1 if v is white under col. From the definition of $(a_F)_{F\in E(H)}$ it readily follows that $\sum_{F\in E(H)} a_F = |\mathrm{col}^{-1}(\mathbf{W})| = \mu$. This is because every vertex $v \in \gamma(t)$ coloured white under col contributes 1 to exactly one summand a_F: if $v \in \beta(t)$ then v contributes to $a_{\{v\}}$ for the unique type-(a) hyperedge $\{v\}$, and if $v \in \gamma(t') \setminus \sigma(t')$ for some child t' of t, then v contributes to a_F, where F is the hyperedge of type (c) corresponding to t'.

Next, we verify that $\mathrm{col}|_{\beta(t)}$ and $(a_F)_{F\in E(H)}$ certify that $x_{t,\mathrm{col}_0,\mu} \geq w_\mu$. For this, we split the contributions of hyperedges of $E(H)$ to the sum $\sum_{F\in E(H)} f_F(\mathrm{col}|_F, a_F)$ into three summands, according to the types of hyperedges of $E(H)$. The hyperedges of type (a) do not contribute to the sum at all. The hyperedges of type (b) contribute exactly $|Z \cap E(G[\beta(t)])|$, while the hyperedges of type (c) contribute exactly

$$\sum_{t'} x_{t',\mathrm{col}|_{\sigma(t')},a_F+|\mathrm{col}^{-1}(\mathbf{W})\cap\sigma(t')|} - |Z \cap E(G[\sigma(t')])|,$$

which is not larger than

$$\sum_{t'} |Z \cap (E(G[\gamma(t')]) \setminus E(G[\sigma(t')]))|;$$

here, the sum is over all children t' of t and the inequality follows from (5). This means that each hyperedge of Z is counted exactly once, so the total contribution is at most $|Z| = x_{t,\mathrm{col}_0,\mu}$.

In the other direction, we want to show $x_{t,\mathrm{col}_0,\mu} \leq w_\mu$. As in the previous case, for $w_\mu = \infty$ the inequality holds trivially. Hence, we assume $w_\mu \leq k$. Let $\mathrm{col} : \beta(t) \to \{\mathbf{B}, \mathbf{W}\}$ be a colouring and $\sum_{F \in E(H)} a_F$ be a partition of μ that, in conjunction, witness the value of w_μ, i.e., they satisfy

- $\mathrm{col}|_{\sigma(t)} = \mathrm{col}_0$,
- $\sum_{F \in E(H)} a_F = \mu$,
- $w_\mu = \sum_{F \in E(H)} f_F(\mathrm{col}|_F, a_F)$.

Our goal is to extend the colouring col on $\gamma(t) \setminus \beta(t)$, so that the total number of white vertices equals μ and the number of bichromatic edges equals w_μ. Initially set $\mathrm{col}' = \mathrm{col}$ and consider children t' of t in the tree decomposition one by one. Let $F = \sigma(t') \in E(H)$ be a type (c) hyperedge of H. Since $w_\mu \leq k$, we have $f_F(\mathrm{col}|_F, a_F) \leq k$, and by the definition of f_F

$$f_F(\mathrm{col}|_F, a_F) = x_{t',\mathrm{col}|_F, a_F - \mu_0} - x_0\,,$$

where $\mu_0 = |\mathrm{col}^{-1}(\mathbf{W}) \cap \sigma(t')|$, and $x_0 = |\{uv \in E(G[\sigma(t')]) : \mathrm{col}(u) \neq \mathrm{col}(v)\}|$. Let $\mathrm{col}_F : \gamma(t') \to \{\mathbf{B}, \mathbf{W}\}$ be the colouring witnessing the value $x_{t',\mathrm{col}|_F, a_F - \mu_0}$. Note that col_F is consistent with col on $F = \sigma(t')$, so we can update col' by setting $\mathrm{col}' = \mathrm{col}' \cup \mathrm{col}_F$.

Observe that the hyperedges of $E(H)$ of type (a) together with shifting by μ_0 ensure that col' colours exactly μ vertices white. Finally, the hyperedges of $E(H)$ of type (b) together with shifting by x_0 ensure that col' has exactly w_μ bichromatic edges, which shows $x_{t,\mathrm{col}_0,\mu} \leq w_\mu$. As $\mathrm{col}'|_{\beta(t)} = \mathrm{col}$ the last part of the claim follows as well. ⌟

The previous claim shows that solving the Hypergraph Painting instance I is enough to find the values $x_{t,\mathrm{col}_0,\cdot}$, however in the previous section we have only shown how to solve q-proper instances of Hypergraph Painting, for bounded q. Therefore, we show that there is a small enough value of q, such that I becomes a q-proper instance.

Claim 36. *There is some q with $q = 2^{\mathcal{O}(k)}$ and $q \geq k$, such that I is a q-proper instance of the* Hypergraph Painting *problem.*

Proof. For the hyperedges $F \in E(H)$ of size at most two the local unbreakability property is trivially satisfied, while for all the other hyperedges $F = \sigma(t')$ local unbreakability follows directly from the definition of f_F.

By Theorem 10 each $G[\gamma(t')] \setminus \sigma(t')$ is connected and $N(\gamma(t') \setminus \sigma(t')) = \sigma(t')$, which means that the graph $G[\gamma(t')] \setminus E(G[\sigma(t')])$ is connected, and consequently $x_{t',\mathrm{col}_F,\cdot} > 0$ for any colouring col_F which uses both colours (as we need to remove at least one edge). This proves the connectivity property.

Recall, that by Theorem 10 the set $\beta(t)$ is (τ', k)-unbreakable in $G[\gamma(t)]$ for some $\tau' = 2^{\mathcal{O}(k)}$. Let $q = \tau' + k$ and let $(w_\mu)_{0 \leq \mu \leq n}$ be a solution for the instance I of the Hypergraph Painting problem. Consider an arbitrary $0 \leq \mu \leq n$ such that $w_\mu \leq k$. We want to show, that there exists a witnessing colouring $\mathrm{col} : \beta(t) \to$

$\{\mathbf{B}, \mathbf{W}\}$ certifying the global unbreakability. In fact we will show that any colouring $\mathrm{col} : \beta(t) \to \{\mathbf{B}, \mathbf{W}\}$ witnessing w_μ satisfies

$$\min(|\mathrm{col}^{-1}(\mathbf{B})|, |\mathrm{col}^{-1}(\mathbf{W})|) \le q = \tau' + k\,.$$

By Claim 35[2] there is an extension col' of col, having $w_\mu \le k$ bichromatic edges of $G[\gamma(t)]$. Note that $(X = N_{G[\gamma(t)]}[\mathrm{col}'^{-1}(\mathbf{B})], Y = \mathrm{col}'^{-1}(\mathbf{W}))$ is a separation of $G[\gamma(t)]$ of order at most k, hence by (τ', k)-unbreakability of $\beta(t)$ we have

$$\min(|(X \setminus Y) \cap \beta(t)|, |(Y \setminus X) \cap \beta(t)|) \le \tau'\,.$$

However $|\mathrm{col}^{-1}(\mathbf{B})| \le |(X \setminus Y) \cap \beta(t)| + k$ and $|\mathrm{col}^{-1}(\mathbf{W})| \le |(Y \setminus X) \cap \beta(t)| + k$, which implies

$$\min(|\mathrm{col}^{-1}(\mathbf{B})|, |\mathrm{col}^{-1}(\mathbf{W})|) \le k + \tau' = q\,,$$

proving the global unbreakability property. ⌟

By Claim 36 we can use Lemma 27 and in $q^{\mathcal{O}(k)} \cdot \eta^{\mathcal{O}(k^2)} \cdot |I|^{\mathcal{O}(1)} = \mathcal{O}^\star(2^{\mathcal{O}(k^3)})$ time compute the values w_μ for each $0 \le \mu \le n$. At the same time Claim 35 shows that $x_{t,\mathrm{col}_0,\mu} = w_\mu$ for each $0 \le \mu \le n$. Since the number of nodes of T is at most n and the number of colourings obeying (4) is $2^{\mathcal{O}(k^2)}$ the whole dynamic programming routine takes $\mathcal{O}^\star(2^{\mathcal{O}(k^3)})$ time. Consequently Theorem 1 follows, apart from the precise dependency on the size of G in the running time. We discuss how to make this dependency as low as $\mathcal{O}(n^3 \log^3 n)$ in the next section. □

4.3. Dependency on the size of G in the running time. Here we argue about the factors polynomial in the size of G in the running time of the algorithm.

We first note that we may assume that $m = |E(G)| = \mathcal{O}(kn)$, by applying the sparsification technique of Nagamochi and Ibaraki [26].

LEMMA 37 ([26]). *Given an undirected graph G and an integer k, in $\mathcal{O}(k(|V(G)|+|E(G)|))$ time we can obtain a set of edges $E_0 \subseteq E(G)$ of size at most $(k+1)(|V(G)|-1)$, such that for any edge $uv \in E(G) \setminus E_0$ in the graph $(V(G), E_0)$ there are at least $k+1$ edge-disjoint paths between u and v.*

Proof. The algorithm performs exactly $k+1$ iterations. In each iteration it finds a spanning forest F of the graph G, adds all the edges of F to E_0 and removes all the edges of F from the graph G.

Observe that for any edge uv remaining in the graph G, the vertices u and v are in the same connected components in each of the forests found. Hence in each of those forests we can find a path between u and v; thus, we obtain $k+1$ edge-disjoint paths between u and v. □

The above lemma allows us to sparsify the graph, so that it contains $\mathcal{O}(kn)$ edges, and any edge cut of size at most k remains in the graph, while any edge cut with at least $k+1$ edges after sparsification still has at least $k+1$ edges. Therefore applying Lemma 37 gives us an equivalent instance $(V(G), E_0)$ and consequently the construction of the decomposition takes $2^{\mathcal{O}(k^2)} n^3$ time.

There are at most n bags of the decomposition, which adds a $\mathcal{O}(n)$ factor to the running time. In each bag t, we consider $\eta^{\mathcal{O}(k)}$ colourings of the adhesion $\sigma(t)$; hence, there are $\eta^{\mathcal{O}(k)} n$ calls to the procedure solving HYPERGRAPH PAINTING.

In each call, we have $V(H) = \beta(t)$ and $|E(H)| = \mathcal{O}(n+m) = \mathcal{O}(kn)$, as we have a hyperedge for each vertex and edge of $\beta(t)$ as well as a hyperedge for each child of t in

[2] Note that to use Claim 35 we do not require that I is proper.

the decomposition. The hypergraph H is stored in a standard way: for each hyperedge we store a list of its members, and for each vertex we store a list of hyperedges in which it participates. As discussed in Section 4.1, each function f can be represented by giving $\eta^{\mathcal{O}(k)}n$ values different than ∞.

Note that we do not need to perform the entire algorithm for Hypergraph Painting for each value of μ independently. Instead, we may perform it only once, and return w_μ to be the minimum $t(\mu)$ among all branches of the algorithm.

By Lemma 25 and the construction of perfect families of [27], there are $2^{\mathcal{O}(k^3)}\log^2 n$ choices of the assignment p, and they can be enumerated in $2^{\mathcal{O}(k^3)}n\log^2 n$ time. More precisely, by Lemma 25 we can enumerate the splitter $\mathcal{F}$ in time $2^{\mathcal{O}(k^3)}|E(H)|\log|E(H)| \leq 2^{\mathcal{O}(k^3)}n\log n$. This splitter has at most $2^{\mathcal{O}(k^3)}\log n$ elements. For each element S of the splitter $\mathcal{F}$, we enumerate an $(|S|,k)$-perfect family in time $2^{\mathcal{O}(k)}|S|\log|S| \leq 2^{\mathcal{O}(k)}n\log n$, and this perfect family has $2^{\mathcal{O}(k)}\log n$ members. Thus, we enumerate at most $2^{\mathcal{O}(k^3)}\log^2 n$ assignments p in time $2^{\mathcal{O}(k^3)}n\log^2 n$.

For each assignment p, we need to perform the two cleaning operations exhaustively. To speed them up, instead of maintaining the entire graph L_p, we keep only its connected components: each vertex of H knows its connected component, and the connected component knows its size and its vertices. In this manner, by enumerating the smaller component, we can merge two connected components in amortized $\mathcal{O}(\log n)$ time, as each vertex changes the connected components it belongs to $\mathcal{O}(\log n)$ times. Consequently we may initiate the graph L_p in $\mathcal{O}(\eta kn + n\log n)$ time, as we have to iterate over $\mathcal{O}(kn)$ hyperedges of size η each, and the total time needed to merge connected components is $\mathcal{O}(n\log n)$.

To apply the operations, we maintain the following auxiliary information. Each hyperedge F stores the set of the connected components of L_p it intersects (note that this set is of size at most 2). Once this set changes its cardinality from 2 to 1, the first operation starts to be applicable on F. As each vertex changes its connected component $\mathcal{O}(\log n)$ times, each list is updated at most $\mathcal{O}(\log n)$ times, which gives $\mathcal{O}(kn\log n)$ time in total.

For the second operation, we need to maintain, for each connected component D of L_p, a list $T(D,\mathbf{B})$ of hyperedges F such that $F\cap D\neq\emptyset$, $F\setminus D\neq\emptyset$, $p(F)>0$ and $\mathrm{col}_{F,p(F)}(v)=\mathbf{B}$ for some $v\in F\cap D$; analogously we define a list $T(D,\mathbf{W})$. Once both lists are non-empty, the second operation is applicable. As each hyperedge is of size at most η, all lists can be recomputed in $\mathcal{O}(\eta kn)$ time, whenever the set of the connected components of the graph L_p changes: each hyperedge F inserts itself into at most 2 lists.

We infer that the operations can be exhaustively applied in $2^{\mathcal{O}(k)}n^2$ time for a fixed assignment p. Also, including the constraints imposed by the colouring col_0, i.e., obtaining the functions $\hat{f}_F(\mathrm{col},\mu)$, can be done in $2^{\mathcal{O}(k^2)}n^2$ time.

We now move to the analysis of the final knapsack-type dynamic programming routine. We first show that the $\oplus$ operation can be performed using $\mathcal{O}(k^2)$ applications of the Fast Fourier Transform, taking total time $\mathcal{O}(k^2 b\log b)$, instead of the naive $\mathcal{O}(b^2)$ time bound. Consider two functions $t_1,t_2\in\{0,\ldots,b\}\to\{0,\ldots,k,\infty\}$. For $i\in\{1,2\}$ and $0\leq j\leq k$ by $p_{i,j}$ we define the polynomial

$$p_{i,j}(x)=\sum_{0\leq\mu\leq b}[t_i(\mu)=j]x^\mu\,.$$

Note that if $(t_1\oplus t_2)(\mu)\neq\infty$, then $(t_1\oplus t_2)(\mu)$ is equal to the smallest j, such that for some partition $j=j_1+j_2$ the coefficient in front of the monomial x^μ in the polynomial

$p_{1,j_1} \cdot p_{2,j_2}$ is non-zero. Therefore we can compute $t_1 \oplus t_2$ in $\mathcal{O}(k^2 b \log b)$ time. There are $\mathcal{O}(1)$ such operations per each hyperedge of $E(H)$. Consequently, the final dynamic programming algorithm takes $2^{\mathcal{O}(k)} n^2 \log n$ time.

Concluding, the total running time is $2^{\mathcal{O}(k^3)} n^3 \log^3 n$, as promised in Theorem 1.

5. Weighted variant. In this section we sketch how using our approach one can solve the following weighted variant of the Minimum Bisection problem:

Theorem 38. *Given a graph G with edge weights $w : E(G) \to \mathbb{R}$ and an integer k, one can in $\mathcal{O}^\star(2^{\mathcal{O}(k^3)})$ time find a partition of $V(G)$ into sets A and B minimizing $\sum_{e \in E(A,B)} w(e)$ subject to $||A| - |B|| \leq 1$ and $|E(A,B)| \leq k$, or correctly state that such a partition does not exist.*

Proof sketch. Essentially, we follow the same approach as in the previous section, except that in all dynamic programming tables we need to add an additional dimension to control the size of the constructed cut $E(A,B)$, and store the weight of the cut as the value of the entry in the DP table.

In some more details, for a fixed bag t, a colouring $\mathrm{col}_0 : \sigma(t) \to \{\mathbf{B}, \mathbf{W}\}$ satisfying (4), and integers $0 \leq \mu \leq n$ and $0 \leq \xi \leq k$ we consider a variable $x_{t,\mathrm{col}_0,\mu,\xi} \in \mathbb{R} \cup \{+\infty\}$ that equals the minimum possible value of $\sum_{e \in E(\mathrm{col}^{-1}(\mathbf{B}), \mathrm{col}^{-1}(\mathbf{W}))} w(e)$ among colourings $\mathrm{col} : \gamma(t) \to \{\mathbf{B}, \mathbf{W}\}$ satisfying:

- $\mathrm{col}|_{\sigma(t)} = \mathrm{col}_0$,
- $|\mathrm{col}^{-1}(\mathbf{W})| = \mu$, and
- $|E(\mathrm{col}^{-1}(\mathbf{B}), \mathrm{col}^{-1}(\mathbf{W}))| = \xi$.

The value $+\infty$ is attained if no such colouring exists.

Analogously, we modify the Hypergraph Painting problem to match the aforementioned definition of the values $x_{t,\mathrm{col}_0,\mu,\xi}$. That is, it takes as an input functions $f_F : \{\mathbf{B}, \mathbf{W}\}^F \times \{0, \ldots, b\} \times \{0, \ldots, k\} \to \mathbb{R} \cup \{+\infty\}$, where we require value $+\infty$ for any colouring that violates the local unbreakability constraint. For each $0 \leq \mu \leq b$ and $0 \leq \xi \leq k$ we seek for a value $w_{\mu,\xi} \in \mathbb{R} \cup \{+\infty\}$ defined as a minimum, among all colourings col extending col_0, and all possible sequences $(a_F)_{F \in E(H)}$ and $(b_F)_{F \in E(H)}$ such that $\sum_F a_F = \mu$ and $\sum_F b_F = \xi$, of

$$\sum_{F \in E(H)} f_F(\mathrm{col}|_F, a_F, b_F).$$

The knapsack-type dynamic programming of Section 4.1 is adjusted in a natural way, and the remaining reasoning of Section 4.1 remains unaffected by the weights. Consequently, the adjusted Hypergraph Painting problem can be solved in time $\mathcal{O}^\star(q^{\mathcal{O}(k)} \cdot d^{O(k^2)})$.

It is straightforward to check that the adjusted Hypergraph Painting problem corresponds again to the task of handling one bag in the tree decomposition of the input graph. To finish the proof note that the value we are looking for equals $\min_{0 \leq \xi \leq k} x_{r,\emptyset,\lfloor n/2 \rfloor,\xi}$, where r is the root of the tree decomposition. ∎

6. α-edge-separators. In this section we argue that the algorithm of Section 4 can be extended to show the following:

Theorem 39. *Given an n-vertex graph G, a real $\alpha \in (0,1)$ and an integer k, one can in $2^{\mathcal{O}(k^3)} n^{\mathcal{O}(1/\alpha)}$ time decide whether there exists a set X of at most k edges of G such that each connected component of $G \setminus X$ has at most αn vertices.*

To prove Theorem 39, we need the following lemma. A slight variation of this lemma appeared as problem C1 on the shortlist of the 54$^{\mathrm{th}}$ International Mathematical

Olympiad, IMO 2013 [30].

LEMMA 40. *Let $\alpha \in (0,1)$ be a real constant and let $a_1, a_2, \ldots, a_n \in [0,\alpha]$ be reals such that $\sum_{\ell=1}^{n} a_\ell = 1$. Then one can partition numbers $a_1, a_2, \ldots, a_n$ into $2\left\lceil \frac{1}{\alpha} \right\rceil - 1$ groups (possibly empty), such that the sum of numbers in each group is at most α.*

Proof. Let $q = \lceil \frac{1}{\alpha} \rceil$. For $\ell = 0, 1, \ldots, n$, let $b_\ell = \sum_{i=1}^{\ell} a_i$. For $j = 1, 2, \ldots, q-1$, let i_j be the unique index such that $b_{i_j - 1} \le j \cdot \alpha$ and $b_{i_j} > j \cdot \alpha$. Let us denote also $i_0 = 0$ and $i_q = n+1$; then also $b_{i_0} \ge 0 \cdot \alpha$ and $b_{i_q - 1} \le q \cdot \alpha$. Define the following groups:

$$\begin{aligned}
\{ \qquad & \{a_1, a_2, \ldots, a_{i_1 - 1}\}, \\
& \{a_{i_1}\}, \\
& \{a_{i_1+1}, a_{i_1+2}, \ldots, a_{i_2-1}\}, \\
& \{a_{i_2}\}, \\
& \ldots \\
& \{a_{i_{q-2}+1}, a_{i_{q-2}+2}, \ldots, a_{i_{q-1}-1}\}, \\
& \{a_{i_{q-1}}\}, \\
& \{a_{i_{q-1}+1}, a_{i_{q-1}+2}, \ldots, a_n\} \qquad \}.
\end{aligned}$$

For every group of form $\{a_{i_j+1}, a_{i_j+2}, \ldots, a_{i_{j+1}-1}\}$ we have that

$$\sum_{\ell=i_j+1}^{i_{j+1}-1} a_\ell = b_{i_{j+1}-1} - b_{i_j} \le (j+1)\alpha - j\alpha = \alpha.$$

On the other hand, for every group of form $\{a_{i_j}\}$ we have that $a_{i_j} \le \alpha$ by the assumption that $a_{i_j} \in [0,\alpha]$. Hence, the formed groups satisfy the required properties. □

The following corollary is immediately implied by Lemma 40.

COROLLARY 41. *Let $\alpha \in (0,1)$ be a real constant and let H be a graph on n vertices. Then the following conditions are equivalent:*
(a) Each connected component of H has at most αn vertices.
(b) There exists a partition of $V(H)$ into ζ possibly empty sets $A_1, A_2, \ldots, A_\zeta$, where $\zeta = 2\left\lceil \frac{1}{\alpha} \right\rceil - 1$, such that $|A_i| \le \alpha n$ for each $i = 1, 2, \ldots, \zeta$ and no edge of H connects two vertices from different parts.

Equipped with Corollary 41, we may now describe how to modify the algorithm of Section 4 to prove Theorem 39. Most of the modifications are straightforward, hence we just sketch the consecutive steps.

Proof sketch of Theorem 39. By Corollary 41, we may equivalently seek for a colouring of $V(G)$ into $\zeta = 2\left\lceil \frac{1}{\alpha} \right\rceil - 1$ colours, such that at most k edges connect vertices of different colours. Essentially, we now proceed as in Section 4, but, instead of colouring vertices into black and white, we use ζ colours, and we keep track of the number of vertices coloured in each colour.

In some more details, for a fixed bag t, a colouring $\mathrm{col}_0 : \sigma(t) \to \{1, \ldots, \zeta\}$ satisfying

$$\exists_{1 \le c \le \zeta}\ |\mathrm{col}_0^{-1}(c)| \ge |\sigma(t)| - 3k \tag{6}$$

and a function $\mu : \{1, \ldots, \zeta\} \to \{0, \ldots, n\}$, we consider a variable $x_{t,\mathrm{col}_0,\mu}$ taking value in $\{0, 1, \ldots, k, \infty\}$, which is equal to the minimum possible number of edges with endpoints coloured by different colours by a colouring col, among all colourings $\mathrm{col} : \gamma(t) \to \{1, 2, \ldots, \zeta\}$ satisfying:

- $\mathrm{col}|_{\sigma(t)} = \mathrm{col}_0$, and
- for each $1 \le c \le \zeta$, $|\mathrm{col}^{-1}(c)| = \mu(c)$.

The value ∞ is attained if all such colourings yield more than k edges with endpoints of different colours.

Recall that, for any bag t, the adhesion $\sigma(t)$ is $(2k, k)$-unbreakable. Similarly as in Claim 35, we infer that if in a colouring $\mathrm{col} : \gamma(t) \to \{1, \ldots, \zeta\}$ at most k edges have endpoints painted in different colours, it needs to colour all but at most $3k$ vertices of $\sigma(t)$ with a single colour. This motivates condition (6). Note that this requirement is only needed to obtain $2^{\mathrm{poly}(k)}$ dependency on k, and, if it is omitted, the dependency will become doubly-exponential.

We now modify the Hypergraph Painting problem to match the aforementioned definition of the values $x_{t,\mathrm{col}_0,\mu}$. That is, the problem takes as an input functions $f_F : \{1, \ldots, \zeta\}^F \times \{0, \ldots, b\}^\zeta \to \{0, 1, \ldots, k, \infty\}$. For each $\mu : \{1, \ldots, \zeta\} \to \{0, \ldots, b\}$ we seek for a value $w_\mu \in \{0, 1, \ldots, k, \infty\}$ defined as a minimum, among all colourings $\mathrm{col} : V(H) \to \{1, \ldots, \zeta\}$ extending col_0, and all possible sequences $(a_F^c)_{F \in E(H), 1 \le c \le \zeta}$ such that $\sum_F a_F^c = \mu(c)$ for each $1 \le c \le \zeta$, of

$$\sum_{F \in E(H)} f_F(\mathrm{col}|_F, (a_F^c)_{1 \le c \le \zeta}).$$

The value of ∞ is attained whenever the sum exceeds k.

In the local unbreakability constraint we require that a value different than ∞ can be attained only if all but at most $3k$ elements of F are coloured in a single colour. This corresponds to the previously discussed condition (6) on valid colourings col_0 of an adhesion $\sigma(t)$. The connectivity requirement states that $f_F(\mathrm{col}, \alpha)$ is non-zero whenever col uses at least two colours: the corresponding colouring of the subgraph $\gamma(t)$ (as in the proof of Claim 36) needs to colour the endpoints of at least one edge with different colours, as $\gamma(t)$ is connected. The global unbreakability constraint requires that whenever $w_\mu < \infty$, there is a witnessing colouring col that colours all but at most $\tau' + k$ vertices with a single colour. This follows from the fact that, in our decomposition, $\beta(t)$ is (τ', k)-unbreakable, so any colouring of $\gamma(t)$ that colours endpoints of at most k edges with different colours needs to paint all but at most $\tau' + k$ vertices of $\beta(t)$ with the same colour.

The core spirit of the reasoning of Section 4.1 remains in fact unaffected by this change. However, for sake of clarity, we now describe the changes in more details. We apply colour-coding to paint the hyperedges with assignment $p : E(H) \to \{0, \ldots, \ell\}$, where $p(F) = 0$ means "definitely monochromatic" and $p(F) = i > 0$ means "monochromatic or coloured according to the colouring $\mathrm{col}_{F,i} : F \to \{1, 2, \ldots, \zeta\}$". The colourings $\mathrm{col}_{F,i}$ are required to comply with the (new) unbreakability constraint, thus there are $\eta^{\mathcal{O}(k)} \zeta^{\mathcal{O}(k)}$ such colourings. We guess a colour—call it *black*—that will be the dominant colour in $\beta(t)$. For the assignment p to be good, i.e., leading to the discovery of an optimum solution, we require that for all hyperedges that are not monochromatic in the solution col_{opt} we have $p(F) = i > 0$ and $\mathrm{col}_{F,i} = \mathrm{col}_{opt}|_F$ (i.e., we have guessed the correct colouring of F), and the "definitely monochromatic" hyperedges contain a spanning hyperforest of the hypergraph $(V(H), E_{not\ black})$, where $E_{not\ black}$ consists of all not-black monochromatic hyperedges in the colouring col_{opt}.

For a hyperedge F, we insert to L_p all possible edges with both endpoints in F if $p(F) = 0$ and all edges between the vertices of the same colour of $\mathrm{col}_{F,i}$ if $p(F) = i > 0$.

It is straightforward to verify that, if p is good, then the following holds:

1. all connected components of L_p are painted monochromatically in col_{opt} (cf. Claim 28);
2. if a connected component D of L_p is not painted black in col_{opt}, then all hyperedges F such that $F \cap D \neq \emptyset$ and $F \setminus D \neq \emptyset$ are not coloured monochromatically by col_{opt} and, consequently, their colourings $\mathrm{col}_{F,p(F)}$ conforms with col_{opt} (cf. Claim 29).

We can now perform adjusted cleaning operations on p as follows, again preferring the first operation to the second:

(1) For any hyperedge F completely contained in some component D of L_p, F is monochromatic in col_{opt}, so set $p(F) = 0$.

(2) If for any connected component D of L_p we have two hyperedges F_1, F_2 (possibly $F_1 = F_2$) such that $F_1 \cap D \neq \emptyset$, $F_2 \cap D \neq \emptyset$, $p(F_1) > 0$, $p(F_2) > 0$, and for some $v_1, v_2 \in D$ (possibly $v_1 = v_2$) we have $\mathrm{col}_{F_1,p(F_1)}(v_1) \neq \mathbf{B}$ and $\mathrm{col}_{F_1,p(F_1)}(v_1) \neq \mathrm{col}_{F_2,p(F_2)}(v_2)$, then set $p(F_1) = 0$. This is because the guess on $\mathrm{col}_{F_1,p(F_1)}$ is clearly incorrect.

Again, it is straightforward to see that a good assignment p remains good under the cleaning operations. After the cleaning operations are applied exhaustively, we may now classify any connected component D of L_p as

- "definitely black": there exists a hyperedge F with $p(F) > 0$ such that $\mathrm{col}_{F,p(F)}$ colours at least one vertex of D black; or
- "potentially not black": otherwise.

A reasoning analogous to that of Claim 31 shows that a definitely black component is painted black by col_{opt}, provided p is good.

On the other hand, consider any connected component D of L_p that is potentially not black. If there exists at least one hyperedge F with $p(F) > 0$ and $F \cap D = \emptyset$, then $\mathrm{col}_{F,p(F)}$ colours some vertex of D with some non-black colour, say blue. Since the second cleaning procedure was not applicable, for every other hyperedge F' with $p(F') > 0$ and $F' \cap D = \emptyset$ we have that $\mathrm{col}_{F',p(F')}$ colours all the vertices of $D \cap F'$ blue. Then it is straightforward to verify that an analogue of Claim 32 holds: for every potentially not black component D, either col_{opt} colours D with a non-black colour, being consistent with all hyperedges F with $p(F) > 0$ that intersect D, or col_{opt} colours D and all intersecting hyperedges black. Observe that in the first option, if there is at least one hyperedge F with $p(F) > 0$ and $F \cap D = \emptyset$, then there is exactly one non-black colour in which D may be coloured. However, there might be components D for which there is no such hyperedge (they form separate connected components in the hypergraph H); we call them *colorful*, as choosing every non-black colour is admissible for them.

We may now proceed to the final knapsack-type dynamic programming. Unfortunately, we cannot use the previous approach verbatim, as it is no longer true that the components of L_p may be considered independently. Indeed, if there exists a hyperedge F that intersects two potentially not black components D_1 and D_2 (and, hence, $p(F) > 0$), then col_{opt} paints D_1 black if and only if it paints D_2 black as well.

Consequently, we need to adjust the knapsack-type DP as follows. A black component is painted black, so we can proceed with them as previously. Two potentially not black components D_1 and D_2 are *entangled* if there exists a hyperedge F intersecting both of them. To avoid confusion, we call connected components of the entanglement relation *blocks*. Observe that all components within any block, apart from trivial blocks consisting of one colorful component, make a joint decision on whether they are all painted black or they are all painted with the unique non-black color admissible for them (which may be different for each component). These decisions are independent between each other: the decision in one block does not influence the decision in another one. On the other hand, each colorful component is not entangled with any other component, so it just chooses to be painted with any of the ζ colors.

This allows us to adjust the knapsack-type dynamic programming of Section 4.1. In a single step we consider either all hyperedges intersecting all connected components of L_p contained in a single block, or all hyperedges intersecting a single colorful component. For a block the DP chooses the minimum out of two options (black or non-black), while for a colorful component it chooses the minimum out of ζ options, one for each colour. It is the straightforward to check that the analogues of Claims 32 and 34 hold. Namely, on one hand, if p is good, then some computation path of the dynamic programming procedure will lead to the discovery of an optimum solution, because colouring the blocks and colorful components as in col_{opt} is among the considered computation paths. On the other hand, even if p is not necessarily good, the value computed by the dynamic programming procedure is equal to the cost of some colouring of the vertices of the hypergraph, computed implicitly along the way and defined in the same way as in the proof of Claim 34. Therefore, no p gives rise to a value smaller than the optimum, while any good p gives rise to a value not larger than the optimum.

It is straightforward to check that the adjusted Hypergraph Painting problem corresponds again to the task of handling one bag in the tree decomposition of the input graph. To finish the proof note that the minimum size of the cut we are looking for equals

$$\min_{\mu:\{1,\ldots,\zeta\}\to\{0,\ldots,\lfloor\alpha n\rfloor\}} x_{r,\emptyset,\mu},$$

where r is the root of the tree decomposition, as long as the cut has size at most k.□

7. Conclusions. In this paper we have settled the parameterized complexity of Minimum Bisection. Our algorithm also works in the more general setting when the edges are weighted, the vertex set is to be partitioned into a constant number of parts rather than only two, and the cardinality of each of the parts is given on input.

The core component of our algorithm is a new decomposition theorem for general graphs. Intuitively, we show that it is possible to partition any graph in a tree-like manner using small separators so that each of the resulting pieces cannot be broken any further. The uncovered structure is very natural in the context of cut problems, and we believe that our decomposition theorem will find further algorithmic applications.

Having settled the parameterized complexity of Minimum Bisection it is natural to ask whether the problem also admits a *polynomial kernel*, i.e. a polynomial-time preprocessing algorithm that would reduce the size of the input graph to some polynomial of the budget k. This question, however, has been already resolved by van Bevern et al. [3], who showed that Minimum Bisection does not admit a polynomial kernel unless coNP $\subseteq$ NP/poly. We conclude with a few intriguing open questions.

(a) Can the running time of our algorithm be improved? In particular, does there exist an algorithm for Minimum Bisection with running time $2^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$, that is, with linear dependence on the parameter in the exponent?

(b) The running time dependence of our algorithm on the input size is roughly cubic. Is it possible to obtain a fixed-parameter tractable algorithm with quadratic, or even nearly-linear running time dependence on input size? Note that the best known algorithm for graphs of bounded treewidth has quadratic dependence on the input size [20].

(c) Are the parameters in the decomposition theorem tight? For example, is it possible to lower the adhesion size from $2^{\mathcal{O}(k)}$ to polynomial in k? Similarly, can one make the bags $(k^{\mathcal{O}(1)}, k)$-unbreakable rather than $(2^{\mathcal{O}(k)}, k)$-unbreakable? Is it possible to achieve both simultaneously? We remark that if the latter question has a positive answer, this would improve the parameter dependence in the running time of our algorithm for MINIMUM BISECTION to $k^{\mathcal{O}(k)}$.

(d) Is it possible to compute our decomposition faster, say in $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ or even in $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time? Currently the main bottleneck is the very simple Lemma 9.

We believe that one possible way to achieve progress on the questions above is via the notion of *linked* (or *lean*) tree decompositions, see e.g. [2, 13, 31]. This graph-theoretic notion has a very similar flavour to our concept of the unbreakability of bags, however its algorithmic aspects seem unexplored so far. Connections between linked tree decompositions and tree decompositions of the kind introduced in this work will be explored in future work.

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