Dominating Set in Weakly Closed Graphs is Fixed Parameter Tractable

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Abstract

In the Dominating Set problem the input is a graph G and an integer k, the task is to determine whether there exists a vertex set S of size at most k so that every vertex not in S has at least one neighbor in S. We consider the parameterized complexity of the Dominating Set problem, parameterized by the solution size k, and the *weak closure* of the input graph G. Weak closure of graphs was recently introduced by Fox et al. [*SIAM J. Comp. 2020*] and captures sparseness and triadic closure properties found in real world graphs. A graph G is *weakly c-closed* if for every induced subgraph G' of G, there exists a vertex $v \in V(G')$ such that every vertex u in $V(G')$ which is non-adjacent to v has less than c common neighbors with v. The weak closure of G is the smallest integer γ such that G is weakly γ-closed. We give an algorithm for Dominating Set with running time $k^{O(\gamma^2 k^3)} n^{O(1)}$, resolving an open problem of Koana et al. [ISAAC 2020].

One of the ingredients of our algorithm is a proof that the VC-dimension of (the set system defined by the closed neighborhoods of the vertices of) a weakly γ-closed graph is upper bounded by 6γ. This result may find further applications in the study of weakly closed graphs.

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1 Introduction

A dominating set of a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices of G such that every vertex in $V \setminus S$ is adjacent to at least one vertex in S. In the Dominating Set problem, the input is a graph G and a positive integer k and the task is to determine whether G has a dominating set of size at most k. Dominating Set is NP-complete and has been extensively studied within all established paradigms for coping with NP-hardness such as parameterized complexity, approximation algorithms and exact exponential time algorithms [9, 13, 19, 31]. In fact, it is hard to overstate the pivotal role that Dominating Set has played in the development of parameterized complexity; it was, together with Clique, one of the first examples of natural parameterized problems that were proved intractable [13] as well as FPT-inapproximable [6, 8, 18].

While, on the one hand, Dominating Set on general graphs has been a driver of parameterized intractability, on the other hand, the study of Dominating Set on restricted graph classes has been a treasure trove of algorithmic techniques. For instance, the subexponential time algorithms for Dominating Set on planar graphs [1, 7], and the linear kernel [2] on planar graphs led to the celebrated bidimensionality theory [11]. These algorithms and kernels have been extended to much wider classes of graphs, such as, (topological) minor free graphs [20], nowhere dense graphs [10, 14], d-degenerate graphs [3, 27], $K_{i,j}$-free graphs [27] and induced ladder-free graphs [17]. In this article we study the Dominating Set problem on c-closed graphs and weakly γ-closed graphs, which were recently introduced by Fox et al. [21].

▶ **Definition 1** ([21]). *A graph G is said to be c-**closed** if for every pair of non-adjacent vertices u and v in G, $|N_G(u) \cap N_G(v)| < c$. A graph G is said to be **weakly** γ-**closed** if for every induced subgraph G' of G there exists a vertex v in G' such that for every vertex u in G' not adjacent to v, $|N_{G'}(u) \cap N_{G'}(v)| < \gamma$. The **closure** of a graph G is the smallest c such that G is c-closed. The **weak closure** of a graph G is the smallest γ such that G is weakly γ-closed.*

The class of c-closed and weakly γ-closed graphs contains the class of graphs of maximum degree at most c and graphs with degeneracy at most γ, respectively. Additionally they capture the triadic closure principle, namely that two people who have many common friends in a social network are likely to be friends themselves. From an application viewpoint, the weak closure is typically found to be small for large real-world social network graphs [21, 23]. In addition, the parameters also have the appealing feature that they are computable in polynomial time [21].

Motivated by the salient features of (weakly) closed graphs, Koana et al. [24] initiated a systematic study of the parameterized complexity of computational problems on c-closed graphs, closely followed by Husic and Roughgarden [22]. Koana et al. [24] show that a number of problems, including Dominating Set, are FPT on closed graphs. In a follow up work Koana et al. [23] show that a number of problems remain FPT even on weakly closed graphs. Very recently, the same set of authors [25] provide polynomial kernels and kernel lower bounds for various problems including Connected Vertex Cover and Capacitated Vertex Cover on weakly closed graphs. They also obtain polynomial kernels for Dominating Set on weakly closed split graphs and weakly closed bipartite graphs. However, they were not able to obtain an FPT algorithm for Dominating Set on weakly closed graphs, leading them to pose the existence of such an algorithm as an open problem. Specifically, Koana et al. [23] asked whether the following parameterized problem is FPT or not.

Dominating Set in weakly γ-closed graph	**Parameter:** γ, k
Input: Weakly γ-closed graph G and a non-negative integer k.	
Question: Does there exist a set $X \subseteq V(G)$ of size at most k such that $N_G[X] = V(G)$.	

In this work, we give an algorithm with running time $k^{O(\gamma^2 k^3)} n^{O(1)}$, resolving the problem in the affirmative. We now state our main result.

▶ **Theorem 2.** *There exists a deterministic algorithm that given as input a weakly γ-closed graph G and an integer k determines in time $k^{O(\gamma^2 k^3)} n^{O(1)}$ whether G has a dominating set of size at most k and outputs one if it exists.*

Methods. Our algorithm is based on domination cores, first defined by Dawar and Kreutzer [10] and then later employed in multiple settings [14, 15, 17]. A *k-domination core* of a graph is a set X of vertices of the graph such that every set of size at most k that dominates X dominates the whole graph. Observe that the set of all vertices of a graph is a domination core. It is well known (for example see [10] Lemma 4.1) that if one can efficiently compute a domination core whose size is upper bounded by a function of k, then we can obtain an FPT algorithm for Dominating Set. Thus our main technical contribution is an algorithm that given a graph produces a k-domination core of the graph of size $k^{O(\gamma k^2)}$.

We now give a very rough sketch of the proof for our main technical claim – that every domination core W of size at least b, where $b = k^{O(\gamma k^2)}$ contains at least one vertex w such that $W \backslash \{w\}$ is also a domination core, and that such a vertex w can be found efficiently. In this exposition we focus only on the claim of existence of w. Suppose such a vertex w does not exist. Then, for every vertex $w \in W$ there must exist a set X_w of size at most k that dominates all of $W \setminus \{w\}$, but does not dominate w – otherwise $W \setminus \{w\}$ is still a domination core. We call a set W that has this property a *k-threshold set*[1] and prove that a weakly γ-closed graph can not contain a k-threshold set of size at least b.

The advantage of shifting our attention from k-domination cores to k-threshold sets is that k-threshold sets are closed under subsets – every subset of a k-threshold set is also a k-threshold set. This allows us to "dig for structure", that is, prove results of the form "if G has a sufficiently large k-threshold set W then W contains a large (as a function of k and $|W|$) k-threshold set W' with some additional property".

By invoking a (multi-color version of the) Ramsey Theorem [4] on an appropriately constructed auxiliary graph, we extract from W a sufficiently large and sufficiently symmetric threshold set $W' \subseteq W$. The existence of a large and symmetric threshold set W' in turn implies that G must contain as an induced subgraph one of three simple pattern graphs (such as a complete bipartite graph with $\gamma + 1$ vertices on both sides). Each one of these three pattern graphs can easily be shown not to be weakly γ-closed, contradicting that G was weakly γ-closed in the first place.

We remark that the actual proof proceeds in a different order of the exposition above. First, in Section 3 we define the pattern graphs that we will use and show that they are not weakly γ-closed. In Section 5 we prove that a purely existential upper bound on the size of k-threshold sets implies both an FPT algorithm to find a small k-domination core, and an FPT algorithm for Dominating Set. In Section 6 we obtain the aforementioned upper bound on the size of k-threshold sets in weakly γ-closed graphs by showing that a k-threshold set of size at least $b = k^{O(\gamma k^2)}$ implies that G must contain one of the forbidden pattern graphs from Section 3.

Efficiently computing a domination core W of size $k^{O(\gamma k^2)}$ immediately leads to a $2^{k^{O(\gamma k^2)}} n^{O(1)}$ time algorithm for Dominating Set on weakly γ-closed graphs. Indeed, to find a dominating set for G of size k (if one exists), it is sufficient to find a set S of size at most k that dominates all of W. This can be done by trying all possible partitions of W into k parts $P_1, \ldots, P_k$, and then determining whether there exists for every part P_i a single vertex $s_i \in V(G)$ that dominates P_i. This algorithm already resolves the open problem of Koana et al. [23] in the affirmative. At the same time the double exponential running time dependence on k is unsatisfactory.

We are able to improve the running time of our algorithm for Dominating Set to $k^{O(\gamma^2 k^3)} n^{O(1)}$ by proving an additional purely graph-theoretic result regarding the structure of weakly γ-closed graphs. A *set system* $(U, \mathcal{F})$ consists of a universe U along with a collection $\mathcal{F}$ of subsets of U. A subset containing $A \subseteq U$ is *shattered* by $\mathcal{F}$ if each subset of A can be expressed as the intersection of A with a set in $\mathcal{F}$. The *Vapnik-Chervonenkis* dimension (*VC-dimension*) of a set system is the cardinality of the largest subset A of U that is shattered by $\mathcal{F}$. The VC-dimension of a graph is defined as the VC-dimension of the set system induced by the closed neighbourhoods of its vertices. We prove in Section 4 that weakly γ-closed graphs have VC-dimension at most 6γ.

▶ **Theorem 3.** *Every weakly γ-closed graph has VC-dimension at most 6γ.*

Theorem 3 is tight up to the constant factor 6 (see Section 4 for a simple construction of a weakly γ-closed graph with VC-dimension γ).

[1] Note that a k-threshold set is not necessarily a k-domination core, however every inclusion minimal k-domination core is a k-threshold set.

Theorem 3 (together with our bound on the size of k-threshold sets) quite directly leads to a $k^{O(\gamma^2 k^3)} n^{O(1)}$ time algorithm for Dominating Set on weakly γ-closed graphs. Indeed, the double exponential running time of the previous algorithm came from the algorithm to determine whether there exists a set S of size at most k that dominates the entire domination core W. The size of the k-domination core W is assumed to be upper bounded by $k^{O(\gamma k^2)}$. Our improved algorithm to find S is remarkably simple: if two vertices u and v not in W have exactly the same set of neighbors in W, we remove u from the graph (since we can always pick v in its place). After this reduction, the Sauer-Shelah Lemma [28, 29] (See Lemma 8) implies that there are at most $k^{O(\gamma^2 k^2)}$ vertices left in G. Then a brute force algorithm that tries all possibilities for S takes time $k^{O(\gamma^2 k^3)} n^{O(1)}$.

We believe that Theorem 3 will find further uses in the design of algorithms for problems on weakly γ-closed graphs. For an example Theorem 3 also immediately implies that the improved approximation algorithm for Dominating Set on graphs of bounded VC-dimension [5, 16] applies to weakly γ-closed graphs (see Section 4 for details).

2 Notation and Preliminaries

In this section we give notations, and definitions that we use throughout the paper. Unless specified we will be using all general graph terminologies from the book of Diestel [12].

Given a graph G, we use $V(G)$ and $E(G)$ to denote the set of vertices and edges, respectively. We denote the open neighbourhood of a vertex v in G by $N_G(v) = \{u : u \in V(G), (u,v) \in E(G)\}$ and closed neighbourhood by $N_G[v] = \{v\} \cup N_G(v)$. Further, we denote the non-neighbourhood of v by $\overline{N_G[v]} = V(G) \setminus N[v]$. We extend this notation to a set $S \subseteq V(G)$ as well, that is $N_G(S) = \bigcup_{v \in S} N_G(v)$, $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $\overline{N_G[S]} = V(G) \setminus N_G[S]$. Whenever the graph G is clear from the context, we will omit the subscript. A *dominating set* of G is a set of vertices $S \subseteq V(G)$ such that $N[S] = V(G)$. For any $X \subseteq V(G)$, we use the notation $G[X]$ to denote the subgraph induced by X in G.

We use the symbol $\uplus$ to denote the disjoint union operation on sets. Let l be a positive integer. We use the notation $[l]$ to denote the set $\{1, \ldots, l\}$. A graph G having vertex set $V(G) = A \uplus B$ is called a *split graph* if A is a clique and B is an independent set. A graph G is *d-degenerate* if every subgraph G' of G has a vertex having degree at most d. We will need the notion of weak ordering of a weakly γ-closed graph. It is very similar to notion of degeneracy ordering for degenerate graphs [12].

▶ **Definition 4** ([21]). *A **weak ordering** O of a weakly γ-closed graph G is an ordering $O = \{v_1, \ldots, v_n\}$ of $V(G)$ such that for each $v_i \in V(G)$ and for each $u \in \overline{N_{G_i}[v_i]}$, it holds that $|N_{G_i}(u) \cap N_{G_i}(v_i)| < \gamma$, where $G_i = G[\{v_i, \ldots, v_n\}]$. A **forward neighbour** of v_i is a vertex adjacent to v_i in G_i.*

3 Obstructions to Weak Closure

In this section, we define a few simple pattern graphs and proceed to show that they (except *split half-graphs*, which are weakly 1-closed) are not weakly γ-closed. Many of our proofs are of the form "every weakly γ-closed graph G either has some desirable property or contains one of these patterns. The second case contradicts that G is weakly γ-closed, so we conclude that G has the desirable property".

▶ **Definition 5.** [2] *Given a positive integer n, let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be disjoint vertex sets. We define the following graphs:*

1. *A bipartite graph G with vertex set $V(G) = A \cup B$ and bipartition A and B is called a* ***complete bipartite graph*** *of order n if $\forall i, j \in [n]$, $(a_i, b_j) \in E(G)$.*
2. *A graph G with vertex set $V(G) = A \cup B$ is called a* ***semi split co-matching*** *of order n if A is a clique and $\forall i, j \in [n]$, $(a_i, b_j) \in E(G)$ iff $i \neq j$. The edges between B can be arbitrary.*
3. *A graph G with vertex set $V(G) = A \cup B$ is called a* ***split half graph*** *of order n if G is a split graph with B being the independent set and $\forall i, j \in [n]$, $(a_i, b_j) \in E(G)$ iff $j > i$.*
4. *A graph G with vertex set $V(G) = A \cup B \cup C$ is called a* ***double split half graph*** *of order n if $G[A \cup B]$ and $G[B \cup C]$ are split half graphs with B being the independent set. That is $\forall i, j \in [n]$, $(a_i, b_j) \in E(G)$ iff $j > i$ and $(b_i, c_j) \in E(G)$ iff $j > i$. The edges between A and C can be arbitrary.*

▶ **Lemma 6.** [3] *If G is weakly γ-closed, then it does not contain any of the following graphs as an induced subgraph.*

(i) Complete bipartite graph of order $n \geq \gamma$.
(ii) Semi split co-matching of order $n > \gamma$.
(iii) Double split half graphs of order $n \geq 3\gamma$.

4 VC-dimension of Weakly Closed Graphs

In this section we prove Theorem 3, that is we show that the VC dimension of weakly γ-closed graphs is at most 6γ. Recall that the VC-dimension of a graph is defined as the VC-dimension of the set system induced by the closed neighbourhoods of its vertices.

Proof of Theorem 3. Suppose that the VC dimension of G is greater than 6γ. We will show that G is not weakly closed, thus contradicting our assumption. Since we assumed that the VC dimension is at least $6\gamma + 1$, there is a set $X \subseteq V(G)$ of size $6\gamma + 1$ that is shattered in G. Since X is shattered, for each $x \in X$, there exists a vertex y that dominates all vertices in X except x. We note that for each $x \in X$, there can be more than one such vertex but we need only one for our proof. We will call y the partner of x and x the partner of y. Observe that no two vertices in X can have the same partner. Let Y be the set of partners of all vertices in X. Also observe that every $x \in X$ dominates all vertices in Y except its partner and every $y \in Y$ dominates all vertices in X except its partner. We start by extracting a sufficiently large clique from X or Y.

▷ Claim 7. There exists a clique Z of size at least $\gamma + 1$ such that $Z \subseteq X$ or $Z \subseteq Y$.

Proof. Let X_1 be an arbitrary subset of X of size 3γ, and let Y_1 be an arbitrarily chosen set of 3γ vertices in Y that have no partner in X_1. If $|X_1 \cap Y_1| > \gamma$ then $Z = X_1 \cap Y_1$ is a clique that satisfies the conclusion of the lemma since every vertex in Y_1 dominates X_1.

We proceed with the case that $|X_1 \cap Y_1| \leq \gamma$. Define $X' = X_1 \backslash Y_1$ and $Y' = Y_1 \backslash X_1$ (i.e. remove common vertices from X_1 and Y_1). Note that $|X'| \geq 2\gamma$ and $|Y'| \geq 2\gamma$, that X' and Y' are disjoint, and that every vertex in X' is adjacent to every vertex in Y'. Let $O_{X' \cup Y'}$ be the order induced by a weak ordering O of G on $X' \cup Y'$. There must be $\gamma + 1$ vertices all from either X' or Y' among the first $2\gamma + 1$ vertices in $O_{X' \cup Y'}$. Let Z be the set of these $\gamma + 1$

[2] Refer to Figure 1 in Appendix A
[3] Proof in Appendix A

vertices all from X' or Y'. Since all vertices in X' are adjacent to all vertices in Y', every pair of vertices in Z must have at least γ common forward neighbours in the ordering O. Thus, since G is weakly γ-closed, Z must be a clique. ◀

Let Z be a clique as provided by Claim 7. Let P_Z be the set of partners of all vertices in Z. Observe that Z and P_Z are disjoint: for every $z \in Z$ its partner z' does not dominate z (by definition of partners) and therefore cannot be in Z, since Z is a clique. Next, we observe that every vertex z in Z is adjacent to every vertex in P_Z except its partner by the definition of X and Y and the fact that $Z \subseteq X$ or $Z \subseteq Y$. The induced subgraph $G[Z \cup P_Z]$ is a semi split co-matching (Definition 5) because Z is a clique, Z and P_Z are disjoint and every vertex z in Z is adjacent to every vertex in P_Z except its partner. This contradicts Lemma 6, concluding the proof. ◀

Theorem 3 is tight up to the constant factor 6, since there exists a weakly γ-closed graph having VC-dimension γ: Consider the bipartite graph G with $V(G) = A \uplus B$ where A has γ vertices and for each set $S \subseteq A$, B has one vertex whose neighbourhood is S. The graph G is weakly γ-closed and has VC-dimension at least γ since A is shattered by the closed neighborhood of vertices in B.

4.1 Set Cover and graphs of bounded VC-dimension

In the Set Cover problem, we are given a universe U, a family $\mathcal{F}$ of sets over U, and a positive integer k and the task is to determine whether there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k such that $\bigcup_{X \in \mathcal{F}'} X = U$. It is known [28, 29] that if the VC-dimension of a set system $(U, \mathcal{F})$ is bounded, then the size of the family $\mathcal{F}$ must be bounded.

▶ **Lemma 8** (Sauer-Shelah lemma [28, 29]). *If the VC-dimension of a set system $(U, \mathcal{F})$ is bounded by d, then $\mathcal{F}$ can consist of at most $\sum_{i=0}^{d} \binom{|U|}{i} = O(|U|^d)$ sets.*

We will exploit the fact that weakly closed graphs have bounded VC-dimension in the following way. Dominating Set on a graph of bounded VC-dimension corresponds to Set Cover on the set system $(U, \mathcal{F})$ where $U = V(G)$ and $\mathcal{F} = \{N[v] \ : \ v \in U\}$.

For a general set system $(U, \mathcal{F})$, there is a naive algorithm that goes over all families $\mathcal{F}'$ of size at most k in $\mathcal{F}$ and checks whether $\mathcal{F}'$ is a set cover in time $|\mathcal{F}|^k |U|^{O(1)}$. However if the VC-dimension of $(U, \mathcal{F})$ is bounded by d, then by Lemma 8, $|\mathcal{F}| = O(|U|^d)$ and therefore this algorithm solves Set Cover in $O(|U|^{kd})$ time.

▶ **Theorem 9.** *There exists a deterministic algorithm that given a* Set Cover *instance $(U, \mathcal{F}, k)$ such that the VC-dimension of $(U, \mathcal{F})$ is bounded by d determines in time $O(|U|^{kd})$ whether the instance has a set cover of size at most k and outputs one if it exists.*

We remark that this is not an FPT algorithm parameterized by k and d. However we will be invoking Theorem 9 with $|U|$ bounded by $2^{poly(k)}$ and d bounded by 6γ in our algorithm for Dominating Set.

An upper bound on the VC-dimension of G also leads to an improved approximation algorithm for Dominating Set. Indeed Brönnimann and Goodrich [5] give a $O(d \log(dk))$ approximation algorithm for set systems of VC-dimension d, where k is the size of the optimal solution. This, together with Theorem 3 directly yields a $O(\gamma \log(\gamma k))$-approximation for Dominating Set on weakly γ-closed graphs.

5 Dominating Set in Weakly Closed Graphs

Our algorithm is based on *domination cores*, which have been used for several algorithms for the Dominating Set problem [14, 15, 17].

▶ **Definition 10.** *Given a graph G, an integer k, a set $S \subseteq V(G)$ is called a **k-domination core** of G if $\forall X \subseteq V(G)$ such that $|X| \leq k$ and $S \subseteq N[X]$, it holds that $N[X] = V(G)$.*

It is easy to see that the set of all vertices in a graph is a trivial domination core. We wish to prove that weakly γ-closed graphs contain k-domination cores whose size is upper bounded by a function of k and γ. This naturally leads our attention to inclusion minimal k-domination cores.

▶ **Definition 11.** *A k-domination core W is called a **minimal k-domination core** if $\forall w \in W$, $W \backslash \{w\}$ is not a k-domination core.*

We note that whenever k is clear from the context, we will omit k while referring to domination cores. In the following lemma, we provide a bound on the size of minimal domination cores in weakly γ-closed graphs.

▶ **Lemma 12.** *Every minimal k-domination core of a weakly γ-closed graph G has size at most b, where $b = k^{O(\gamma k^2)}$.*

Lemma 12 leads to the following intuitive algorithm - Start with the trivial domination core $D = V$ and as long as $|D| > b$ keep discarding a vertex x from D such that D remains a domination core (we will soon discuss how to algorithmically identify the vertex x, for now ignore this issue).

Finally use D to construct a Set Cover instance having universe D and family $\mathcal{F} = \{N[v] \cap D : v \in V(G)\}$. Since G is weakly γ-closed, by Theorem 3, G has VC-dimension at most 6γ. Thus, the Set Cover instance also has VC-dimension at most 6γ and so we use Theorem 9 to find a set cover of size at most k if exists from which a dominating set for G can easily be recovered.

We now turn to the issue of identifying a vertex x to remove from D when $|D| > b$. To this end, we will use the following property of every minimal k-domination core W: for each $w \in W$, there is a set X_w of size at most k that dominates all of $W \backslash \{w\}$ but not w. Indeed, suppose there is a $w \in W$ for which no such X_w exists, and consider a set X of size at most k which dominates $W \backslash \{w\}$. Then X also dominates w (by the non-existence of X_w) and by extension all of G (since W is a k-domination core). But then $W \backslash \{w\}$ is also a domination core, contradicting minimality. We capture this property in the following definition.

▶ **Definition 13.** *A vertex set S is a **k-threshold set** if for every $v \in S$ there exists a set X_v of size at most k so that $N[X_v] \cap S = S \backslash \{v\}$.*

Also note that every subset S' of a k-threshold set S is also a k-threshold set because for every $v \in S'$, a set X_v of size at most k such that $N[X_v] \cap S = S \backslash \{v\}$ also satisfies $N[X_v] \cap S' = S' \backslash \{v\}$ as $S' \subseteq S$. We will use this property explicity in the next section. For now, the discussion leading up to Definition 13 immediately leads to the following observation.

▶ **Observation 14.** *Every minimal k-domination core of a graph G is also a k-threshold set of G.*

Since every minimal k-domination core is a k-threshold set, we will bound the size of k-threshold sets in weakly γ-closed graphs, proving Lemma 12 and leading to an algorithm.

▶ **Lemma 15.** *Every k-threshold set of a weakly γ-closed graph G has size at most b, where $b = k^{O(\gamma k^2)}$*

We now outline how Lemma 15 can be used to identify a vertex x to be removed from a domination core D having size more than b such that $D\backslash\{x\}$ still remains a domination core. No subset of D having size $b+1$ can be a threshold set because of Lemma 15. Thus, we can pick an arbitrary subset X of D having size $b+1$ and for each $x \in X$, test whether $X\backslash\{x\}$ has a dominating set of size at most k without dominating x. Since X is not a threshold set, we will find a vertex $x \in X$ for which such a dominating set does not exist. Thus, we can remove x from D and $D\backslash\{x\}$ will still remain a domination core.

We are now ready to patch up our ideas and provide the full algorithm to prove Theorem 2 assuming Lemma 15 is true. We dedicate the next section solely for the proof of Lemma 15.

Proof of Theorem 2 (assuming the statement of Lemma 15). We first provide the algorithm: Initialize $D = V(G)$. As long as $|D| > b$, arbitrarily pick a subset X of D having size $b+1$. For each $x \in X$, construct a Set Cover instance $I_x = (U_x, \mathcal{F}_x, k)$ with universe $U_x = X\backslash\{x\}$ and family $\mathcal{F}_x = \{N[y] \cap X\backslash\{x\} : y \in \overline{N[x]}\}$. Solve I_x using Theorem 9. If I_x is a no instance, set $D = D\backslash\{x\}$ and proceed to start of the loop.

After the loop terminates, construct the Set Cover instance $I = (U, \mathcal{F}, k)$ where $U = D$ and $\mathcal{F} = \{X_v = N[v] \cap D : v \in V(G)\}$. Use Theorem 9 to find a set cover $\mathcal{S} \subseteq \mathcal{F}$ having size at most k if exists for I. Return no and terminate the algorithm if I is a no instance. If I is a yes instance, return the set $D' = \{v : X_v \in \mathcal{S}\}$.

▷ Claim 16. During each iteration of the loop, the algorithm finds a vertex x to remove from D.

Proof. Consider an arbitrary iteration of the loop. It is clear that $|D| > b$ since the algorithm enters the loop. Observe that no subset of D having size $b+1$ can be a k-threshold set by Lemma 15. Let X be the subset of D picked by the algorithm in that iteration, it is clear that X is not a k-threshold set. Thus, by definition of a k-threshold set, there exists a vertex $x \in X$ for which $X\backslash\{x\}$ does not have a dominating set of size at most k that does not dominate x. It is also easy to see that $X\backslash\{x\}$ has a dominating set of size at most k not dominating x if and only if I_x has a set cover of size at most k. Thus, there is a vertex $x \in X$ for which I_x does not have a set cover of size at most k. Therefore, the algorithm would have removed at least one element from D in that iteration. ◀

▷ Claim 17. In each iteration of the algorithm, D is a domination core.

Proof. Since the set of all vertices of G is itself a trivial domination core, the algorithm starts with a domination core $D = V(G)$. Let X be the subset of D of size $b+1$ picked by the algorithm in that iteration. Also let $x \in X$ be the vertex removed from D in that iteration. By the previous claim, such an x exists. Since the algorithm removed x from D, the set cover instance I_x must have been a no instance. It is easy to see that I_x is a no instance if and only if $X\backslash\{x\}$ does not have a dominating set of size at most k without dominating x. Thus, since every set of size at most k dominating $D\backslash\{x\}$ will dominate $X\backslash\{x\}$ which in turn will dominate x, $D\backslash\{x\}$ is a k-domination core.

Thus, in all iterations of the algorithm, D is a k-domination core. ◀

Now consider D in the last step of the algorithm. The algorithm reaches this step because of the first claim. It is easy to see that I is an yes instance if and only if D has a dominating set of size at most k. Since D is a domination core by the previous claim, this implies that G has dominating set of size at most k if and only if I is a yes instance. Thus, the algorithm returns a dominating set of G of size at most k if one exists, otherwise returns no. Namely, the recovered set D' is a dominating set of G.

For the runtime, the time taken to identify a vertex to remove from D when $|D| > b$ is $b^{O(\gamma k)}$ using Theorem 9 as $|U_x| = b$ and the VC-dimension of the set system $(U_x, \mathcal{F}_x)$ is bounded by 6γ by Theorem 3. This step is repeated at most $n - b$ times. The final step to find the dominating set again takes $b^{O(\gamma k)}$ time since in the last step D has size at most b. Thus, in total the algorithm takes $b^{O(\gamma k)} n^{O(1)}$ time which is $k^{O(\gamma^2 k^3)}$. ◀

6 Threshold Sets in Weakly Closed Graphs

In this section, we prove the crux of our algorithm, namely Lemma 15 which bounds the size of threshold sets in weakly γ-closed graphs. We first begin by stating that the graph induced by any k-threshold set of a weakly γ-closed graph is sparse.

▶ **Lemma 18.** [4] *Given any weakly γ-closed graph G and k-threshold set S of G, $G[S]$ is $(\gamma - 1)k$-degenerate.*

Since every d-degenerate graph on n vertices has an independent set of size at least $n/(d + 1)$ [12], any large k-threshold set will also have a large independent set. This leads us to define the following notion.

▶ **Definition 19.** *A k-threshold set S of a graph G is called an **independent k-threshold set** of G if S is an independent set.*

Further, since every k-threshold set S of a weakly γ-closed graph has an independent set of size at least $\frac{|S|}{(\gamma-1)k+1}$ and since every subset of a k-threshold set is also a k-threshold set, we obtain the following result.

▶ **Lemma 20.** *Every k-threshold set S of a weakly γ-closed graph has an independent k-threshold set of size at least $\frac{|S|}{(\gamma-1)k+1}$.*

By the previous lemma, it is clear that to bound the size of threshold sets in weakly closed graphs, it is enough to bound the size of independent threshold sets. This fact along with Lemma 21 stated below combined prove Lemma 15[5].

▶ **Lemma 21.** *Every independent k-threshold set of a weakly γ-closed graph G has size at most $k^{O(\gamma k^2)}$.*

We prove Lemma 21 by contradiction. Assuming that G has a large independent k-threshold set, we first use results from Ramsey theory to extract a sufficiently large and highly symmetric independent 2-threshold set (this is never proved explitly in the argument). The highly structured independent 2-threshold set implies that G contains one of the obstructions from Lemma 6, contradicting that G is weakly γ-closed.

[4] Proof in Appendix B
[5] proof in Appendix C.

Proof of Lemma 21. Let W be an independent k-threshold set of a weakly γ-closed graph G having size greater than $(3^{15}k^2)^{(3^{16}\gamma k^2)}$. As a first step, we will use results from Ramsey theory to obtain three subsets of vertices of G having useful properties. We will then use these sets to show that G has one of the graphs listed in Lemma 6 as an induced subgraph. By Lemma 6, this will imply that G is not a weakly γ-closed graph, contradicting our assumption and thus completing the proof.

Since W is a k-threshold set of G, for every vertex $w \in W$ there exists a set $X_w \subseteq V(G)$ of size at most k that dominates all vertices in W except w. For each $w \in W$, order the vertices in X_w arbitrarily. Let $X_w = \{x_w^1, \dots, x_w^{p_w}\}$ be the ordering. Also order the vertices in W arbitrarily. Let $W = \{w_1, \dots, w_q\}$ be the ordering.

We now create an auxiliary edge-colored complete graph H with vertex set W. Each color will be a tuple[6] whose size and possible values will become clear in the next step where we assign colors to the edges.

For every pair $i, j \in [|W|]$ such that $i < j$, we color the edge (w_i, w_j) in H as follows:

1. One entry for the number r such that $x_{w_i}^r$ dominates w_j (if more than one such r exists, choose one arbitrarily)
2. One entry for the number s such that $x_{w_j}^s$ dominates w_i (if more than one such s exists, choose one arbitrarily)
3. For each pair[7] of vertices in the multi-set $\{w_i, w_j, x_{w_i}^r, x_{w_j}^s, x_{w_j}^r, x_{w_i}^s\}$ one entry from $\{0, 1, 2\}$ to denote whether those two vertices are (0) the same vertex (1) different and adjacent vertices or (2) different and non-adjacent vertices.

From the definition of H, it follows that the number of possible distinct edge-colors of H is at most $3^{15}k^2$. Let $B \subseteq W$ be a monochromatic clique of maximum size in H and let τ be the color of all the edges in the clique. We will now use the well-known fact (from Ramsey theory [4]) that every edge-colored complete graph on n vertices colored with t colors has a monochromatic clique of size at least $\log_t(n)/t$ to lower bound the size of B. Since the number of possible distinct edge-colors of H is at most $3^{15}k^2$ and the size of W is greater than $(3^{15}k^2)^{(3^{16}\gamma k^2)}$, the size of B is at least 3γ.

Let $B = \{b_1, \dots, b_l\}$ be the ordering of vertices of B in W. Let r and s be the two entries in τ that denote the numbers such that for every pair $i, j \in [l]$ having $i < j$, $x_{b_i}^r$ dominates b_j and $x_{b_j}^s$ dominates b_i. Let $A = \{x_{b_1}^r, \dots, x_{b_l}^r\}$ and $C = \{x_{b_1}^s, \dots, x_{b_l}^s\}$ be ordered multi-sets. For now, we will assume that A and C could be multi-sets but we will soon prove that it is not the case. We now capture some desired properties of A, B and C.

▷ Claim 22. The multi-sets $B = \{b_1, \dots, b_l\}$, $A = \{x_{b_1}^r, \dots, x_{b_l}^r\}$, and $C = \{x_{b_1}^s, \dots, x_{b_l}^s\}$ satisfy the following properties:

1. B is an independent set in G.
2. A, B and C are sets.
3. $A \cap B = \emptyset$, $B \cap C = \emptyset$ and either $A \cap C = \emptyset$ or $A = C$.
4. $\forall i \in [l]$, $(b_i, x_{b_i}^r) \notin E(G)$ and $(b_i, x_{b_i}^s) \notin E(G)$.
5. $\forall i, j \in [l]$ such that $j > i$, $(x_{b_i}^r, b_j) \in E(G)$ and $(b_i, x_{b_j}^s) \in E(G)$.
6. A and C are each either an independent set or a clique in G.
7. $\forall i, j \in [l]$, such that $j < i$, $(x_{b_i}^r, b_j) \in E(G)$ or $\forall i, j \in [l]$, such that $j < i$ $(x_{b_i}^r, b_j) \notin E(G)$.
8. $\forall i, j \in [l]$, such that $j < i$, $(b_i, x_{b_j}^s) \in E(G)$ or $\forall i, j \in [l]$, such that $j < i$, $(b_i, x_{b_j}^s) \notin E(G)$.

[6] When comparing equality of two edge colors, we compare corresponding entries of the two tuples in the order they are defined. Thus the order of the entries in the tuples matter.

[7] We will not need all 15 pairs in our arguments. The colors are defined in this way to keep the description simple.

Proof. Since $B \subseteq W$ and W is a independent threshold set, it follows that B is an independent set (property 1). We now prove property 2. By definition, B is a subset of W which is a set. Now we prove that for each pair $i, j \in [l]$ such that $i < j$, $x^r_{b_i} \neq x^r_{b_j}$ and $x^s_{b_i} \neq x^s_{b_j}$. Since r and s are entries in the coloring τ, $x^r_{b_i}$ dominates b_j and $x^s_{b_j}$ dominates b_i. But by definition $x^r_{b_j}$ does not dominate b_j and $x^s_{b_i}$ does not dominate b_i. Therefore $x^r_{b_i} \neq x^r_{b_j}$ and $x^s_{b_i} \neq x^s_{b_j}$.

For property 3, we first show that $A \cap B = \emptyset$. We prove that $\forall i, j \in [l]$, $b_i \neq x^r_{b_j}$. If $i = j$, then $b_i \neq x^r_{b_j}$ because by definition $x^r_{b_i}$ belongs to X_i and thus does not dominate b_i. Let $b_i = x^r_{b_j}$ for some $i > j$, then in the coloring τ the entry corresponding to the pair of vertices b_i and $x^r_{b_j}$ must be 0 since they are the same. Thus, since all edges in clique B in H have color τ, it means that $x^r_{b_1} = b_2$ and $x^r_{b_1} = b_3$ but $b_2 \neq b_3$. Thus, $b_i \neq x^r_{b_j}$. Similarly, we can prove that $b_i \neq x^r_{b_j}$ in the case when $i < j$. The proof that $B \cap C = \emptyset$ is symmetric and therefore omitted.

Now, we show that $A \cap C = \emptyset$ or $A = C$. If $r = s$, then $A = C$. If $r \neq s$, we will show that $\forall i, j \in [l]$, $x^r_{b_i} \neq x^s_{b_j}$. If $i = j$, by the definition of X_{b_i}, it follows that $x^r_{b_i} \neq x^s_{b_i}$. If $i \neq j$, without loss of generality let us consider the case when $i < j$ and a similar argument will hold for the case when $i > j$. If $x^r_{b_i} = x^s_{b_j}$, then by our coloring τ, $x^r_{b_1} = x^s_{b_2}$ and $x^r_{b_1} = x^s_{b_3}$. But $x^s_{b_2} \neq x^s_{b_3}$ by property 2. Thus, $x^r_{b_i} \neq x^s_{b_j}$.

Property 4 is true because $A \cap B = \emptyset$, $B \cap C = \emptyset$ and for each b_i in B, $x^r_{b_i}$ and $x^s_{b_i}$ are in X_i and thus do not dominate b_i. Property 5 follows because $A \cap B = \emptyset$, $B \cap C = \emptyset$ and the fact that r and s are entries in the coloring τ such that $\forall i, j \in [l]$, having $j > i$, $x^r_{b_i}$ dominates b_j and $x^s_{b_j}$ dominates b_i.

Since $A \cap B = \emptyset$ and $B \cap C = \emptyset$, $\forall i, j \in [l]$ such that $j < i$ the coloring τ has an entry with value either 1 or 2 corresponding to each pair in $\{(x^r_{b_i}, x^r_{b_j}), (x^s_{b_i}, x^s_{b_j}), (x^r_{b_i}, b_j), (b_i, x^s_{b_j})\}$. Since (1) denotes that the pair of vertices are adjacent and (2) denotes that the pair of vertices are non-adjacent, properties 6-8 are true. This completes the proof.

◀

We now use the sets (Claim 22 Property 2) A, B, and C to show that G has one of the graphs listed in Lemma 6 as an induced subgraph. For this, we will use the properties listed in Claim 22. We remark that we will directly refer to them as properties rather than referring to the claim each time. Recall that $l = |A| = |B| = |C|$. Firstly, we divide into two cases based on whether $A = C$ or not.

Case (i) $A = C$ **:** By property 3, A and B are disjoint. We divide this case further into two cases based on property 6 - A is either an independent set or a clique.

(a) A is a clique: Let $G' = G[A \cup B]$. Then, B is an independent set (by property 1) and $\forall i, j \in [l]$ $(x^r_i, b_i) \notin E(G')$ if $i = j$ (by property 4) and $(x^r_i, b_j) \in E(G')$ otherwise (by properties 5 and $A = C$). Thus G' is a semi split co-matching of order $l \geq 3\gamma$.

(b) A is an independent set: Let $A' = \{x^r_{b_1}, \dots, x^r_{b_\gamma}\}$, $B' = \{b_{\gamma+1}, \dots, b_{2\gamma}\}$, and $G' = G[A' \cup B']$. Observe that we can define sets A' and B' since $l \geq 3\gamma$. Again, by property 5, $\forall i \in \{1, \dots, \gamma\}, j \in \{\gamma+1, \dots, 2\gamma\}$, $(x^r_{b_i}, b_j) \in E(G')$. Thus G' is a complete bipartite graph of order γ.

Case (ii) $A \neq C$ **:** Since $A \neq C$, by property 3 the sets A, B, and C are disjoint. We divide this case further based on properties 6-8.

(a) A is an independent set: Let $A' = \{x^r_{b_1}, \dots, x^r_{b_\gamma}\}$, $B' = \{b_{\gamma+1}, \dots, b_{2\gamma}\}$ and $G' = G[A' \cup B']$. We can show that G' is a complete bipartite graph by the same argument as case (i.b).

(b) C is an independent set: Same argument as the previous case with sets $B' = \{b_1, \dots, b_\gamma\}$, $C' = \{x^s_{b_{\gamma+1}}, \dots, x^s_{b_{2\gamma}}\}$ and graph $G' = G[B' \cup C']$.

(c) A is a clique and $\forall i \in [l]$, $\forall j < i$, $x^r_{b_i}$ is adjacent to b_j: Similar to case (i.a), $G' = G[A \cup B]$ is a semi split co-matching of order $l \geq 3\gamma$.

(d) C is a clique and $\forall i \in [l]$, $\forall j < i$, b_i is adjacent to $x^s_{b_j}$: Same argument as previous case with $G' = G[B \cup C]$.

(e) A and C are cliques, $\forall i \in [l]$, $\forall j < i$, $x^r_{b_i}$ is not adjacent to b_j and $\forall i \in [l]$, $\forall j < i$, b_i is not adjacent to $x^s_{b_j}$: Let $G' = G[A \cup B \cup C]$. By the case we are in and property 5, it follows that G' is a double split half graph with B being the independent set (property 1).

Thus, in all cases $G[A \cup B \cup C]$ is not weakly γ-closed by Lemma 6, contradicting the assumption that G is weakly γ-closed, and completing the proof of the lemma. ◀

7 Conclusion and Barriers to Further Improvements

In this work we gave an algorithm for Dominating Set with running time $2^{O(\gamma^2 k^3)} n^{O(1)}$. This resolves affirmatively an open problem of Koana et al. [23] who asked whether the problem is fixed-parameter tractable when parameterized by k and the weak closure γ of the input graph. Our running time hides a large constant in the exponent. We made no effort to optimize this constant because, at this point, it is not even clear that the form $O(\gamma^2 k^3)$ of the exponent in the running time is near-optimal.

On the way to obtaining our main result, we proved that every minimal k-domination core of G has size at most $k^{O(\gamma k^2)}$. We also showed that the VC-dimension of a weakly γ-closed graph G is at most 6γ and used this result in our FPT algorithm for Dominating Set and to obtain an $O(\gamma \log(\gamma k))$-approximation for Dominating Set. The bound on VC-dimension might be interesting for other problems on weakly-closed graphs.

Our work leaves the following natural open problem: *does* Dominating Set *admit a kernel of size $k^{f(\gamma)}$ for some function f?* One natural approach would be to improve the bound in Lemma 12 by obtaining a polynomial upper bound for the size of minimal domination cores in weakly closed graphs. Unfortunately, this is not possible: for every positive integer k, there exists a weakly 1-closed graph with a minimal k-domination core of size 2^{k+1} (see Appendix D). Notice that the argument only shows an obstacle for using this approach for getting polynomial kernels and does not rule out the existence of polynomial kernels.

In light of the $O(\gamma \log(\gamma k))$ approximation algorithm from Section 4, it is natural to ask whether Dominating Set could admit for every fixed constant γ a constant factor approximation algorithm on weakly γ-closed graphs. It is known from [30] Theorem 2 that there exists a c such that a polynomial time $c\frac{\log n}{\log \log n}$-approximation algorithm for Dominating Set in $K_{3,3}$-free graphs would imply that $\mathsf{NP} \subseteq \mathsf{DTIME}(2^{n^{1-\varepsilon}})$ for some $0 < \varepsilon < 1/2$. The graphs constructed in the reduction[8] are also weakly 3-closed and hence we get the same result for weakly 3-closed graphs.

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[8] The reduction is from Set Cover on a set system in which the maximum intersection between any two sets in the family is 1 [30, 26].

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A Proof of Lemma 6 (Obstructions to Weak Closure)

In order to prove Lemma 6, we will first prove that complete bipartite graphs, semi split co-matchings and double split half graphs (see Figure 1) having order more than γ, γ, and 3γ respectively are not weakly γ-closed.

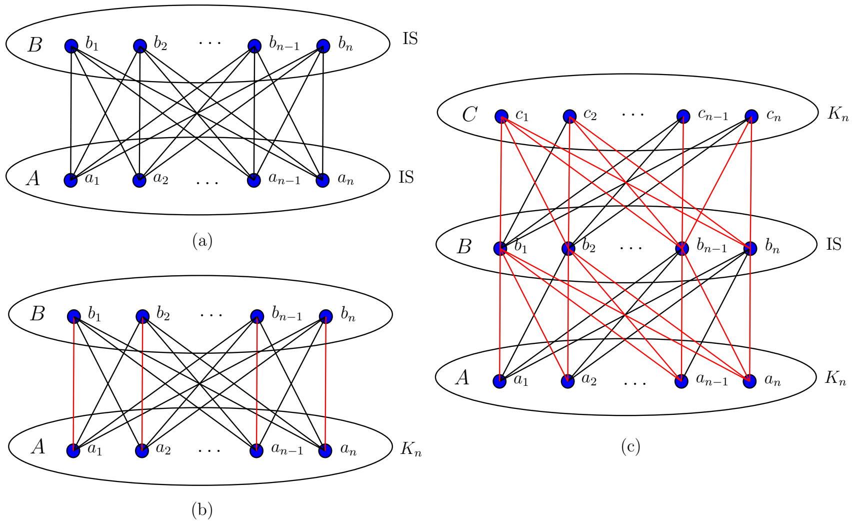

Figure 1 *Sufficiently large (a) complete bipartite graph, (b) semi split co-matching and (c) double split half graph are not weakly γ-closed. Edges are colored black and non-edges are colored red. Note that there may be arbitrary edges between the vertices in B in a semi split co-matching and between the two cliques (A and C) in a double split half graph. On the other hand split half graphs are weakly 1-closed.*

Let G be a graph, if for a vertex v in G, there exists a non-neighbour u in G such that $|N(u) \cap N(v)| \geq \gamma$, we will refer to u as a *weak-pair* of v. Observe that if u is a weak-pair of v, then v is also a weak-pair of u. To prove that G is not weakly γ-closed, it is enough to show that every vertex in G has a weak-pair.

▶ **Lemma 23.** *If G is a complete bipartite graph of order $n \geq \gamma$, it is not weakly γ-closed.*

Proof. Let $V(G) = \{a_1, \dots, a_n\} \uplus \{b_1, \dots, b_n\}$. First, we show that $\forall i, j \in [n]$, having $i \neq j$, a_i is a weak-pair of a_j. It holds that $(a_i, a_j) \notin E(G)$ and $|N(a_i) \cap N(a_j)| \geq \gamma$ since it is a complete bipartite graph and $n \geq \gamma$. Similarly $\forall i, j \in n$, having $i \neq j$, b_i is a weak-pair of b_j. Thus G is not weakly γ-closed. ◀

▶ **Lemma 24.** *If G is a semi split co-matching of order $n > \gamma$, it is not weakly γ-closed.*

Proof. Let $V(G) = \{a_1, \dots, a_n\} \uplus \{b_1, \dots, b_n\}$. We show that $\forall i \in [n]$, b_i is a weak-pair of a_i and thus a_i is a weak-pair of b_i. Since G is a semi-split co-matching, a_i is not adjacent to b_i and both a_i and b_i are adjacent to all a_j, $j \neq i$. Because $n > \gamma$, $|N(a_i) \cap N(b_i)| \geq \gamma$. Since all vertices in G have a weak pair, it is not weakly γ-closed. ◀

▶ **Lemma 25.** *If G is a double split half graph of order $n \geq 3\gamma$, it is not weakly γ-closed.*

Proof. Let $V(G) = \{a_1, \dots, a_n\} \uplus \{b_1, \dots, b_n\} \uplus \{c_1, \dots, c_n\}$.

First, we will prove that $\forall i \in [n]$, b_i has a weak-pair. Since G is a double split half graph, observe that $\forall i \in [n]$, b_i is adjacent to all c_j, $j > i$ and to all a_j, $j < i$. Thus, since both $G[\{a_1,\ldots,a_n\}]$ and $G[\{c_1,\ldots,c_n\}]$ are cliques and $n \geq 3\gamma$, it follows that either $|N[b_i] \cap N[a_i]| \geq \gamma$ or $|N[b_i] \cap N[c_i]| \geq \gamma$. Hence, since b_i is not adjacent to both a_i and c_i, either a_i or c_i is a weak pair of b_i.

Second, we will prove that $\forall i \in [n]$, a_i has a weak-pair. We divide the proof into two cases: (a) $i > \gamma$ and (b) $i \leq \gamma$.

For case (a), we will show that b_i is a weak-pair of a_i. Since G is a double split half graph, a_i is not adjacent to b_i, b_i is incident to all a_j, $j < i$ and $G[\{a_1,\ldots,a_n\}]$ is a clique. Thus, as we are in the case when $i > \gamma$, it follows that $|N(a_i) \cap N(b_i)| \geq \gamma$. This proves that b_i is a weak-pair of a_i.

For case (b), we will show that either b_i or some c_j, $j > n - \gamma$ is a weak-pair of a_i. If a_i is not adjacent to some c_j, $j > n - \gamma$, then since G is a double split half graph, both a_i and c_j are adjacent to all b_k, $i < k < j$. Since, $n \geq 3\gamma$, $i \leq \gamma$ and $j > n - \gamma$, $|N(a_i) \cap N(c_j)| \geq \gamma$ and thus c_j is a weak-pair of a_i. If a_i is adjacent to all c_j, $j > n - \gamma$. Then again since G is a double split half graph, a_i is not adjacent to b_i and b_i is adjacent to all c_j, $j > n - \gamma$. Thus, it follows that $|N(a_i) \cap N(b_i)| \geq \gamma$ since $n \geq 3\gamma$. This proves that b_i is a weak-pair of a_i.

Finally, we can use a very similar argument to that used for a_is to prove that $\forall i \in [n]$, c_i has weak-pair. But here the two cases will be (a) $i \leq n - \gamma$ and (b) $i > n - \gamma$.

Therefore, since all vertices have a weak-pair, G is not weakly γ-closed. ◀

We give a short proof for lemma 6 using the previous lemmas.

Proof of Lemma 6. By the definition of weakly γ-closed graphs any graph having an induced subgraph that is not weakly γ-closed graph is also not weakly γ-closed. Thus, Lemma 6 follows from all the previous lemmas in this section. ◀

B Proof of Lemma 18

We now prove Lemma 18 which says that given a weakly γ-closed graph G and a k-threshold set S of G, $G[S]$ is $(\gamma - 1)k$-degenerate.

Proof of Lemma 18. Given a weak ordering O of a weakly γ-closed graph G, let the order induced by O on a subset S of vertices of G be denoted by O_S. To complete the proof, it is enough to prove the following claim.

▷ Claim 26. Given any weakly γ-closed graph G, weak ordering O of G and k-threshold set S of G, every vertex in S has forward degree at most $(\gamma - 1)k$ in O_S.

Proof. Suppose the claim was not true. Let u be the first vertex in O_S having more than $(\gamma - 1)k$ forward neighbours in O_S. Let F be the set of forward neighbours of u in O_S. Also, let X be a dominating set of $S \setminus \{u\}$ having size at most k and not dominating u. Since S is a k-threshold set of G, such a set X exists.

Firstly we prove that every vertex $v \in X$ that is not adjacent to u can dominate at most $\gamma - 1$ vertices in F since G is weakly γ-closed. If v is ahead of u in the ordering O, then since no non-neighbour of u can have more than $\gamma - 1$ forward common neighbours with u, v is adjacent to at most $\gamma - 1$ vertices in F. Similarly, if u is ahead of v in O, the same argument holds with respect to v.

Now, since $|F| > (\gamma - 1)k$ and $|X| \leq k$, by pigeon hole principle there is a vertex $v \in X$ that is adjacent to more than $\gamma - 1$ vertices in F. Therefore u must be equal to or adjacent to v as G is weakly γ-closed. Thus, we have reached a contradiction to the fact that X did not dominate u. This completes the proof. ◀

Let O be a weak ordering of G, then by the above claim, O_S is a degeneracy ordering of $G[S]$ with degeneracy $(\gamma - 1)k$. Thus $G[S]$ is a $(\gamma - 1)k$-degenerate graph. ◀

C Proof of Lemma 15

We now give a short proof for Lemma 15, that is we prove that the size of k-threshold sets in weakly γ-closed graphs is at most $k^{O(\gamma k^2)}$.

Proof of Lemma 15. Lemma 20 shows that every k-threshold set S of a weakly γ-closed graph must have an independent k-threshold set of size at least $\frac{|S|}{(\gamma-1)k+1}$. Lemma 21 shows that every independent k-threshold set of a weakly γ-closed graph has size at most $k^{O(\gamma k^2)}$. Combining these two results, we can infer that every k-threshold set of a weakly γ-closed graph must have size at most $k^{O(\gamma k^2)}$. ◀

D Minimal k-domination cores of size 2^k in weakly 1-closed graphs

Consider the graph G obtained by taking a complete binary tree T of depth $k+1$ and making every node adjacent to all its ancestors. The set S of all the nodes in level $k+1$ is a minimal k-domination core.

S is a k-domination core because any vertex adjacent to any vertex v in S is adjacent to all vertices adjacent to v. Thus since $N[S] = V(G)$, any set of size at most k dominating S will dominate $V(G)$ as well.

For every vertex $v \in S$, let A_v be the set of ancestors of v in T and let C_v be the set of all children of all the nodes in A_v in T. Then for each $v \in S$, the set $C_v \setminus (A_v \cup \{v\})$ is a dominating set of $S \setminus \{v\}$ of size k that does not dominate v. Therefore S is minimal.

It is natural to ask whether the example can be strengthened to give a c-closed graph with an exponential size minimal k-domination core. However, it is possible to upper bound the size of minimal k-domination cores in c-closed graphs by ck^{c+1}. We omit the proof of this statement, as it is out of scope for this paper.