Erdős-Pósa Property of Obstructions to Interval Graphs

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Abstract

The duality between packing and covering problems lies at the heart of fundamental combinatorial proofs, as well as well-known algorithmic methods such as the primal-dual method for approximation and win/win-approach for parameterized analysis. The very essence of this duality is encompassed by a well-known property called the Erdős-Pósa property, which has been extensively studied for over five decades. Informally, we say that a class of graphs $\mathcal{F}$ admits the Erdős-Pósa property if there exists f such that for any graph G, either G has k vertex-disjoint "copies" of the graphs in $\mathcal{F}$, or there is a set $S \subseteq V(G)$ of $f(k)$ vertices that intersects all copies of the graphs in $\mathcal{F}$. In the context of any graph class $\mathcal{G}$, the most natural question that arises in this regard is as follows—do obstructions to $\mathcal{G}$ have the Erdős-Pósa property? Having this view in mind, we focus on the class of interval graphs. Structural properties of interval graphs are intensively studied, also as they lead to the design of polynomial-time algorithms for classic problems that are NP-hard on general graphs. Nevertheless, about one of the most basic properties of such graphs, namely, the Erdős-Pósa property, nothing is known. In this paper, we settle this anomaly: we prove that the family of obstructions to interval graphs—namely, the family of chordless cycles and ATs—admits the Erdős-Pósa property. Our main theorem immediately results in an algorithm to decide whether an input graph G has k vertex-disjoint ATs and chordless cycles, or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all ATs and chordless cycles.

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1 Introduction

Packing and covering problems are ubiquitous in both graph theory and computer science. The duality between packing and covering problems lies at the heart of not only fundamental combinatorial proofs, but also well-known algorithmic methods such as the primal-dual method for approximation and win/win-approach for parameterized analysis. The very essence of this duality is encompassed by a well-known property called the Erdős-Pósa property. This property, being both simple and powerful, has been extensively studied for over five decades. In the context of any graph class $\mathcal{G}$, the most natural question that arises in this regard is as follows—do obstructions to $\mathcal{G}$ have the Erdős-Pósa property? Having this view in mind, we focus on the class of interval graphs. Arguably, this is the most basic class of graphs that can be viewed as geometric inputs—indeed, an interval graph is the intersection graph of a family of intervals on the real lines. Interval graphs are among the most well-studied classes of graphs in the literature. In particular, the usage of interval graphs as models is relevant to a wide variety of applications, ranging from resource allocation in operations research and scheduling theory to assembling contiguous subsequences in DNA mapping. From an algorithmic point of view, the structural properties of interval graphs are also intensively studied as they allow to design polynomial-time algorithms for well-known problems in computer science, such as Independent Set and Hamiltonian Path, that are NP-hard on general graphs. Nevertheless, about one of the most basic properties of such graphs, namely, the Erdős-Pósa property, nothing is known! Our main contribution settles this anomaly: we prove that obstructions to interval graphs admit the Erdős-Pósa property.

Before we turn to consider our contribution in more detail, we present a gentle introduction to the rich realm of studies of Erdős-Pósa properties. For this purpose, we first define packing and covering problems. Let $\preceq$ be a containment relation (of a graph into another graph), and let $\mathcal{F}$ be a family of graphs. For example, we can define the containment relationship $\preceq$ as follows: for graphs G and H, $H \preceq G$ if and only if H is an induced subgraph/subgraph/minor/topological minor of G. In this setting, $(\mathcal{F}, \preceq)$-Packing is the problem whose input consists of a graph G and an integer k, and the objective is to decide if G has k vertex-disjoint subsets, $S_1, S_2, \ldots, S_k \subseteq V(G)$, where for each $i \in [k]$, there exists $F \in \mathcal{F}$ such that $F \preceq G[S_i]$. For example, if $\mathcal{F} = \{F\}$ and the relation refers to induced subgraphs, then we simply ask whether G has k vertex-disjoint "exact copies" of F. The $(\mathcal{F}, \preceq)$-Covering problem has the same input, but its objective is to decide if there is a set $S \subseteq V(G)$ of size at most k such that there does not exist $F \in \mathcal{F}$ that satisfies $F \preceq G - S$. Some well-known examples of packing problems (and their corresponding covering problems) are Maximum Matching (Vertex Cover), Vertex-Disjoint s-t Paths (s-t-Separator), Cycle Packing (Feedback Vertex Set), P_3-Packing (Cluster Vertex Deletion), and Triangle Packing (Triangle Free Deletion).

Kőnig's and Menger's theorems are cornerstones of Graph Theory in general, and of the study of packing and covering problems in particular, which have also found a wide variety of applications in computer science. For example, Menger's theorem is particularly relevant to survivable network design (see, e.g., [5, 46]) and combinatorial optimization (see, e.g., [43, 19]). Formally, Kőnig's theorem states that in bipartite graphs, the maximum size of a matching *equals* the minimum size of a vertex cover [30, 13]. Menger's theorem also exhibits an equality—it states that for a given graph G and a pair of vertices s and t, either G has k vertex-disjoint paths between s and t or there is a set $S \subseteq V(G) \setminus \{s, t\}$ of size k such that $G - S$ has no path between s and t [33, 13]. Both theorems relate a packing problem to a covering problem,[1] by exhibiting equality between the size of a maximum packing and the size of a minimum covering. However, most natural packing and covering problems are not known to exhibit such an equality; in fact, frequently such an equality is *proven* not to exist. By simply relaxing the notion of equality, we enter the rich realm of the Erdős-Pósa properties.

The Erdős-Pósa Property. A celebrated theorem by Erdős and Pósa [14] states that for any graph G, either there is a set of k vertex-disjoint cycles in G, or there is a set $S \subseteq V(G)$ of $f(k) = \mathcal{O}(k \log k)$ vertices that intersects (covers) all cycles of G.[2] Notably, Erdős and Pósa [14] also showed that there exists a constant c and infinitely many pairs (G, k) such

[1] For example, Kőnig's theorem addresses the class $\mathcal{F} = \{F\}$ such that F is the graph on a single edge, where $\preceq$ refers to induced subgraphs/subgraphs.

[2] In the terminology of packing and covering, we address the class $\mathcal{F}$ of all cycles, where $\preceq$ refers to induced subgraphs/subgraphs.

that G has neither k vertex-disjoint cycles nor a set $S \subseteq V(G)$ of $ck \log k$ vertices that covers all cycles of G. That is, not only equality cannot be expected, but also any function $f(k) = o(k \log k)$. We remark that later, Simonovits [45] provided concrete examples which realize the lower bound. The result of Erdős and Pósa [14] initiated a flurry of extensive study of the so called "Erdős-Pósa property" for various families of graphs as well as containment relationships. Formally, a family of graphs $\mathcal{F}$ and a containment relation $\preceq$ are said to admit the Erdős-Pósa property if there exists a function $f(\cdot)$ such that given a graph G and an integer k, either there are k vertex-disjoint subsets $S_1, \ldots S_k \subseteq V(G)$ so that for each $i \in [k]$, there is $F \in \mathcal{F}$ satisfying $F \preceq G[S_i]$, or there is a set $S \subseteq V(G)$ of size at most $f(k)$ so that there is no $F \in \mathcal{F}$ satisfying $F \preceq G - S$. Here, the first question that comes to mind is—do all families of graphs $\mathcal{F}$ and containment relationships $\preceq$ exhibit the Erdős-Pósa property?

The answer to this question is negative. For example, consider a fixed graph H, and let $\mathcal{F}(H)$ be the family of graphs that contain H as a minor. Robertson and Seymour [42] showed that $\mathcal{F}(H)$ with the containment relation referring to subgraphs admits the Erdős-Pósa property if and only if H is a planar graph. This result generalizes the result in [14]. However, the function $f(\cdot)$ given by [42] is exponential—can it be made polynomial? A few years ago, the bound was improved to $\mathcal{O}(k \log^c k)$ by Chekuri and Chuzhoy [12] following a more general approach which is applicable to other families as well. A well-known example of a different flavor conerns odd cycles. Specifically, for $\mathcal{F}$ being the family of odd length cycles, Reed [39] showed that $\mathcal{F}$ (for subgraphs and induced subgraphs) does not admit the Erdős-Pósa property.

Since the emergence of the result of Erdős and Pósa [14], a multitude of studies on the Erdős-Pósa property have appeared in the literature for several combinatorial objects beyond graphs. This includes extensions to digraphs [32, 44, 40, 22, 20], rooted graphs [9, 26, 35, 24], labeled graphs [29], signed graphs [23, 3], hypergraphs [1, 6, 7], matroids [16], helly-type theorems [21], H-minors [41], H-immersions [17, 31], and H-butterfly directed minors [2] (also see [38]). This list is not comprehensive but rather illustrative. We refer to surveys such as [37] for more information. Even for subfamilies of cycles alone, there is a vast literature devoted to the Erdős-Pósa property. Studies of the Erdős-Pósa property for subfamilies of cycles include, for example, long cycles (subgraphs) [4, 34], directed cycles (subgraphs and induced subgraphs) [40, 20], chordless cycles (induced subgraphs) [25] and cycles intersecting a prescribed vertex set [27, 35]. Not all subfamiles of cycles admit the Erdős-Pósa property. For example, recall the result stated earlier regarding the family of odd cycles [39]. For this subfamily of cycles alone, there has been a sequence of research about finding classes of graphs for which the family of odd cycles (subgraphs and induced subgraphs) admits the Erdős-Pósa property. This includes planar graphs [15], or graphs with certain connectivity constraints [48, 36, 28, 24]. Not only the family of odd cycles does not admit the Erdős-Pósa property, but also subfamilies such as the family of chordless cycles of length at least 5 [25].

A large number of the results above can be viewed as the question of packing or covering obstructions to a class of graphs. In some of these papers, this view is explicitly stated as the motivation behind the conducted studies. For example, the classic result by Erdős and Pósa [14] regards the question of packing and covering obstructions to forests. The results concerning odd cycles address obstructions to bipartite graphs. The setting of the work about packing and covering chordless cycles, as presented by [25], addresses obstructions to chordal graphs. Furthermore, Kőnig's theorem relates to obstructions to edgeless graphs, and the work by Robertson and Seymour [42] relates to obstructions to subfamilies of minor free graphs. We remark that other results can also be interpreted in this manner. Given that the class of interval graphs is among the most basic, well-studied families of graphs, we find it important to study the Erdős-Pósa [14] property with respect to it. Let $\mathcal{F}$ be the family of chordless cycles and asteroidal triples (ATs), see Section 2. It is well known that the class of interval graphs is precisely the class of graphs that exclude every graph in $\mathcal{F}$ as an induced subgraph [18, 8]. Given this clean characterization, the following question naturally arises:

> Does the family of chordless cycles and ATs—that is, obstructions to interval graphs—admit the Erdős-Pósa property?

Our Contribution. We provide an affirmative answer to the question above. Moreover, the dependency of the size of the covering set on k in our result is only $\mathcal{O}(k^2 \log k)$.[3] Specifically, we obtain the following theorem, where from now on, "obstructions" refer to ATs and chordless cycles.

▶ **Theorem 1.** *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph.*

As a consequence of our main theorem, we also derive an algorithm to decide whether an input graph G has k vertex-disjoint obstructions (to interval graphs), or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all such obstructions.

We conclude the introduction with a high-level (informal) overview of our proof. We begin by easily "getting rid" of all chordless cycles due to the work by [25], as well as all small ATs. Now, the heart of our proof consists of two main components. First, we exhibit the Erdős-Pósa property of the family of ATs on graphs that have a clique caterpillar (that is, a tree decomposition that is a caterpillar, where every bag is a clique). Second, we show how this result can be utilized to derive our main theorem by analyzing "conflict-free sets" (defined below) with respect to a modular tree decomposition of the graph. Let us now elaborate on each component.

To analyze the case of a clique caterpillar, we present a procedure that at each iteration, finds an AT $\mathbb{O}$ with specific properties, inserts a set S of $\mathcal{O}(k)$ new vertices into a set $S^\star$ initialized to be empty, and removes the vertices in S from the graph (only for the sake of the execution of the procedure). Specifically, the set S consists of the terminals, centers and a few base vertices of $\mathbb{O}$, as well as all of the vertices of a "small" separator between the non-shallow terminals of $\mathbb{O}$ that we push as much as possible to the right of the caterpillar. The procedure terminates once the graph becomes an interval graph. Hence, it is clear that if at most $\mathcal{O}(k)$ iterations take place, then $S^\star$ is a set of size $\mathcal{O}(k^2)$ that intersects all ATs, which implies that our job is done. Otherwise, we require an intricate analysis to establish the existence of k vertex-disjoint ATs. Roughly, the two main components here are (1) from the sequence of ATs encountered by our procedure, we can extract a sequence of the same length (of possibly *different* ATs) where each AT has the property that the subpath of its base that lies after the separator does not intersect any AT positioned after it in the sequence, and (2) from the modified sequence, we can extract a *subsequence* of ATs where disjointess is also guaranteed with respect to base vertices that lie before the separator.

Towards the proof of the second item, we first show that for every sequence "resembling" the one encountered by our procedure, and for all ATs $\mathbb{O}$ and $\mathbb{O}'$ in that sequence such that

[3] In fact, all of our arguments achieve the dependency $\mathcal{O}(k^2)$, but we gain an extra $\log k$ factor due to an invocation of a result by Kwon and Kim [25]. Shaving off the $\log k$ factor in the result by [25] will automatically also shave it off from our result.

$\mathbb{O}'$ comes before $\mathbb{O}$, we have the following property: only the leftmost terminal and base vertex of $\mathbb{O}$ can belong to the base path of $\mathbb{O}'$ that lies before the separator associated with $\mathbb{O}'$, and even that is only possible under certain conditions. This result then allows us to further argue about the relation between every *three* ATs in the sequence with respect to the "left sides of separators". Having established this relation, the argument about a complete sequence is derived. Towards the proof of the first item, we first show that for any AT $\mathbb{O}$ in the sequence, we can find a path between a vertex in the separator associated with $\mathbb{O}$ and the right terminal of $\mathbb{O}$ that avoids all ATs coming after $\mathbb{O}$ in the sequence. Then, by relying on structural results by Cao and Marx [11], we argue that this path can be used to replace part of $\mathbb{O}$ so that the result is yet another AT.

Let us now turn to our analysis of the general case—specifically, we explain how it is reduced to instances of the case of a clique caterpillar. We define "problematic" nodes in the modular tree decomposition of the input graph as the nodes associated with subgraphs that contain at least one AT that is not present in any of the subgraphs associated with their children. This definition also immediately gives rise to an association between nodes and ATs, so that each AT is associated with exactly one node. We observe that maximal modules of problematic nodes are vertex disjoint, and that each problematic node has "many" children. It is also easily shown that the set of all problematic nodes can be partitioned into two sets that have no "conflict"—that is, on the unique path between every two nodes of one set, there exists a node of the other set. The point in analyzing each conflict-free set P separately is that for each problematic node in such a set, we prove that there exist at least k vertices in the subgraph associated with that node that do not belong to any subgraph associated with its problematic descendants from P. In particular, this allows us to examine each problematic node *individually*, and associate an instance of the clique caterpillar case with it (the construction of the caterpillar decomposition itself partially follows from structural results by Cao and Marx [11]). Specifically, we are able to collect the sets of ATs found in each instance, and argue that (after some modification) all of these ATs across all the sets are in fact vertex disjoint. This result then allows us to handle the "packing perspective" of the proof. We remark that although we can create $\mathcal{O}(k)$ instances of the clique caterpillar case, and each individual instance can create a gap of $\mathcal{O}(k^2)$, we eventually get a gap of only $\mathcal{O}(k^2)$ rather than $\mathcal{O}(k^3)$ as we argue that the sum of the contributions to the gap of all individual instances is $\mathcal{O}(k^2)$.

Due to lack of space, proofs of statements marked by '*' are relegated to the appendix.

2 Preliminaries

For $n \in \mathbb{N}$, we use $[n]$ as a shorthand for $\{1, 2, \ldots, n\}$. Given a function $f : A \to B$ and a subset $A' \subset A$, we use $f|_{A'}$ to denote the restriction of f to A'.

Basic Graph Theory. We refer to standard terminology from the book of Diestel [13] for those graph-related terms that are not explicitly defined here. Additional details may also be found in Appendix A. When the graph G is clear from context, denote $n = |V(G)|$ and $m = |E(G)|$. We say that a vertex v in G is *simplicial* if $N_G(v)$ induces a clique. A *caterpillar* is a tree T for which there exists a subpath P of T, called a *central path*, such that the removal of the vertices of P from T results in an edgeless graph. Given a rooted tree T and a vertex $v \in V(T)$, we use $T|_v$ to denote the subtree of T rooted at v. Moreover, $\mathsf{child}(v)$ denotes the set of children of v in T. We do not treat a vertex as a descendant of itself. A *chordal graph* is a graph that has no chordless cycle on at least four vertices.

Interval Graphs. An *interval graph* is a graph G that does not contain any of the following graphs, called *obstructions*, as an induced subgraph (see Figure 1 in the appendix).

- **Long Claw.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_3\}, \{t, b_2\}, \{c, b_1\}, \{c, b_2\}, \{c, b_3\}\}$.
- **Whipping Top (or Umbrella).** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_2\}, \{c, t\}, \{c, b_1\}, \{c, b_2\}, \{b_3, t_\ell\}, \{b_3, b_1\}, \{b_3, c\}, \{b_3, b_2\}, \{b_3, t_r\}\}$.
- **†-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c\} \cup \{b_1, b_2, \dots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c), \{t_\ell, b_1\}, \{t_r, b_z\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 2$. A †-AW where $z = 2$ is called a *net*.
- **‡-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c_1, c_2\} \cup \{b_1, b_2, \dots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c_1\}, \{t, c_2\}, \{c_1, c_2\}, \{t_\ell, b_1\}, \{t_r, b_z\}, \{t_\ell, c_1\}, \{t_r, c_2\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ is called a *tent*.
- **Hole.** A chordless cycles on at least four vertices.

Long claws and whipping tops are also called ATs, but we shall reserve this name for †-AWs and ‡-AWs (AW stand for Asteroidal Witness).[4] An obstruction $\mathbb{O}$ is *minimal* if there does not exist an obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subset V(\mathbb{O})$. In each of the first four obstructions, the vertices t_ℓ, t_r, and t are called *terminals*, the vertices c, c_1, and c_2 are called *centers*, and the other vertices are called *base vertices.* To simplify notation, when we consider a †-AW, we use c_1 and c_2 to refer to c (this allows us to refer to a †-AW and a ‡-AW in a unified manner). Furthermore, the vertex t is called the *shallow terminal.* The induced path on the set of base vertices is called the *base* of the AT, and it is denoted by $\mathsf{base}(\mathbb{O})$. Moreover, we say that the induced path on the set of base vertices, t_ℓ and t_r is the *extended base* of the AT, and it is denoted by $P(\mathbb{O})$. Given a graph G, a vertex v is *shallow in G* if G has at least one AT where v is the shallow terminal.

Tree Decomposition. For a tree decomposition (T, β) of a graph G, if T is a path, then (T, β) is also called a *path decomposition*, and if T is a caterpillar then (T, β) is also called a *caterpillar decomposition.* A *clique path* (*clique caterpillar*) of a graph G is a path decomposition (resp. *caterpillar decomposition*) of G where every bag is a distinct maximal clique. We remark that not every graph admits a clique caterpillar.

Modules. Let G be a graph. A subset $M \subseteq V(G)$ is a *module* if for all $u, w \in M$ and $v \in V(G) \setminus M$, either both u and w are adjacent to v or both u and w are not adjacent to v. A module is *nontrivial* if neither $V(M) = \emptyset$ nor $V(M) = V(G)$.

A *modular tree decomposition* of a graph $G = (V, E)$ is a linear-size representation of all its modules. It consists of a rooted tree T, a function $f : V(T) \to 2^{V(G)}$ and a function $g : V(T) \to \{0, 1\}$, which in particular satisfy the following properties:

1. M is a module of G if and only if there is a node $v \in V(T)$ for which, either $M = f(v)$, or both $g(v) = 1$ and there is a subset U of the set of children of v such that $M = \bigcup_{u \in U} f(u)$.
2. Every $v, u \in V(T)$ that have the same parent in T satisfy $f(v) \cap f(u) = \emptyset$.
3. For every $v \in V(T)$, $\bigcup_{u \in \mathsf{child}(v)} f(u) = f(v)$.
4. $|V(T)| \leq 2n - 1$.

Furthermore, no node in T has exactly one child. Every graph G admits a modular tree decomposition, which can be constructed in $O(n^2)$ time and $O(n)$ space [47].

[4] Like other papers on this topic, we abuse the standard usage of the term AT in the literature, which refers to a triple of vertices such that each pair is joined by a path that avoids the neighborhood of the other vertex. Our usage and the standard one are "almost equivalent" (see, e.g., [10]).

Hitting Chordless Cycles and Small Obstructions. We first state the following corollary of the results of Kim and Kwon [25]. (See Proposition B.1 and Lemma 33 in the appendix.)

▶ **Corollary 2.** *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices.*

3 The Case of a Clique Caterpillar

This section analyzes the Erdős-Pósa Property of ATs on graphs with a clique caterpillar. Let us begin with a definition.

▶ **Definition 3.** Let G be a graph. A clique caterpillar (T, β) of G is *nice* if every shallow vertex belongs to the bag of only one node of T and that node is a leaf.

The objective of this section is to prove the following lemma.

▶ **Lemma 4.** *Let $k \in \mathbb{N}$, and let G be a graph with a nice clique caterpillar (T, β), such that G is chordal and has no obstruction on at most ten vertices.*[5] *Then, at least one of the following conditions holds:* **(i)** *G has k vertex-disjoint ATs;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ is an interval graph.*

To simplify statements in this section, let us fix $k \in \mathbb{N}$ and a chordal graph G with a nice clique caterpillar (T, β), which has no obstruction on at most ten vertices. Thus, whenever we discuss an obstruction in G, that obstruction is necessarily an AT on more than ten vertices. Moreover, let us fix a central path of T, and call it P. We denote $P = p_1 - p_2 - \cdots - p_d$ for $d = |V(P)|$. We think of P as a path oriented from p_1 to p_d. For a vertex $v \in V(G)$, we let $\mathsf{first}(v)$ be the first node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define $\mathsf{first}(v) = \mathsf{nil}$), and we let $\mathsf{last}(v)$ be the last node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define $\mathsf{last}(v) = \mathsf{nil}$). The notation $p_i < p_j$ means that $i < j$ (similarly, we define $\leq$). Note that as non-terminal vertices of an AT have non-adjacent neighbors, we have the following observation.

▶ **Observation 3.1.** *Let $\mathbb{O}$ be an AT in G. For every non-terminal vertex v of $\mathbb{O}$, there exists $p \in V(P)$ such that $v \in \beta(p)$.*

Observation 3.1 implies that the notation presented next is well defined. In what follows, when we consider an AT $\mathbb{O}$, we index the base vertices $b_1^{\mathbb{O}}, b_2^{\mathbb{O}}, \ldots, b_{\eta^{\mathbb{O}}}^{\mathbb{O}}$ such that $\mathsf{first}(b_1^{\mathbb{O}}) \leq \mathsf{first}(b_{\eta^{\mathbb{O}}}^{\mathbb{O}})$. When $\mathbb{O}$ is clear from context, we simplify the notation, also in the context of terminal and center vertices.[6] Note that $\eta \geq 5$, as G does not have ATs on at most ten vertices (we use this observation implicitly throughout, e.g. to assume that $b_1, b_2, b_{\eta-2}, b_{\eta-1}$ and b_η are distinct vertices). We remark that clearly, for all $i \in \{2, 3, \ldots, \eta - 1\}$, $\mathsf{first}(b_i) \leq \mathsf{last}(b_{i-1}) < \mathsf{first}(b_{i+1})$ (also stated as Proposition 8.4 in [11]).

Our analysis relies on a notion of a special type of obstruction, defined by Cao and Marx [11], to exploit the "almost linear nature" of a caterpillar. To this end, we have the following notation. Given an AT $\mathbb{O}$, $\widehat{N}(\mathbb{O})$ denotes the set of vertices $v \in V(G)$ such that v is adjacent to every vertex in $\mathsf{base}(\mathbb{O})$. We also need to give three definitions.

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[5] We remark that the existence of the clique caterpillar already implies that G is chordal [18, 8].

[6] For example, if we consider an AT denoted by $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}^i$, then we use b_1 (b_η), b'_1 ($b'_{\eta'}$), b^i_1 ($b^i_{\eta^i}$) to refer to the first (last) base vertex of $\mathbb{O}$, $\mathbb{O}'$ and $\mathbb{O}^i$, respectively.

▶ **Definition 5** ([11]). (i) An AT $\mathbb{O}$ in G is *minimal* if there does not exist an AT $\mathbb{O}'$ such that $\mathsf{last}(b_1) \leq \mathsf{last}(b'_1) \leq \mathsf{first}(b'_{\eta'}) \leq \mathsf{first}(b_\eta)$, and $\mathsf{last}(b_1) < \mathsf{last}(b'_1)$ or $\mathsf{first}(b'_{\eta'}) < \mathsf{first}(b_\eta)$.

(ii) An AT $\mathbb{O}$ in G is *short* if $P(\mathbb{O})$ is a shortest path between t_ℓ and t_r in $G[\beta(p_i) \cup \beta(p_{i+1}) \cup \ldots \cup \beta(p_j) \cup \{t_\ell, t_r\}] - \widehat{N}(\mathsf{base}(\mathbb{O}))$, where $p_i = \mathsf{last}(b_1)$ and $p_j = \mathsf{first}(b_\eta)$.

(iii) An AT $\mathbb{O}$ in G is *first* if there does not exist an AT $\mathbb{O}'$ such that $\mathsf{first}(b'_{\eta'}) < \mathsf{first}(b_\eta)$.

We say that an AT is *good* if it is first, minimal and short. The following proposition asserts that a good AT exists. In this context, recall that we implicitly assume that G is not an arbitrary graph, but in particular it is a graph that has a nice clique caterpillar.

▶ **Proposition 3.1** (Lemma 8.8 & Proof of Theorem 2.4 (Page 31) [11]). *If G is not an interval graph, then it has a good AT.*

▶ **Proposition 3.2** (Claim 5 [11]). *Let $\mathbb{O}$ be a good AT. For any vertex $v \in (\beta(p_1) \cup \beta(p_2) \cup \ldots \beta(p_i)) \setminus \widehat{N}(\mathbb{O})$, where $p_i = \mathsf{first}(b_{\eta-2})$, it holds that v is not adjacent to any vertex that is shallow in G.*

Procedure `SeparateProcedure`. Let us consider the following procedure, which we call `SeparateProcedure`. Initialize $G^1 = G$ and $i = 1$. Now, as long as G^i is not an interval graph, we execute the following procedure:

1. Let $\mathbb{O}^i$ be a good AT in G^i, whose existence is guaranteed by Proposition 3.1.
2. Denote $p_j = \mathsf{first}(b^i_{\eta^i-2})$ and $p_q = \mathsf{last}(b^i_1)$. For all $\delta \in [d]$, denote $\beta^i(p_\delta) = \beta(p_\delta) \cap V(G^i)$. Let $\gamma^i = \gamma$ be the index in $\{q, q+1, \ldots, j-1\}$ such that,
   - there does not exist an index $\delta \in \{\gamma+1, \gamma+2, \ldots, j-1\}$ such that $|(\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$, and
   - $|(\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$.

   If such an index γ does not exist, define $\gamma = \mathsf{nil}$. Intuitively, γ is the largest index of a "small" separator in $G^i \setminus \widehat{N}(\mathbb{O})$ between b^i_1 and $b^i_{\eta-2}$.
3. Denote $S^i = (\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1})) \setminus \widehat{N}(\mathbb{O}^i)$ if $\gamma \neq \mathsf{nil}$, and $S^i = \emptyset$ otherwise.
4. Define $G^{i+1} = G^i - ((V(\mathbb{O}^i) \setminus \mathsf{base}(\mathbb{O}^i)) \cup \{b^i_1, b^i_2, b^i_3, b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}, b^i_{\eta^i}\} \cup S^i)$.
5. Increment i by 1.

Let $i^\star$ denote the last index i considered by `SeparateProcedure`. In particular, $G^{i^\star}$ is an interval graph. Let us denote $S^\star = V(G) \setminus V(G^{i^\star})$. Then, $G - S^\star$ is an interval graph. Furthermore, note that $|S^\star| = \mathcal{O}(i^\star \cdot k)$.

▶ **Observation 3.2.** *If $i^\star \leq 2k$, then $S^\star \subseteq V(G)$ is a set of size $\mathcal{O}(k^2)$ such that $G - S^\star$ is an interval graph.*

Thus, to prove Lemma 4, it is sufficient to prove the following claim.

▶ **Lemma 6.** *If $i^\star > 2k$, then G has k vertex-disjoint obstructions.*

In what follows, we suppose that $i^\star > 2k$. To prove this lemma, we first need to introduce the following definitions.

▶ **Definition 7.** Let $i \in [2k]$. We say that an AT $\mathbb{O}$ in G is *i-relevant* if it is an AT in G^i, $t = t^i$, $t_\ell = t^i_\ell$, $t_r = t^i_r$, $c_1 = c^i_1$, $c_2 = c^i_2$, $b_1 = b^i_1$, $b_2 = b^i_2$, $b_{\eta-2} = b^i_{\eta^i-2}$, $b_{\eta-1} = b^i_{\eta^i-1}$ and $b_\eta = b^i_{\eta^i}$.[7] If in addition $b_3 = b^i_3$ and $b_{\eta-3} = b^i_{\eta-3}$, then we say that $\mathbb{O}$ is *highly i-relevant.*

[7] That is, $\mathbb{O}$ and the AT $\mathbb{O}^i$ considered in the i-th iteration of `SeparateProcedure` have the same terminals, centers and two first and three last base vertices.

▶ **Definition 8.** A tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ is *relevant* if for all $i \in [2k]$, $\widehat{\mathbb{O}}^i$ is i-relevant.

We further need the following notation. For every $i \in [2k]$, $\mathsf{before}(i) = \beta(p_1) \cup \beta(p_2) \cup \ldots \cup \beta(p_{\gamma^i})$ if $\gamma^i \neq \mathsf{nil}$ and $\mathsf{before}(i) = \emptyset$ otherwise. The heart of the proof of Lemma 6 is given by two statements. Towards the first one, let us first prove the following claim.

▶ **Lemma 9 (*).** *For all $i, i' \in [2k]$ where $i > i'$, i-relevant AT $\mathbb{O}$ and i'-relevant AT $\mathbb{O}'$, it holds that,* (1) $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i') \subseteq \{t_\ell, b_1\}$*;* (2) $|V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')| \leq 1$*; and* (3) *if $b_1 \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')$ then $t_\ell \notin (\bigcup_{i \in [d]} \beta(p_i))$.*

We now present the first statement that lies at the heart of the proof.

▶ **Lemma 10 (*).** *For all $i, i', \widehat{i} \in [2k]$ such that $i > i' > \widehat{i}$, i-relevant AT $\mathbb{O}$, i'-relevant AT $\mathbb{O}'$ and $\widehat{i}$-relevant AT $\widehat{\mathbb{O}}$, for at least one index $j \in \{i', \widehat{i}\}$, the following condition holds: $V(\mathbb{O}) \cap V(\mathbb{O}^j) \cap \mathsf{before}(j) = \emptyset$.*

▶ **Corollary 11 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. There exist k indices, $i_1 < i_2 < \ldots < i_k$, so that for every two indices $x, y \in \{i_1, i_2, \ldots, i_k\}$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \mathsf{before}(x) = \emptyset$.*

Towards the statement of the second lemma that lies at the heart of our proof, let us first state an immediate observation and one additional lemma.

▶ **Observation 3.3.** *An AT in G can contain at most four vertices of a clique in G.*

▶ **Lemma 12 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. For all $i \in [2k-1]$, there exists a path in $G^i - \widehat{N}(\mathbb{O}^i)$ from a vertex in $S^i \cup \{b_1^i\}$ to $b_{\eta-2}^i$ that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.*

We are now ready to prove the second statement central to the proof of Lemma 6.

▶ **Lemma 13 (*).** *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. For all $i \in [2k-1]$ such that $\mathbb{O}^i$ is a highly i-relevant good AT in G^i, there exists an i-relevant AT $\mathbb{O}'$ such that the following condition holds: the base path of $\mathbb{O}'$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.*

▶ **Corollary 14 (*).** *There exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.*

We are now ready to prove Lemma 6.

Proof of Lemma 6. By Corollary 14, there exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$. By Corollary 11, there exist k indices, $i_1 < i_2 < \ldots < i_k$, such that for every two indices $x, y \in \{i_1, i_2, \ldots, i_k\}$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \mathsf{before}(x) = \emptyset$. Without loss of generality, suppose that $i_1 = 1, i_2 = 2, \ldots, i_k = k$ (the arguments to follow hold for any $i_1 < i_2 < \ldots < i_k$).

We claim that $\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^k$ are vertex disjoint, which would complete the proof. To prove this claim, we arbitrarily choose $i, j \in [k]$ such that $i < j$. First note that as $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are i-relevant and j-relevant, we have that the terminals and centers of $\widehat{\mathbb{O}}^i$ do not belong to $\widehat{\mathbb{O}}^j$. Moreover, the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex $v^\star$ in $S^i \cup \{b_1^i\}$ to $b_{\eta^i}^i$ that has no vertex of $\widehat{\mathbb{O}}^j$. Let W denote the subpath of the base path of $\widehat{\mathbb{O}}^i$ from b_1^i to $v^\star$. Hence, to conclude that $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are vertex disjoint, it remains to show that no vertex of W belongs to $\widehat{\mathbb{O}}^j$. Notice that $V(W) \subseteq V(\widehat{\mathbb{O}}^i) \cap \mathsf{before}(i)$. By our choice of $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^k)$, it holds that $V(\widehat{\mathbb{O}}^i) \cap \mathsf{before}(i)$ does not have any vertex of $\widehat{\mathbb{O}}^j$. Thus, the proof is complete. ◀

4 Decomposition of Modules

Let us begin with the following simple observation, on which we rely implicitly in our arguments, and which follows immediately from the definition of a modular tree decomposition. For simplicity, we use the abbreviations $f|_v = f|_{V(T|_v)}$ and $g|_v = g|_{V(T|_v)}$.

▶ **Observation 4.1.** *Let G be a graph with a modular tree decomposition (T, f, g), and let $v \in V(T)$. Then, $(T|_v, f|_v, g|_v)$ is a modular tree decomposition of $G[f(v)]$.*

We proceed by introducing the definition of a *problematic set* and a *problematic node*.

▶ **Definition 15.** Let G be a graph with a modular tree decomposition (T, f, g). The *set of problematic obstructions* of a node $v \in V(T)$, denoted by $\mathsf{prob}_G(v)$, is the set of all obstructions $\mathbb{O}$ in $G[f(v)]$ such that for every child u of v in T, $\mathbb{O}$ is not an obstruction in $G[f(u)]$, that is, $V(\mathbb{O}) \setminus f(u) \neq \emptyset$. When G is clear from context, it is omitted.

▶ **Definition 16.** Let G be a graph with a modular tree decomposition (T, f, g). A node $v \in V(T)$ is *problematic* if $\mathsf{prob}(v) \neq \emptyset$. The set of problematic nodes is denoted by $\mathsf{prob}_G(T)$. When G is clear from context, it is omitted.

▶ **Observation 4.2.** *Let G be a graph with a modular tree decomposition (T, f, g). The sets $\mathsf{prob}(v)$, $v \in \mathsf{prob}(T)$, define a partition of the set of obstructions of G. That is, for all $u, v \in V(T)$, $\mathsf{prob}(u) \cap \mathsf{prob}(v) = \emptyset$, and the set of obstructions of G is precisely $\bigcup_{v \in \mathsf{prob}(T)} \mathsf{prob}(v)$.*

We argue that nodes assigned 1 by g are non-problematic. And further, a problematic node should have "many" children.

▶ **Lemma 17 (*).** *G be a graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ such that $g(v) = 1$. Then, v is not a problematic node.*

▶ **Lemma 18 (*).** *Let G be a graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node. Then, v has at least $\max\{2k, 10\} + 1$ children in T.*

In order to proceed, we need the following definition and notation.

▶ **Definition 19.** Let G be a graph with a modular tree decomposition (T, f, g). A subset $P \subseteq \mathsf{prob}(T)$ has a conflict if there exist $u, v \in P$ such that v is a descendant of u in T and on the (unique) path between u and v in T no vertex belongs to $\mathsf{prob}(T) \setminus P$.

▶ **Definition 20.** Let G be a graph with a modular tree decomposition (T, f, g). For a node $v \in V(T)$, $\mathsf{pack}_G(v)$ is the maximum number of vertex-disjoint obstructions in $\mathsf{prob}(v)$. When G is clear from context, it is omitted.

Note that a problematic node is precisely a node such that $\mathsf{pack}(v) \geq 1$.

▶ **Lemma 21 (*).** *Let G be a graph that has no obstruction on at most $\max\{2k, 10\}$ vertices, and which does not have k vertex-disjoint obstructions. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathsf{prob}(T)$ with no conflicts. Then, for each $v \in P$ and each child u of v in T such that u has a problematic descendant, there exist at least k vertices in $f(u)$ that do not belong to $\bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of u in T.*

▶ **Lemma 22 (*).** *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathsf{prob}(T)$ with no conflicts. Then, G has $\min\{k, \sum_{v \in P} \mathsf{pack}(v)\}$ vertex-disjoint obstructions.*

We also show that $\mathsf{prob}(T)$ can be divided into two sets with no conflicts.

▶ **Lemma 23 (*).** *Let G be a graph with a modular tree decomposition (T, f, g). There exists a partition (P_1, P_2) of $\mathsf{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict.*

Specific classes of graphs, called *reduced graphs* and *nice interval graphs*, were defined by Cao and Marx as follows.

▶ **Definition 24** ([11])**.** A graph G is *reduced* if it satisfies the following properties: (i) Every non-trivial module of G is a clique, and (ii) G does not have any obstruction on at most ten vertices.

▶ **Definition 25** ([11])**.** A graph G is *nice* if it satisfies the following properties: (i) G is chordal; (ii) G does not have any obstruction on at most ten vertices; and (iii) every vertex in G that is a shallow terminal of at least one obstruction is simplicial.

These definitions were in particular used to derive the following results.

▶ **Proposition 4.1** (Theorem 2.1 [11])**.** *Let G be a reduced graph. Every vertex in G that is a shallow terminal of at least one obstruction is simplicial.*

▶ **Proposition 4.2** (Proposition 8.3 [11])**.** *Any nice graph has a nice clique caterpillar (T, β).*

▶ **Corollary 26.** *Any chordal reduced graph has a nice clique caterpillar (T, β).*

Let us derive a consequence of Corollary 26 with respect to a modular tree decomposition.

▶ **Lemma 27 (*).** *Let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node such that for every child u of v in T, $G[f(u)]$ is a clique. Then, $G[f(v)]$ has a nice clique caterpillar.*

Towards the proof of the main result of this section, we need one additional notation.

▶ **Definition 28.** Let G be a graph with a modular tree decomposition (T, f, g), and let $v \in V(T)$. Then, $\mathsf{clique}(G, v)$ denotes the graph obtained from G by turning each $G[f(u)]$, $u \in \mathsf{child}(v)$, into a clique. That is, $V(\mathsf{clique}(G, v)) = V(G)$ and $E(\mathsf{clique}(G, v)) = E(G) \cup (\bigcup_{u \in \mathsf{child}(v)} \{\{x, y\} : x, y \in f(u)\})$.

▶ **Lemma 29 (*).** *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$. Then, the set of obstructions in $\mathsf{clique}(G, v)[f(v)]$ is precisely $\mathsf{prob}_G(v)$.*

▶ **Lemma 30 (*).** *Let $k \in \mathbb{N}$, and let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$. Then, at least one of the following conditions holds:* **(i)** $\mathsf{pack}(v) \geq k$*;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ that intersects the vertex set of every obstruction in $\mathsf{prob}(v)$.*

We are now ready to prove the main result of this section.

▶ **Lemma 31.** *Let $k \in \mathbb{N}$, and let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Then, at least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ is an interval graph.*

Proof. Suppose that G does not have k vertex-disjoint obstructions, else we are done. By Lemma 23, there exists a partition (P_1, P_2) of $\mathsf{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict. By Lemma 22, for each $i \in [2]$, G has $\sum_{v\in P_i} \mathsf{pack}(v)$ vertex-disjoint obstructions. Thus, by Observation 4.2, for each $i \in [2]$, $|\sum_{v\in P_i} \mathsf{pack}(v)| < k$. This means that $\sum_{v\in \mathsf{prob}(T)} \mathsf{pack}(v) < 2k$.

By Lemma 30, for all $v \in \mathsf{prob}(T)$, there exists a subset $D_v \subseteq V(G)$ of size $\mathcal{O}((\mathsf{pack}(v)+1)^2)$ that intersects the vertex set of every obstruction in $\mathsf{prob}(v)$. Denote $D = \bigcup_{v\in \mathsf{prob}(T)} D_v$. Then, $|D| = \mathcal{O}(\sum_{v\in \mathsf{prob}(T)}(\mathsf{pack}(v) + 1)^2)$. By Observation 4.2, we have that $G - D$ is an interval graph. Thus, to conclude the proof, it remains to show that $\sum_{v\in \mathsf{prob}(T)}(\mathsf{pack}(v) + 1)^2 = \mathcal{O}(k^2)$. Since for all $v \in \mathsf{prob}(T)$, $\mathsf{pack}(v) \geq 1$, it is sufficient to show that $\sum_{v\in \mathsf{prob}(T)}(\mathsf{pack}(v))^2 = \mathcal{O}(k^2)$. Recall that $\sum_{v\in \mathsf{prob}(T)} \mathsf{pack}(v) < 2k$. Thus, $\sum_{v\in \mathsf{prob}(T)} (\mathsf{pack}(v))^2 \leq (\sum_{v\in \mathsf{prob}(T)} \mathsf{pack}(v)) \cdot (\sum_{v\in \mathsf{prob}(T)} \mathsf{pack}(v)) < 2k \cdot 2k = \mathcal{O}(k^2)$. This completes the proof. ◀

5 Putting It All Together

Finally, we are ready to prove our main theorem.

Proof of Theorem 1. By Corollary 2, at least one of the following conditions hold: **(i)** G has k vertex-disjoint obstructions; **(ii)** there exists a subset $D' \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D'$ is a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. In the first case, our proof is complete, and thus we next suppose that the second case applies. Then, by Lemma 31, at least one of the following conditions hold: **(i)** $G - D'$ has k vertex-disjoint obstructions; **(ii)** there exists a subset $\widehat{D} \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $(G - D') - \widehat{D}$ is an interval graph. In the first case, our proof is complete. In the second case, we have that $D = D' \cup \widehat{D}$ is a set of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph, which again completes the proof. ◀

Before we turn to prove a corollary of our main theorem, we need one more proposition.

▶ **Proposition 5.1** ([10]). *There exists an $\mathcal{O}(nm)$-time algorithm that, given a graph G, outputs an integer d' such that the following conditions hold:* **(i)** *there exists a subset $D' \subseteq V(G)$ of size at most d' such that $G - D'$ is an interval graph;* **(ii)** *$d' \leq 8d$ for the integer d that is the minimum size of a subset $D \subseteq V(G)$ such that $G - D$ is an interval graph.*

As a consequence of Theorem 1 and Proposition 5.1, we derive the following corollary.

▶ **Corollary 32** (*). *There exist a constant $c \in \mathbb{N}$ and an $\mathcal{O}(nm)$-time algorithm that, given a graph G and an integer $k \in \mathbb{N}$, correctly concludes which one of the following conditions holds:*[8] **(i)** *G has k vertex-disjoint obstructions;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $ck^2 \log k$ such that $G - D$ is an interval graph.*

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A Detailed Preliminaries

We denote the set of natural numbers by $\mathbb{N}$. For $n \in \mathbb{N}$, we use $[n]$ as a shorthand for $\{1, 2, \ldots, n\}$. Given a function $f : A \to B$ and a subset $A' \subset A$, we use $f|_{A'}$ to denote the restriction of f to A'.

Basic Graph Theory. We refer to standard terminology from the book of Diestel [13] for those graph-related terms that are not explicitly defined here. Given a graph G, we denote its vertex set and its edge set by $V(G)$ and $E(G)$, respectively. Given a set $\mathcal{C}$ of subgraphs of G, denote $V(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} V(C)$. The disjoint union of (vertex-disjoint) graphs $H_1, H_2, \ldots, H_t$ is the graph of the vertex set $V(H_1) \cup V(H_2) \cup \cdots \cup V(H_t)$ and edge set $E(H_1) \cup E(H_2) \cup \ldots \cup E(H_t)$. Moreover, when the graph G is clear from context, denote $n = |V(G)|$ and $m = |E(G)|$. Given a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. Moreover, a graph H is an *induced subgraph* of G if there exists $U \subseteq V(G)$ such that $G[U] = H$. For a set of vertices $X \subseteq V(G)$, $G - X$ denotes the induced subgraph $G[V(G) \setminus X]$, i.e. the graph obtained by deleting the vertices in X from G. We say that G is a *clique* if for all distinct vertices $u, v \in V(G)$, we have that $\{u, v\} \in E(G)$, and that $V(G)$ is an independent set if for all distinct vertices $u, v \in V(G)$, we have that $\{u, v\} \notin E(G)$. Given a vertex $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G. We say that v is *simplicial* if $N_G(v)$ induces a clique.

A *path* $P = x_1 - x_2 - \ldots - x_\ell$ in G is a subgraph of G where $V(P) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(P) = \{\{x_i, x_{i+1}\} \mid i \in [\ell - 1]\} \subseteq E(G)$, where $\ell \in [n]$. The *length* of P is the number of edges on P. The vertices x_1 and x_ℓ are the *endpoints* of P, and the remaining vertices in $V(P)$ are the *internal vertices* of P. A *walk* $W = x_1 - x_2 - \ldots - x_\ell$ in G is a sequence of (not necesarily distinct) vertices of G such that for all $i \in [\ell - 1]$, either $x_i = x_{i+1}$ or $\{x_i, x_{i+1}\} \in E(G)$. A *cycle* $C = x_1 - x_2 - \ldots - x_\ell - x_1$ in G is a subgraph of G where $V(C) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(C) = \{\{x_i, x_{i+1}\} \mid i \in [\ell - 1]\} \cup \{\{x_1, x_\ell\}\} \subseteq E(G)$. We say that $\{u, v\} \in E(G)$ is a *chord* of P if $u, v \in V(P)$ but $\{u, v\} \notin E(P)$. Similarly, we say that $\{u, v\} \in E(G)$ is a *chord* of C if $u, v \in V(C)$ but $\{u, v\} \notin E(C)$. A path P or cycle C is said to be *induced* (or, alternatively, *chordless*) if it has no chords. A *caterpillar* is a tree T for which there exists a subpath P of T, called a *central path*, such that the removal of the vertices of P from T results in an edgeless graph. Given a rooted tree T and a vertex $v \in V(T)$, we use $T|_v$ to denote the subtree of T rooted at v. Moreover, $\mathsf{child}(v)$ denotes the set of children of v in T. A *chordal graph* is a graph that has no chordless cycle on at least four vertices.

Interval Graphs. An *interval graph* is a graph G that does not contain any of the following graphs, called *obstructions*, as an induced subgraph (see Figure 1).

- **Long Claw.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_3\}, \{t, b_2\}, \{c, b_1\}, \{c, b_2\}, \{c, b_3\}\}$.
- **Whipping Top.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_2\}, \{c, t\}, \{c, b_1\}, \{c, b_2\}, \{b_3, t_\ell\}, \{b_3, b_1\}, \{b_3, c\}, \{b_3, b_2\}, \{b_3, t_r\}\}$.
- **†-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c), \{t_\ell, b_1\}, \{t_r, b_z\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 2$. A †-AW where $z = 2$ is called a *net*.
- **‡-AW.** A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c_1, c_2\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t, c_1\}, \{t, c_2\}, \{c_1, c_2\}, \{t_\ell, b_1\}, \{t_r, b_z\}, \{t_\ell, c_1\}, \{t_r, c_2\}\} \cup \{\{c, b_i\} \mid i \in [z]\} \cup \{\{b_i, b_{i+1}\} \mid i \in [z-1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ is called a *tent*.
- **Hole.** A chordless cycles on at least four vertices.

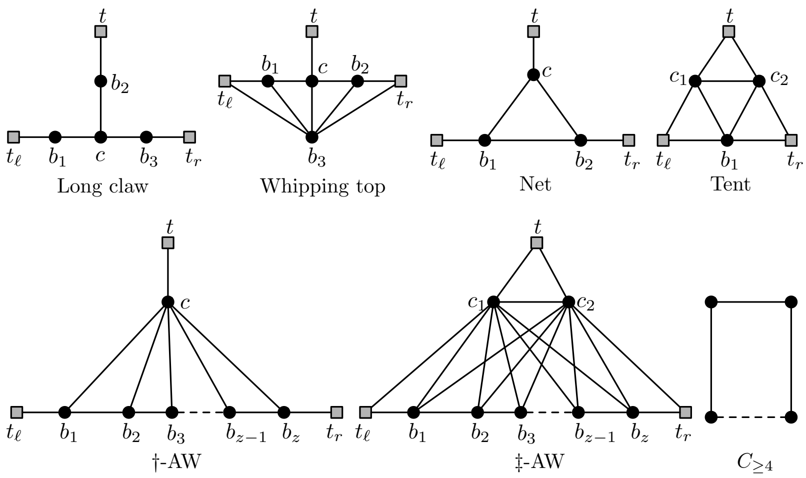

Figure 1 The set of obstructions for an interval graph.

Long claws and whipping tops are also called ATs, but we shall reserve this name for †-AWs and †-AWs. An obstruction $\mathbb{O}$ is *minimal* if there does not exist an obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subset V(\mathbb{O})$. In each of the first four obstructions, the vertices t_ℓ, t_r, and t are called *terminals*, the vertices c, c_1, and c_2 are called *centers*, and the other vertices are called *base vertices*. To simplify notation, when we consider a †-AW, we use c_1 and c_2 to refer to c (this allows us to refer to a †-AW and a ‡-AW in a unified manner). Furthermore, the vertex t_s is called the *shallow terminal*. In the case of an AT $\mathbb{O}$, the induced path on the set of base vertices is called the *base* of the AT, and it is denoted by $\mathsf{base}(\mathbb{O})$. Moreover, we say that the induced path on the set of base vertices, t_ℓ and t_r is the *extended base* of the AT, and it is denoted by $P(\mathbb{O})$. Given a graph G, a vertex v is *shallow in* G if G has at least one AT where v is the shallow terminal.

Tree Decomposition. A *tree decomposition* of a graph G is a pair (T, β) where T is a tree, and $\beta : V(T) \to 2^{V(G)}$ is a function that satisfies the following properties.

(i) $\bigcup_{x \in V(T)} \beta(x) = V(G)$,
(ii) for any edge $\{u, v\} \in E(G)$ there is a node $x \in V(T)$ such that $u, v \in \beta(x)$,
(iii) and for any $v \in V(G)$, the collection of nodes $T_v = \{x \in V(P) \mid v \in \beta(x)\}$ is a subtree of T.

For $v \in V(P)$, we call $\beta(v)$ the *bag* of v. In case T is a path, then (T, β) is also called a *path decomposition*, and in case T is a caterpillar then (T, β) is also called a *caterpillar decomposition*. We refer to the vertices in $V(P)$ as *nodes*. A *clique path* (*clique caterpillar*) of a graph G is a path decomposition (resp. *caterpillar decomposition*) of G where every bag is a distinct maximal clique. We remark that not every graph admits a clique caterpillar.

Modules. Let G be a graph. A subset $M \subseteq V(G)$ is a *module* if for all $u, w \in M$ and $v \in V(G) \setminus M$, either both u and w are adjacent to v or both u and w are not adjacent to v. A module is *nontrivial* if neither $V(M) = \emptyset$ nor $V(M) = V(G)$.

The following simple proposition asserts that a "large" obstruction cannot intersect a module in more than one vertex unless it is not contained in that module.

▶ **Proposition A.1** (Proposition 4.4 [11]). *Let M be a module in G and $\mathbb{O}$ be a minimal obstruction. If $|V(\mathbb{O})| > 4$, then either $V(\mathbb{O}) \subseteq V(M)$ or $|V(\mathbb{O}) \cap V(M)| \leq 1$.*

A *modular tree decomposition* of a graph $G = (V, E)$ is a linear-size representation of all its modules. It consists of a rooted tree T, a function $f : V(T) \to 2^{V(G)}$ and a function $g : V(T) \to \{0, 1\}$, which in particular satisfy the following properties:

1. M is a module of G if and only if there is a node $v \in V(T)$ for which, either $M = f(v)$, or both $g(v) = 1$ and there is a subset U of the set of children of v such that $M = \bigcup_{u \in U} f(u)$.
2. Every $v, u \in V(T)$ that have the same parent in T satisfy $f(v) \cap f(u) = \emptyset$.
3. $|V(T)| \leq 2n - 1$.

Furthermore, no node in T has exactly one child. Every graph G admits a modular tree decomposition. In fact, such a decomposition can be constructed in $O(n^2)$ time and $O(n)$ space:

▶ **Proposition A.2** ([47]). *Given a graph G, a modular tree decomposition can be constructed in $\mathcal{O}(n^2)$ time and $\mathcal{O}(n)$ space.*

B Omitted Results and Proofs

Kim and Kwon [25] obtain the following result on the Erdos-Posa property of chordless cycles of length 4 or more.

▶ **Proposition B.1** ([25]). *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint holes;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is a chordal graph.*

▶ **Lemma 33.** *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds:* **(i)** *G has k vertex-disjoint obstructions on at most $\max\{2k, 10\}$ vertices;* **(ii)** *there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ has no obstruction on at most $\max\{2k, 10\}$ vertices.*

Proof. If the first condition holds, then we are done. Thus, suppose that it does not hold, which means that G has no k vertex-disjoint obstructions on at most $\max\{2k, 10\}$ vertices. Let $\mathcal{O}$ be a set of maximum size of vertex-disjoint obstructions in G on at most $\max\{2k, 10\}$ vertices. Then, $|\mathcal{O}| \leq k - 1$, which implies that $|V(\mathcal{O})| \leq (k - 1) \cdot \max\{2k, 10\} = \mathcal{O}(k^2)$. By the maximality of $\mathcal{O}$, we have that $G - V(\mathcal{O})$ has no obstruction on at most $\max\{2k, 10\}$ vertices. Thus, the second condition holds. ◀

B.1 Proof of Lemma 9

Arbitrarily select $i, i' \in [2k]$ such that $i > i'$, an i-relevant AT $\mathbb{O}$ and an i'-relevant AT $\mathbb{O}'$. For simplicity, denote $\gamma' = \gamma^{i'}$ (recall that $\gamma^{i'}$ is index in `SeparateProcedure`). Suppose that there exists $v^\star \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')$, else we are done. (Note that this implies that $\gamma' \neq \mathsf{nil}$, and in particular $S^{i'} \neq \emptyset$.) Recall that $\mathsf{before}(i') = \beta(p_1) \cup \beta(p_2) \cup \ldots \cup \beta(p_{\gamma'})$ and $p_{\gamma'} < \mathsf{first}(b'_{\eta'-2})$. Now, we have that $\mathsf{first}(b_{\eta'-2}) \leq \mathsf{last}(b_{\eta'-2}) < \mathsf{first}(b_{\eta'})$. Because $\mathbb{O}$ is i-relevant and $\mathbb{O}'$ is i'-relevant where $i' < i$, we have that $\mathsf{first}(b'_{\eta'}) \leq \mathsf{first}(b_\eta)$, and $v^\star$ is neither a center nor a terminal of $\mathbb{O}'$, and it is neither $b'_{\eta'-1}$ nor $b'_{\eta'}$. Thus, $v^\star$ and $b'_{\eta'}$ are not adjacent (specifically, $\mathsf{last}(v^\star) < \mathsf{first}(b'_{\eta'})$), and therefore $v^\star \notin \widehat{N}(\mathbb{O}')$. Since $v^\star \in \mathsf{before}(i')$, we have that $\mathsf{first}(v^\star) \leq p_{\gamma'}$. Since $v^\star \notin S^{i'}$ (as $S^{i'} \cap V(\mathbb{O}) = \emptyset$), we have that $\mathsf{last}(v^\star) \leq p_{\gamma'}$. From this we first derive that $\mathsf{last}(v^\star) < \mathsf{first}(b_\eta)$. This means that $v^\star$ is not a center of $\mathbb{O}$. Moreover, since (T, β) is nice, it is not shallow. Therefore, $v^\star$ belongs to the extended base of $\mathbb{O}$, which we denoted by $P(\mathbb{O})$. Since $\mathsf{last}(v^\star) \leq p_{\gamma'} < \mathsf{first}(b_\eta)$ and $P(\mathbb{O})$ is an induced path, we have that $P(\mathbb{O})$ must contain at least one vertex from $\beta(p_{\gamma'}) \cap \beta(p_{\gamma'+1})$ with one neighbor (on $P(\mathbb{O})$) from $\beta(p_1) \cup \ldots \cup \beta(p_{\gamma'})$ and the other neighbor (on $P(\mathbb{O})$) from $\beta(p_{\gamma'+1}) \cup \ldots \cup \beta(p_d)$. Since $\mathbb{O}$ is i-relevant and $i > i'$, this vertex cannot belong to $S^{i'}$. This means that $\mathbb{O}$ contains as a base vertex $b^\star$ that is adjacent to all the vertices of $\mathsf{base}(\mathbb{O}')$.

Since $\mathbb{O}$ is i-relevant and $i > i'$, and because $v^\star \in V(\mathbb{O}) \cap V(\mathbb{O}')$, we have that $v^\star$ must be a base vertex of $\mathbb{O}'$, and it can be neither b'_1 nor b'_2. In particular, we derive that $b^\star$ is a neighbor of $v^\star$. Recall that we have argued that $v^\star$ and $b^\star$ belong to $P(\mathbb{O})$. As $P(\mathbb{O})$ is an induced path, $b^\star$ cannot be adjacent to the other neighbor of $v^\star$ on $P(\mathbb{O})$ (if one exists). Let us suppose that such a neighbor exists, and denote it by $n^\star$. We claim that $n^\star \notin \beta(p_1) \cup \ldots \cup \beta(p_d)$. To show this, suppose by way of contradiction that this claim is false. Because $b^\star$ is adjacent to all the vertices of $\mathsf{base}(\mathbb{O}')$, this means that $\mathsf{last}(n^\star) < \mathsf{last}(b'_1)$. Moreover, as $v^\star$ can be neither b'_1 nor b'_2 but it belongs to $\mathsf{base}(\mathbb{O}')$, we have that $\mathsf{last}(b'_1) < \mathsf{first}(v^\star)$, which means that $\mathsf{last}(n^\star) < \mathsf{first}(v^\star)$. However, this implies that $n^\star$ and $v^\star$ are not adjacent, which is a contradiction. Thus, $n^\star \notin \beta(p_1) \cup \ldots \cup \beta(p_d)$.

Overall, we have shown that $v^\star \in V(P(\mathbb{O}))$, $v^\star$ is adjacent to $b^\star$ on $P(\mathbb{O})$, and that either $v^\star$ has no other neighbor on $P(\mathbb{O})$ or its other neighbor does not belong to $(\bigcup_{i\in[d]} \beta(p_i))$. By Observation 3.1, this means that either $v^\star = t_\ell$ or both $v^\star = b_1$ and $t_\ell \notin (\bigcup_{i\in[d]} \beta(p_i))$. To conclude the proof, it remains to show that $|V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')| \leq 1$. Suppose, by way of contradiction, that this claim is false, and let $u^\star \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')$ such that $u^\star \neq v^\star$. Then, as the choice of $v^\star$ was arbitrary, we again derive that $u^\star = t_\ell$ or both $u^\star = b_1$ and $t_\ell \notin (\bigcup_{i\in[d]} \beta(p_i))$. Then, without loss of generality, suppose that $v^\star = t_\ell$ and $u^\star = b_1$. However, this means that $v^\star \notin (\bigcup_{i\in[d]} \beta(p_i))$. This is a contradiction because $v^\star \in \mathsf{before}(i')$. ◀

B.2 Proof of Lemma 10

Arbitrarily select $i, i', \widehat{i} \in [2k]$ such that $i > i' > \widehat{i}$, an i-relevant AT $\mathbb{O}$, an i'-relevant AT $\mathbb{O}'$ and an $\widehat{i}$-relevant AT $\widehat{\mathbb{O}}$. Suppose, by way of contradiction, that $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i') \neq \emptyset$ and $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}) \neq \emptyset$. By Conditions 1, 2 and 3 in Lemma 9 with respect to i', we have that either **(a)** $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i') = \{t_\ell\}$, or **(b)** $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i') = \{b_1\}$ and $t_\ell \notin (\bigcup_{i\in[d]} \beta(p_i))$. Furthermore, by Conditions 1, 2 and 3 in Lemma 9 with respect to $\widehat{i}$, we have that either **(c)** $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}) = \{t_\ell\}$, or **(d)** $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}) = \{b_1\}$ and $t_\ell \notin (\bigcup_{i\in[d]} \beta(p_i))$.

Observe that (a) and (d) cannot happen simultaneously, as well as (b) and (c) cannot happen simultaneously. This means that either $t_\ell \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}))$ or $b_1 \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}))$. Let x denote the vertex in $\{t_\ell, b_1\}$ such that $x \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \mathsf{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i}))$. In particular, $x \in V(\mathbb{O}') \cap V(\widehat{\mathbb{O}}) \cap \mathsf{before}(\widehat{i})$. By Condition 1 in Lemma 9, we have that $x \in \{t'_\ell, b'_1\}$. This means that $\{t'_\ell, b'_1\} \cap V(\mathbb{O}) \neq \emptyset$. However, this is a contradiction since $\mathbb{O}$ is i-relevant and $\mathbb{O}'$ is i'-relevant where $i' < i$. This completes the proof. ◀

B.3 Proof of Corollary 11

For all $i \in [2k]$, Lemma 10 implies that there exists at most one index $j \in [i-1]$ such that $V(\mathbb{O}^i) \cap V(\mathbb{O}^j) \cap \mathsf{before}(j) \neq \emptyset$. If such an index exists, denote it by j_i, and else define $j_i = 1$. Let us initialize $I = [2k]$ and $J = \emptyset$. For $i = 2k, 2k-1, \ldots, 1$ (in this order): If $i \in I$, then insert i into J and remove i and j_i from I. Notice that at the end, the indices in J satisfy that for every two indices $x, y \in J$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \mathsf{before}(x) = \emptyset$. Moreover, at each iteration, we disallow at most one new index in $[2k]$ from being inserted in J while we also insert one index into J. Thus, $|J|$ is at least half $2k$, which completes the proof. ◀

B.4 Proof of Lemma 12

Arbitrarily select $i \in [2k-1]$. Denote $U = \bigcup_{j=i+1}^{2k} V(\widehat{\mathbb{O}}^j)$. Moreover, denote $T = S^i$ if $\gamma^i \neq \mathsf{nil}$ and $T = \{b_1^i\}$ otherwise. Let y be some vertex in T. To prove the lemma, it suffices to show that there exists a path in G^i from y to $b_{\eta-2}^i$ that does not contain any vertex from U. In addition, if $\gamma^i \neq \mathsf{nil}$ then denote $\mu = \gamma^i$, and otherwise let μ denote $j-1$ where j is the index such that $p_j = \mathsf{last}(b_1^i)$. Moreover, let α denote the index j such that $p_j = \mathsf{first}(b_{\eta^i-2}^i)$. By the definition of μ, we know that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $|(\beta^i(p_\mu) \cap \beta^i(p_{\mu+1})) \setminus \widehat{N}(\mathbb{O}^i)| \geq 8k$. Notice that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $G[\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})]$ is a clique. Therefore, by Observation 3.3, we have that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $|((\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)) \setminus U| \geq 8k - 4(2k-i) \geq 1$.

We have thus shown that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, there exists at least one vertex, which we denote by x_δ, that belongs to $((\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)) \setminus U$. Observe that $y - x_\delta - x_{\delta+1} - \cdots - x_{\alpha-1} - b_{\eta^i-2}^i$ is a walk in $G^i - \widehat{N}(\mathbb{O}^i)$ from y to $b_{\eta-2}^i$ that does not contain any vertex from U. Clearly, if $G[\{y, x_\delta, x_{\delta+1}, \cdots, x_{\alpha-1}, b_{\eta^i-2}^i\}]$ contains a walk from y to $b_{\eta^i-2}^i$, then it also contains a path from y to $b_{\eta^i-2}^i$. This completes the proof. ◀

B.5 Proof of Lemma 13

Arbitrarily select $i \in [2k-1]$. In case $\{b_{\eta-3}^i, b_{\eta-2}^i, b_{\eta-1}^i\} \cap S^i \neq \emptyset$ or $|\mathsf{base}(\mathbb{O})| \leq 6$, the claim is clearly true, and hence we next assume this is not case. Denote $U = \bigcup_{j=i+1}^{2k} V(\widehat{\mathbb{O}}^j)$. For simplicity of notation, denote $\mathbb{O} = \mathbb{O}^i$. Note that $\{t_s, t_\ell, t_r, c_1, c_2, b_1, b_2, b_3, b_{\eta-3}, b_{\eta-2}, b_{\eta-1}, b_\eta\} \cap U = \emptyset$. Now, notice that to prove the lemma, it suffices to show that there exists an induced path from b_1 to b_η in $G^i - U$ **(a)** whose second vertex is b_2 and two before last vertices are $b_{\eta-2}$ and $b_{\eta-1}$, **(b)** whose internal vertices are all adjacent to c_1 and c_2 and are all neither equal nor adjacent to t_ℓ, t_r and t_s, and **(c)** has a subpath $Q^\star$ from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i with no vertex from U. Indeed, together with the centers and terminals of $\mathbb{O}$, we will thus get the desired AT $\mathbb{O}'$.

Let W be an arbitrary induced path from b_3 to $b_{\eta-3}$ in $G^i - \widehat{N}(\mathbb{O})$ (as $|\mathsf{base}(\mathbb{O})| \geq 7$, $3 < \eta - 3$). Since W is induced, every vertex on W must belong to $(\beta^i(p_\alpha) \cup \beta^i(p_{\alpha+1}) \cup \ldots \beta^i(p_\gamma)) \setminus \widehat{N}(\mathbb{O})$, where $p_\alpha = \mathsf{last}(b_3)$ and $p_\gamma = \mathsf{first}(b_{\eta-3})$. Because $\mathbb{O}$ is a good AT in G^i, by Proposition 3.2, for any vertex $v \in (\beta^i(p_1) \cup \beta^i(p_2) \cup \ldots \beta^i(p_\gamma)) \setminus \widehat{N}(\mathbb{O})$, it holds that v is not adjacent to any vertex that is shallow in G^i and in particular not to t_s. Thus, no vertex on W is adjacent to t_s. Moreover, since c_1 and c_2 are adjacent to all vertices on $\mathsf{base}(\mathbb{O})$, they are adjacent to all vertices in $\beta^i(p_\alpha) \cup \beta^i(p_{\alpha+1}) \cup \ldots \beta^i(p_\gamma)$ (since c_1 and c_2 belong to these bags), and in particular to all vertices on W.

Let w be any internal vertex on W. On the one hand, because W is an induced path, $\mathsf{last}(b_3) < \mathsf{last}(w)$. This means that if w was adjacent to any vertex in $\{t_\ell, b_1\}$, then we could have replaced $b_2 - b_3$ by w in $P(\mathbb{O})$ and obtain a walk (which contains a path) shorter than $P(\mathbb{O})$ between t_ℓ and t_r in $G[V(P(\mathbb{O})) \cup \beta(p_\alpha) \cup \beta(p_{\alpha+1}) \cup \ldots \cup \beta(p_\gamma)] - \widehat{N}(\mathsf{base}(\mathbb{O}))$, which is a contradiction as $\mathbb{O}$ is good and hence it is in particular short. Thus, w is not adjacent to any vertex in $\{t_\ell, b_1\}$. On the other hand, because W is an induced path, $\mathsf{first}(w) < \mathsf{first}(b_{\eta-3})$. Symmetrically to the former case, we deduce that w is not adjacent to any vertex in $\{b_{\eta-1}, b_\eta, t_r\}$.

Up until now, our arguments imply that to prove the lemma, it is sufficient to show that there exists an induced path from b_3 to $b_{\eta-3}$ in $G^i - \widehat{N}(\mathbb{O})$ having a subpath $Q^\star$ from a vertex in $S^i \cup \{b_3^i\}$ to $b_{\eta-3}$ with no vertex from U. Indeed, by appending $b_1 - b_2$ to the beginning of such a path and $b_{\eta-2} - b_{\eta-1} - b_\eta$ to the end of such a path, while removing b_3 ($b_{\eta-3}$) if the first (last) internal vertex of the path is adjacent to b_2 ($b_{\eta-2}$), we derive an induced path satisfying properties (a), (b) and (c) as required.

By Lemma 12, there exists a path Q' in $G^i - (\widehat{N}(\mathbb{O}) \cup U)$ from a vertex $v^\star$ in $S^i \cup \{b_1\}$ to $b_{\eta-2}$. Notice that $b_{\eta-3} \in \beta(\mathsf{first}(b_{\eta-2}))$ and that for every vertex $v \in S^i \cup \{b_1\}$, $\mathsf{first}(v) < \mathsf{first}(b_{\eta-2})$. This means that Q' has at least one vertex adjacent to $b_{\eta-3}$. Moreover, if $v^\star = b_1$, then because $\mathsf{last}(b_1) < \mathsf{first}(b_3) \leq \mathsf{first}(b_{\eta-3})$, Q' has at least one vertex adjacent to b_3. Also recall that $\{b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}\} \cap S^i \neq \emptyset$. We thus derive that there exists a path, and hence also an induced one, in $G^i - (\widehat{N}(\mathbb{O}) \cup U)$ from a vertex $v^\star \in S^i \cup \{b_3\}$, such that either $v^\star = b_3$ or $\mathsf{last}(b_3) < \mathsf{last}(v^\star)$, to $b_{\eta-3}$. Let us denote such a path by Q. Note that if $v^\star \in \{b_3^i\}$, then the proof is already complete.

Suppose now that $v^\star \in S^i$ and $\mathsf{last}(b_3) < \mathsf{last}(v^\star)$. Let b_j be the first vertex on $\mathsf{base}(\mathbb{O})$ that belongs to S^i (note that such a vertex must exist). Then, $G' = G^i[\{b_3, b_4, \ldots, b_j\} \cup V(Q)]$ has an induced path from b_3 to $b_{\eta-3}$ having a subpath from a vertex in S^i (which may or may not be $v^\star$) to $b_{\eta-3}$. Indeed, we derive such a path by taking any shortest path from b_3 to $b_{\eta-3}$ in G'—such a path necessarily contains a subpath from some vertex $u \in S^i$ (where u does not belong to U as $U \cap S^i = \emptyset$) to $\beta_{\eta-3}$ whose internal vertices all belong to $V(Q)$. This completes the proof. ◀

B.6 Proof of Corollary 14

In order to apply induction, we claim that for all $j \in [2k]$, we have a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that **(i)** for all $i \in [2k] \setminus [j-1]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$, and **(ii)** for all $i \in [j-1]$, $\widehat{\mathbb{O}}^i$ is a highly i-relevant good AT in G^i. The proof is by induction j, and it is clear that if it is correct for $j = 1$, then we will derive the lemma. In the base case, where $j = 2k$, the claim is true since $(\mathbb{O}^1, \mathbb{O}^2, \ldots, \mathbb{O}^{2k})$, as compute by `SeparateProcedure`, is a relevant tuple where every AT is highly relevant and good as required (the disjointness condition holds trivially).

Now, suppose that $j < 2k$. By the inductive hypothesis, there exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that **(i)** for all $i \in [2k] \setminus [j]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$, and **(ii)** for all $i \in [j]$, $\widehat{\mathbb{O}}^i$ is a highly i-relevant good AT in G^i. Apply Lemma 13 with $i = j$ to obtain a j-relevant AT $\mathbb{O}'$ such that the following condition holds: the base path of $\mathbb{O}'$ has a subpath Q from a vertex in $S^i \cup \{b_1^j\}$ to $b^j_{\eta^j}$ that does not contain any of the vertices of the ATs $\widehat{\mathbb{O}}^{j+1}, \widehat{\mathbb{O}}^{j+2}, \ldots, \widehat{\mathbb{O}}^{2k}$. Thus, we replace $\widehat{\mathbb{O}}^j$ by $\mathbb{O}'$ in $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \ldots, \ldots, \widehat{\mathbb{O}}^{2k})$ to obtain a tuple as required to prove the claim. ◀

B.7 Proof of Lemma 17

Let $v_1, v_2, \ldots, v_t$ be the children of v. Suppose, by way of contradiction, that the lemma is false. Then, $G[f(v)]$ has an obstruction $\mathbb{O}$ on more than ten vertices such that for all $i \in [t]$, $\mathbb{O}$ is not an obstruction in $G[f(v_i)]$. Note that the leaves ℓ of a modular tree decomposition satisfy $|f(\ell)| = 1$. This means that $t \geq 1$. It is thus well defined to let $i \in [t]$ be the smallest index such that $V(\mathbb{O}) \subseteq f(v_1) \cup f(v_2) \cup \ldots \cup f(v_i)$. Since $G[f(v_1)]$ does not contain $\mathbb{O}$, $i \geq 2$. Moreover, by the choice of i and since $G[f(v_i)]$ does not contain $\mathbb{O}$, $V(\mathbb{O}) \cap f(v_i) \neq \emptyset$ and $V(\mathbb{O}) \setminus f(v_i) \neq \emptyset$. Since $f(v_1) \cup f(v_2) \cup \ldots \cup f(v_{i-1})$ and $f(v_i)$ are both modules, by Proposition A.1, $|V(\mathbb{O}) \cap (f(v_1) \cup f(v_2) \cup \ldots \cup f(v_{i-1}))| = 1$ and $|V(\mathbb{O}) \cap f(v_i)| = 1$. However, this means that $|V(\mathbb{O})| = 2$, which is a contradiction as no obstruction consists of only two vertices. ◀

B.8 Proof of Lemma 18

Since $v \in V(T)$ is a problematic node, $G[f(v)]$ has an obstruction $\mathbb{O}$ such that for every child u of v in T, $\mathbb{O}$ is not an obstruction in $G[f(u)]$. Moreover, since G has no obstruction on at most $\max\{2k, 10\}$ vertices, we have that $|V(\mathbb{O})| \geq \max\{2k, 10\} + 1$. By Proposition A.1, $|V(\mathbb{O}) \cap f(u)| \leq 1$ for every child u of v. Since $f(v) = \bigcup_{u \in \mathsf{child}(v)} f(u)$, this means that v has at least $\max\{2k, 10\} + 1$ children in T. ◀

B.9 Proof of Lemma 21

We arbitrarily select $v \in P$ and a child u of v in T such that u has a problematic descendant. Since P has no conflicts, there exists a problematic node $x \notin P$ that is either u or a descendant of u, such that $x \notin \bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of u in T. (Note that $u \notin P$.) We first claim that among the children of x, there are at most k children that are each problematic or has a problematic descendant. To see this, note that for each child y of x that is problematic or has a problematic descendant, the subgraph $G[f(y)]$ has an obstruction, and that for distinct children y, y' of x, it holds that $f(y) \cap f(y') = \emptyset$. Thus, since G does not have k vertex-disjoint obstructions, the claim follows.

By Lemma 18, we further have that x has at least $2k + 1$ children. To conclude the lemma, note that for every child y of x that is not problematic and has no problematic descendant, we have at least one vertex in $f(y) \subseteq f(u)$ that does not belong to $\bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of u in T. As we have shown that x has less than k children that are problematic or have problematic descendants, and x has at least $2k + 1$ children, we know that x has at least k non-problematic children with no problematic descendant. Thus, we derive that there exist at least k vertices in $f(u)$ that do not belong to $\bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of u in T. ◀

B.10 Proof of Lemma 22

Suppose that G does not have k vertex-disjoint obstructions, else the proof is complete. For the proof, we strengthen the statement of the lemma as follows. We claim that G has $\sum_{v \in P} \mathsf{pack}(v)$ vertex-disjoint obstructions that belong to $\bigcup_{v \in P} \mathsf{prob}(v)$. We prove the claim by induction on $|V(T)|$. In the basis, where $|V(T)| \leq 1$, the claim is trivially true.

Now, suppose that $|V(T)| \geq 2$, and let r denote the root of T. Let $v_1, v_2, \ldots, v_t$ be the children of r in T. For all $i \in [t]$, let us denote $G_i = G[f(v_i)]$, $T_i = T|_{v_i}$, $f_i = f|_{V(T_i)}$ and $g_i = g|_{V(T_i)}$. Note that (T_i, f_i, g_i) is a modular tree decomposition of G_i. Moreover, denote $P_i = P \cap V(T_i)$. By the inductive hypothesis, G_i has $\sum_{v \in P_i} \mathsf{pack}(v)$ vertex-disjoint obstructions that belong to $\bigcup_{v \in P_i} \mathsf{prob}(v)$, and let us denote a set of such vertex-disjoint obstructions by $\mathcal{O}_i$. As the subtrees T_i, $i \in [t]$, are vertex-disjoint, we have that $\mathcal{O} = \bigcup_{i \in [t]} \mathcal{O}_i$ is a set of $\sum_{v \in (P \setminus \{r\})} \mathsf{pack}(v)$ vertex-disjoint obstructions. If $r \notin P$, the proof is complete, and therefore we next suppose that $r \in P$.

Let $\mathcal{O}_r$ denote a set of $\mathsf{pack}(r)$ vertex-disjoint obstructions in $\mathsf{prob}(r)$, such that among all such sets, it minimizes $|V(\mathcal{O}_r) \cap V(\mathcal{O})|$. We claim that $V(\mathcal{O}_r) \cap V(\mathcal{O}) = \emptyset$, which would complete the proof, as then $\mathcal{O}_r \cup \mathcal{O}$ would be a set of $\sum_{v \in P} \mathsf{pack}(v)$ vertex-disjoint obstructions. Suppose, by way of contradiction, that this claim is false. Since then $V(\mathcal{O}_r) \cap V(\mathcal{O}) \neq \emptyset$, there exists $\mathbb{O} \in \mathcal{O}_r$ that contains at least one vertex in $f(u)$ for a descendant $u \in P$ of r in T that belongs to P. Suppose, without loss of generality, that v_1 is the child of r such that u is a descendant of v_1. Then, as G has no obstruction on at most 10 vertices, we have that for all $\mathbb{O}' \in \mathcal{O}_r$, $|V(\mathbb{O}') \cap f(v_1)| \leq 1$. In particular, $|V(\mathbb{O}) \cap f(v_1)| = 1$ and since $|\mathcal{O}_r| \leq k-1$ (because G does not have k vertex-disjoint obstructions), $|V(\mathcal{O}_r) \cap f(v_1)| \leq k-1$. By Lemma 21, there exist at least k vertices in $f(v_1)$ that do not belong to $\bigcup_w f(w)$ where w ranges over all nodes in P that are descendants of v_1 in T. Thus, there exists a vertex $x \in f(v_1)$ that does not belong to any obstruction in $\mathcal{O}_r \cup \mathcal{O}$. Because $f(v_1)$ is a module and $|V(\mathbb{O}) \cap f(v_1)| = 1$, by replacing the vertex in $V(\mathbb{O}) \cap f(v_1)$ by x in $\mathbb{O}$, we obtain another obstruction $\widehat{\mathbb{O}} \in \mathsf{prob}(r)$. However, then $(\mathcal{O}_r \setminus \{\mathbb{O}\}) \cup \{\widehat{\mathbb{O}}\}$ is a set of $\mathsf{pack}(r)$ vertex-disjoint obstructions in $\mathsf{prob}(r)$ that uses less vertices from $V(\mathcal{O})$ than $\mathcal{O}_r$. This contradicts the choice of $\mathcal{O}_r$, and hence the proof is complete. ◀

B.11 Proof of Lemma 23

Let us prove this claim by induction on $|\mathsf{prob}(T)|$. In the basis, where $\mathsf{prob}(T) = \emptyset$, the partition is simply $(\emptyset, \emptyset)$. Now, suppose that the claim holds up to $i-1$, and let us prove it for i. Let $P \subseteq \mathsf{prob}(T)$ contain the set consisting of every problematic node v such that there is no problematic node apart from v on the unique path between v and the root of T (this set may simply consist only of the root of T, if it belongs to P). Let us denote $P = \{r^1, r^2, \ldots, r^t\}$. Notice that $t \geq 1$. Now, for each r^i, $i \in [t]$, let P_i denote the set consisting of every problematic node v that is a descendant of r^i and such that there is no problematic node apart from v on the unique path between v and r^i. Let us denote $P_i = \{v^i_1, v^i_2, \ldots, v^i_{t_i}\}$. Now, we apply the induction hypothesis on each $T_{v^i_j}$, $i \in [t], j \in [t_i]$, to obtain a partition (A^i_j, B^i_j). Without loss of generality, we can assume that $v^i_j \in A^i_j$, else we can swap A^i_j and B^i_j. Now, we define $A = \bigcup_{i \in [t]} \bigcup_{j \in [t_i]} A^i_j$ and $B = \bigcup_{i \in [t]} \bigcup_{j \in [t_i]} B^i_j$. Since for all $i, i' \in [t], j \in [t_i], j' \in [t_{i'}]$, neither v^i_j is an ancestor of $v^{i'}_{j'}$ nor $v^{i'}_{j'}$ is an ancestor of v^i_j, we have that neither A nor B has a conflict. Next, notice that for all $i, i' \in [t]$, neither r^i is an ancestor of $r^{i'}$ nor $r^{i'}$ is an ancestor of r^i. Furthermore, for all $i \in [t]$, all the descendants of r^i that can have a conflict with r^i are in A. Thus, $(A, B \cup \{r^i : i \in [t]\})$ is a partition of $\mathsf{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict. ◀

B.12 Proof of Lemma 27

By Corollary 26, it is sufficient to show that every non-trivial module of $G[f(v)]$ is a clique. By Observation 4.1, Lemma 17 and since v is problematic, we have that $g(v) = 0$. Thus, by Observation 4.1 and the definition of a modular tree decomposition, every non-trivial module of G is a subset of $f(u)$ for a child u of v in T. Since for every child u of v in T, $G[f(u)]$ is a clique, the proof is complete. ◀

B.13 Proof of Lemma 29

In one direction, let $\mathbb{O}$ be an obstruction in $\mathsf{prob}_G(v)$. To show that $\mathbb{O}$ is an obstruction in $\mathsf{clique}(G, v)[f(v)]$, it is sufficient to show that $\mathbb{O}$ does not contain two vertices between whom an edge was added. However, as G has no obstruction on at most ten vertices, we have that $|V(\mathbb{O})| > 10$. Then, as $\mathbb{O}$ is not present in $G[f(u)]$ for any child u of v in T, by Proposition A.1, its has at most one vertex in $f(u)$ for any child u of v in T. This concludes this direction.

In the other direction, let $\mathbb{O}$ be an obstruction in $\mathsf{clique}(G,v)[f(v)]$. Note that $\mathsf{clique}(G,v)[f(u)]$, $u \in \mathsf{child}(v)$, is a clique. Thus, $\mathbb{O}$ is not present in $G[f(u)]$ for any child u of v in T. Thus, to show that $\mathbb{O}$ is an obstruction in $\mathsf{prob}_G(v)$, it is sufficient to show that $\mathbb{O}$ is an obstruction in G. In turn, this means that it is sufficient to show that $\mathbb{O}$ does not contain two vertices between whom an edge was added. Suppose, by way of contradiction, that it does. Then, there exist $u \in \mathsf{child}(v)$ and $x, y \in f(u)$ such that $x, y \in V(\mathbb{O})$. Note that $f(u)$ is also a module in $\mathsf{clique}(G,v)[f(v)]$. Thus, Proposition A.1 implies that $|V(\mathbb{O})| \leq 4$, which means that $\mathbb{O}$ is a chordless cycle on four vertices. However, this is not possible as x, y are adjacent to each other and have the same neighborhood in $\mathsf{clique}(G,v)$. This concludes the proof. ◀

B.14 Proof of Lemma 30

Suppose that $\mathsf{pack}(v) < k$, else the proof is complete. By Lemma 29, the set of obstructions in $\mathsf{clique}(G,v)[f(v)]$ is precisely $\mathsf{prob}_G(v)$. Therefore, $\mathsf{clique}(G,v)[f(v)]$ has less than k vertex-disjoint obstructions. By Lemma 27, $\mathsf{clique}(G,v)[f(v)]$ has a nice clique caterpillar. Thus, by Lemma 4, there exists a subset $D \subseteq f(v)$ of size $\mathcal{O}(k^2)$ such that $\mathsf{clique}(G,v)[f(v)] - D$ is an interval graph. That is, D intersects the vertex set of every obstruction in $\mathsf{clique}(G,v)[f(v)]$, which means that D intersects the vertex set of every obstruction in $\mathsf{prob}_G(v)$. ◀

B.15 Proof of Corollary 32

Let c' be the constant in the $\mathcal{O}$ notation in Theorem 1, and denote $c = 8c'$. Our algorithm is as follows. Given a graph G and an integer $k \in \mathbb{N}$, it calls the algorithm in Proposition 5.1 to obtain an integer d'. If $d' > ck^2 \log k$, then it concludes that the first condition holds, and otherwise it concludes that the second condition holds. Clearly, the algorithm runs in time $\mathcal{O}(nm)$.

If the algorithm concluded that the second condition holds, then by the correctness of the algorithm in Proposition 5.1, this conclusion is correct. Otherwise, by Proposition 5.1, there does not exist a subset $D \subseteq V(G)$ of size at most $c'k^2 \log k$ such that $G - D$ is an interval graph. By Theorem 1, this means that G has k vertex-disjoint obstructions. Thus, the conclusion of the algorithm is again correct. ◀