

# HITTING FORBIDDEN MINORS: APPROXIMATION AND KERNELIZATION\*

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**Abstract.** We study a general class of problems called  $\mathcal{F}$ -DELETION problems. In an  $\mathcal{F}$ -DELETION problem, we are asked whether a subset of at most  $k$  vertices can be deleted from a graph  $G$  such that the resulting graph does not contain as a minor any graph from the family  $\mathcal{F}$  of forbidden minors. We study the problem parameterized by  $k$ , using  $p$ - $\mathcal{F}$ -DELETION to refer to the parameterized version of the problem. We obtain a number of algorithmic results on the  $p$ - $\mathcal{F}$ -DELETION problem when  $\mathcal{F}$  contains a planar graph. We give

- a linear vertex kernel on graphs excluding  $t$ -claw  $K_{1,t}$ , the star with  $t$  leaves, as an induced subgraph, where  $t$  is a fixed integer.
- an approximation algorithm achieving an approximation ratio of  $O(\log^{3/2} OPT)$ , where  $OPT$  is the size of an optimal solution on general undirected graphs.

Finally, we obtain polynomial kernels for the case when  $\mathcal{F}$  only contains graph  $\theta_c$  as a minor for a fixed integer  $c$ . The graph  $\theta_c$  consists of two vertices connected by  $c$  parallel edges. Even though this may appear to be a very restricted class of problems it already encompasses well-studied problems such as VERTEX COVER, FEEDBACK VERTEX SET and DIAMOND HITTING SET. The generic kernelization algorithm is based on a non-trivial application of protrusion techniques, previously used only for problems on topological graph classes.

**1. Introduction.** Let  $\mathcal{F}$  be a finite set of graphs. Throughout the paper we assume that  $\mathcal{F}$  is explicitly given to us. In an  $p$ - $\mathcal{F}$ -DELETION problem<sup>1</sup>, we are given an  $n$ -vertex graph  $G$  and an integer  $k$  as input, and asked whether at most  $k$  vertices can be deleted from  $G$  such that the resulting graph does not contain a graph from  $\mathcal{F}$  as a minor. More precisely, the problem is defined as follows:

$p$ - $\mathcal{F}$ -DELETION  
*Instance:* A graph  $G$  and a non-negative integer  $k$ .  
*Parameter:*  $k$   
*Question:* Does there exist  $S \subseteq V(G)$ ,  $|S| \leq k$ , such that  $G \setminus S$  contains no graph from  $\mathcal{F}$  as a minor?

We refer to such a subset  $S$  as an  $\mathcal{F}$ -hitting set. The  $p$ - $\mathcal{F}$ -DELETION problem is a generalization of several fundamental problems. For example, when  $\mathcal{F} = \{K_2\}$ , a complete graph on two vertices, this is the VERTEX COVER problem. When  $\mathcal{F} = \{C_3\}$ , a cycle on three vertices, this is the FEEDBACK VERTEX SET problem. Other famous cases are  $\mathcal{F} = \{K_{2,3}, K_4\}$ ,  $\mathcal{F} = \{K_{3,3}, K_5\}$  and  $\mathcal{F} = \{K_3, T_2\}$ , which correspond to removing vertices to obtain outerplanar graphs, planar graphs, and graphs of pathwidth one respectively. Here,  $K_{i,j}$  denotes the complete bipartite graph with bipartitions of sizes  $i$  and  $j$ , and  $K_i$  denotes the complete graph on  $i$  vertices. Further, a  $T_2$

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<sup>1</sup>We use prefix  $p$  to distinguish the parameterized version of the problem.

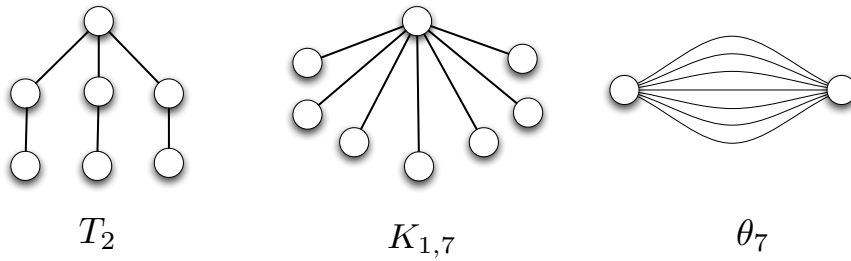


FIG. 1.1. Graphs  $T_2$ ,  $t$ -claw  $K_{1,t}$  with  $t = 7$ , and  $\theta_c$  with  $c = 7$

is a star on three leaves, each of whose edges has been subdivided exactly once. A  $T_2$  structure is depicted in the leftmost graph of Figure 1.1. In the literature, these problems are known as  $p$ -OUTERPLANAR DELETION SET,  $p$ -PLANAR DELETION SET and  $p$ -PATHWIDTH ONE DELETION SET respectively.

Our interest in the  $p$ - $\mathcal{F}$ -DELETION problem is motivated by its generality and the recent developments in kernelization or polynomial time preprocessing. The parameterized complexity of this general problem is well understood. By a celebrated result of Robertson and Seymour, every  $p$ - $\mathcal{F}$ -DELETION problem is non-uniformly fixed-parameter tractable (FPT). That is, for every  $k$  there is an algorithm solving the problem in time  $O(f(k) \cdot n^3)$  [59]. On the other hand, whenever  $\mathcal{F}$  is given explicitly, we have that the excluded minors for the class of graphs that are YES-instances of the  $p$ - $\mathcal{F}$ -DELETION problem can be computed explicitly [2]. This leads to a single algorithm for all  $k$ , making the problem is uniformly FPT. In this paper we study this problem from the view point of polynomial time preprocessing and approximation, when the obstruction set  $\mathcal{F}$  satisfies certain properties.

Preprocessing as a strategy for coping with hard problems is universally applied in practice and the notion of *kernelization* provides a mathematical framework for analyzing the quality of preprocessing strategies. We consider parameterized problems, where every instance  $I$  comes with a *parameter*  $k$ . Such a problem is said to admit a *polynomial kernel* if every instance  $(I, k)$  can be reduced in polynomial time to an equivalent instance with both size and parameter value bounded by a polynomial in  $k$ . The notion of a *linear kernel* is analogous, where we require the size and the parameter of the reduced instance to be linear in  $k$ . The study of kernelization is a major research frontier of Parameterized Complexity and many important recent advances in the area are on kernelization. These include general results showing that certain classes of parameterized problems have polynomial kernels [4, 15, 43, 51]. The recent development of a framework for ruling out polynomial kernels under certain complexity-theoretic assumptions [14, 34, 44] has added a new dimension to the field and strengthened its connections to classical complexity. For overviews of kernelization we refer to surveys [12, 46] and to the corresponding chapters in books on Parameterized Complexity [30, 41, 55].

While the initial interest in kernelization was driven mainly by practical applications, the notion of kernelization turned out to be very important in theory as well. It is well known, see e.g. [35], that a parameterized problem is fixed parameter tractable, or belongs to the class FPT, if and only if it has a (perhaps exponential) kernel. Kernelization enables us to classify problems within the class FPT further, based on the sizes of the problem kernels. So far, most of the work done in the field of kerneliza-

tion is still specific to particular problems and powerful unified techniques to identify classes of problems with polynomial kernels are still in their nascent stage. One of the fundamental challenges in the area is the possibility of characterising general classes of parameterized problems possessing kernels of polynomial sizes. From this perspective, the class of the  $p$ - $\mathcal{F}$ -DELETION problems is very interesting because it contains as special cases the  $p$ -VERTEX COVER and  $p$ -FEEDBACK VERTEX SET problems which are the most intensively studied problems from the kernelization perspective.

*Our contribution and key ideas.* One of the main conceptual contributions of this work is the extension of protrusion techniques employed in [15, 43] for obtaining meta-kernelization theorems for problems on sparse graphs like planar and  $H$ -minor-free graphs, to general graphs. We demonstrate this by obtaining a number of kernelization results for the  $p$ - $\mathcal{F}$ -DELETION problem when  $\mathcal{F}$  contains a planar graph. Our first result is the following theorem for graphs that do not contain  $K_{1,t}$  (a star on  $t$  leaves, see Figure 1.1) as an induced subgraph.

**THEOREM 1.1.** *Let  $\mathcal{F}$  be an obstruction set containing a planar graph. Then  $p$ - $\mathcal{F}$ -DELETION admits a linear vertex kernel on graphs excluding  $K_{1,t}$  as an induced subgraph, where  $t$  is a fixed integer.* Several well studied graph classes do not contain graphs with induced  $K_{1,t}$ . Of course, every graph with maximum vertex degree at most  $(t-1)$  is  $K_{1,t}$ -free. The class of  $K_{1,3}$ -free graphs, also known as claw-free graphs, contains line graphs and de Bruijn graphs. Unit disc graphs are known to be  $K_{1,7}$ -free [26].

Our kernelization is a divide and conquer algorithm which finds large protrusions. A *protrusion* is a subgraph of constant treewidth separated from the remaining part of the graph by a constant number of vertices. Having found protrusions of substantial size, the kernelization algorithm replaces them with smaller, “equivalent” protrusions. Here we use the results from the work by Bodlaender et al. [15] that enable this step whenever the parameterized problem in question “behaves like a regular language”. To prove that  $p$ - $\mathcal{F}$ -DELETION has the desired properties for this step, we formulate the problem in monadic second order logic and show that it exhibits certain monotonicity properties. As a corollary we obtain that  $p$ -FEEDBACK VERTEX SET,  $p$ -DIAMOND HITTING SET,  $p$ -PATHWIDTH ONE DELETION SET, and  $p$ -OUTERPLANAR DELETION SET admit a linear vertex kernel on graphs excluding  $K_{1,t}$  as an induced subgraph. The same methodology applies to  $p$ -DISJOINT CYCLE PACKING, which is the problem of finding at least  $k$  vertex-disjoint cycles (parameterized by  $k$ ). In particular, we obtain an  $O(k \log k)$  vertex kernel for  $p$ -DISJOINT CYCLE PACKING on graphs excluding  $K_{1,t}$  as an induced subgraph. We note that  $p$ -DISJOINT CYCLE PACKING does not admit a polynomial kernel on general graphs [17] unless  $\text{NP} \subseteq \text{CoNP}/\text{poly}$ .

Let  $\theta_c$  be a graph with two vertices and  $c \geq 1$  parallel edges (see Figure 1.1). Our second result is the following theorem on general graphs.

**THEOREM 1.2.** *Let  $\mathcal{F}$  be an obstruction set containing only  $\theta_c$ . Then  $p$ - $\mathcal{F}$ -DELETION admits a kernel of size  $O(k^2 \log^{3/2} k)$ .* A number of well-studied NP-hard combinatorial problems are special cases of  $p$ - $\theta_c$ -DELETION. When  $c = 1$ , this is the classical  $p$ -VERTEX COVER problem [54]. For  $c = 2$ , this is another well studied problem, the  $p$ -FEEDBACK VERTEX SET problem [7, 9, 24, 48]. When  $c = 3$ , this is the  $p$ -DIAMOND HITTING SET problem [40]. Let us note that the size of the best known kernel for  $c = 2$  is  $O(k^2)$ , which is very close to the size of the kernel in Theorem 1.2. Also, Dell and van Melkebeek proved that no NP-hard vertex deletion problem based on a graph property that is inherited by subgraphs can have kernels of size  $O(k^{2-\epsilon})$  unless  $\text{NP} \subseteq \text{CoNP}/\text{poly}$  [34] and thus the sizes of the kernels in Theorem 1.2 are

tight up to a polylogarithmic factor.

The proof of Theorem 1.2 is obtained in a series of non-trivial steps. The very high level idea is to reduce the general case to problem on graphs of bounded degree, which allows us to use the protrusion techniques as in the proof of Theorem 1.1. However, vertex degree reduction is not straightforward and requires several new ideas. One of the new tools is a generic  $O(\log^{3/2} OPT)$ -approximation algorithm for the  $p$ - $\mathcal{F}$ -DELETION problem when the class of excluded minors for  $\mathcal{F}$  contains at least one planar graph. More precisely, we obtain the following result, which is of independent interest.

**THEOREM 1.3.** *Let  $\mathcal{F}$  be an obstruction set containing a planar graph, and let  $OPT$  be the size of the smallest  $\mathcal{F}$ -hitting set. Given a graph  $G$ , in polynomial time we can find a subset  $S \subseteq V(G)$  such that  $G[V \setminus S]$  contains no element of  $\mathcal{F}$  as a minor and  $|S| = O(OPT \cdot \log^{3/2} OPT)$ . The constants in the theorem depend only on the family  $\mathcal{F}$ , and in particular, the size of the smallest planar graph in  $\mathcal{F}$ . While several generic approximation algorithms are known for problems of minimum vertex deletion to obtain subgraphs with property  $P$ , like when  $P$  is a hereditary property with a finite number of minimal forbidden subgraphs [53], or can be expressed as a universal first order sentence over subsets of edges of the graph [50], we are not aware of any generic approximation algorithm for the case when a property  $P$  is characterized by a finite set of forbidden minors.*

We then use the approximation algorithm as a subroutine in a polynomial time algorithm that transforms the input instance  $(G, k)$  into an equivalent instance  $(G', k')$  such that  $k' \leq k$  and the maximum degree of  $G'$  is bounded by  $O(k \log^{3/2} k)$ . After obtaining an equivalent instance with bounded degree, we apply protrusion techniques and prove Theorem 1.2. An important combinatorial tool used in designing this algorithm is the  $q$ -Expansion Lemma. For  $q = 1$  this lemma is Hall's theorem and its usage is equivalent to the application of the Crown Decomposition technique [1, 23]. Applying the Lemma for  $q = 2$  amounts to what is known as the *double crown decomposition*, used by Prieto and Sloper for the first time to obtain a quadratic kernel for packing  $k$  disjoint copies of stars with  $s$  leaves [58]. Prieto [57] used  $q$ -Expansion Lemma in its most general form to obtain a quadratic kernel for finding a maximal matching with at most  $k$  edges. Prieto [57] referred to the  $q$ -Expansion Lemma as the  *$q$ -spike crown decomposition*. We would like to add here that even though the applications of  $q$ -Expansion Lemma is the same as the  $q$ -spike crown decomposition, the  $q$ -Expansion Lemma involves a slightly weaker hypothesis (details can be found in Section 5).

*Related work.* All non-trivial  $p$ - $\mathcal{F}$ -DELETION problems are NP-hard [52]. By one of the most well-known consequences of the celebrated Graph Minor theory of Robertson and Seymour the  $p$ - $\mathcal{F}$ -DELETION problem is non-uniformly fixed parameter tractable. Whenever  $\mathcal{F}$  is given explicitly, the problem is uniformly FPT because the excluded minors for the class of graphs that are YES-instances of the  $p$ - $\mathcal{F}$ -DELETION problem can be computed explicitly [2]. A special case of the problem when  $\mathcal{F}$  contains a planar graph was introduced by Fellows and Langston [38], who gave a non-constructive algorithm running in time  $f(k) \cdot n^2$  for some function  $f(k)$  [38, Theorem 6]. Even more special case of the problem, when the set  $\mathcal{F}$  contains only  $\theta_c$ , has been studied from approximation and parameterized perspectives. In particular, the case of  $p$ - $\theta_1$ -DELETION or, equivalently,  $p$ -VERTEX COVER, is the most well-studied problem in Parameterized Complexity. Different kernelization techniques were applied on the problem, eventually resulting in a  $2k$ -sized vertex kernel [1, 22, 33, 47].

For the kernelization of  $p$ -FEEDBACK VERTEX SET, or  $p$ - $\theta_2$ -DELETION, there has been a sequence of dramatic improvements starting from an  $O(k^{11})$  vertex kernel by Buragge et al. [20], improved to  $O(k^3)$  by Bodlaender [11], and then finally to  $O(k^2)$  by Thomassé [61]. Recently Philip et al. [56] and Cygan et al. [31] obtained polynomial kernels for  $p$ -PATHWIDTH ONE DELETION SET. Constant factor approximation algorithms are known for VERTEX COVER and FEEDBACK VERTEX SET [7, 8]. Very recently, a constant factor approximation algorithm for the DIAMOND HITTING SET problem, or  $p$ - $\theta_3$ -DELETION, was obtained in [40]. Prior to our work, no polynomial kernels were known for  $p$ -DIAMOND HITTING SET or more general families of  $p$ - $\mathcal{F}$ -DELETION problems.

One of the main techniques used in this work is the extension of the protrusion theory employed in [15, 43] for obtaining meta-kernelization theorems for problems on sparse graphs like planar and  $H$ -minor-free graphs, to general graphs. Bodlaender et al. [15] were first to use protrusion techniques (or rather graph reduction techniques) to obtain kernels, but the idea of using graph replacement for algorithms has been there for long time. The idea of graph replacement for algorithms dates back to Fellows and Langston [39]. Arnborg et al. [5] essentially showed that protrusions exist for many problems on graphs of bounded treewidth, and gave safe ways of reducing graphs. Using this, Arnborg et al. [5] obtained a linear time algorithm for MSO expressible problems on graphs of bounded treewidth. Bodlaender and Fluiters [13, 18, 32] generalized these ideas in several ways — in particular, they applied it to some optimization problems. It is also important to mention the work of Bodlaender and Hagerup [16], who used the concept of graph reduction to obtain parallel algorithms for MSO expressible problems on bounded treewidth graphs.

The remaining part of the paper is organised as follows. In Section 2 we provide preliminaries on basic notions from Graph Theory and Logic used in the paper. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we give an approximation algorithms proving Theorem 1.3. The proof of Theorem 1.2 is given in Section 5. We conclude with open questions in Section 6.

**2. Preliminaries.** In this section we give various definitions which we use in the paper. For  $n \in \mathbb{N}$ , we use  $[n]$  to denote the set  $\{1, \dots, n\}$ . We use  $V(G)$  to denote the vertex set of a graph  $G$ , and  $E(G)$  to denote the edge set. The degree of a vertex  $v$  in  $G$  is the number of edges incident on  $v$ , and is denoted by  $d(v)$ . We use  $\Delta(G)$  to denote the maximum degree of  $G$ . A graph  $G'$  is a *subgraph* of  $G$  if  $V(G') \subseteq V(G)$  and  $E(G') \subseteq E(G)$ . The subgraph  $G'$  is called an *induced subgraph* of  $G$  if  $E(G') = \{\{u, v\} \in E(G) \mid u, v \in V(G')\}$ . Given a subset  $S \subseteq V(G)$  the subgraph induced by  $S$  is denoted by  $G[S]$ . The subgraph induced by  $V(G) \setminus S$  is denoted by  $G \setminus S$ . We denote by  $N_G(S)$  the open neighborhood of  $S$ , i.e. the set of vertices in  $V(G) \setminus S$  adjacent to  $S$ . Whenever the graph  $G$  is clear from the context, we omit the subscript in  $N_G(S)$  and denote it only by  $N(S)$ . Given a graph  $G$  and  $S \subseteq V(G)$ , we define  $\partial_G(S)$  as the set of vertices in  $S$  that have a neighbor in  $V(G) \setminus S$ . For a set  $S \subseteq V(G)$  the neighborhood of  $S$  is  $N_G(S) = \partial_G(V(G) \setminus S)$ . When it is clear from the context, we omit the subscripts.

By *contracting* an edge  $(u, v)$  of a graph  $G$ , we mean identifying the vertices  $u$  and  $v$ , keeping all the parallel edges and removing all the loops. A *minor* of a graph  $G$  is a graph  $H$  that can be obtained from a subgraph of  $G$  by contracting edges. We keep parallel edges after contraction since the graph  $\theta_c$  which we want to exclude as a minor itself contains parallel edges.

Let  $G, H$  be two graphs. A subgraph  $G'$  of  $G$  is said to be a *minor-model* of  $H$  in

$G$  if  $G'$  contains  $H$  as a minor. The subgraph  $G'$  is a *minimal minor-model* of  $H$  in  $G$  if no proper subgraph of  $G'$  is a minor-model of  $H$  in  $G$ . A graph class  $\mathcal{C}$  is *minor closed* if any minor of any graph in  $\mathcal{C}$  is also an element of  $\mathcal{C}$ . A minor closed graph class  $\mathcal{C}$  is  *$H$ -minor-free* or simply  *$H$ -free* if  $H \notin \mathcal{C}$ . Let  $\mathcal{F}$  be a finite set of graphs. A vertex subset  $S \subseteq V(G)$  of a graph  $G$  is said to be a  *$\mathcal{F}$ -hitting set* if  $G \setminus S$  does not contain any graphs in the family  $\mathcal{F}$  as a minor.

**2.1. Monadic Second Order Logic (MSO).** The syntax of MSO on graphs includes the logical connectives  $\vee, \wedge, \neg, \Leftrightarrow, \Rightarrow$ , variables for vertices, edges, sets of vertices and sets of edges, the quantifiers  $\forall, \exists$  that can be applied to these variables, and the following five binary relations:

1.  $u \in U$  where  $u$  is a vertex variable and  $U$  is a vertex set variable;
2.  $d \in D$  where  $d$  is an edge variable and  $D$  is an edge set variable;
3.  $\mathbf{inc}(d, u)$ , where  $d$  is an edge variable,  $u$  is a vertex variable, and the interpretation is that the edge  $d$  is incident on the vertex  $u$ ;
4.  $\mathbf{adj}(u, v)$ , where  $u$  and  $v$  are vertex variables  $u, v$ , and the interpretation is that  $u$  and  $v$  are adjacent;
5. equality of variables representing vertices, edges, set of vertices and set of edges.

Many common graph-theoretic notions such as vertex degree, connectivity, planarity, being acyclic, and so on, can be expressed in MSO, as can be seen from introductory expositions [19, 28]. Of particular interest to us are  $p$ -MIN-MSO problems. In a  $p$ -MIN-MSO graph problem  $\Pi$ , we are given a graph  $G$  and an integer  $k$  as input. The objective is to decide whether there is a vertex/edge set  $S$  of size at most  $k$  such that the MSO-expressible predicate  $P_{\Pi}(G, S)$  is satisfied.

**2.2. Parameterized Algorithms and Kernels.** A parameterized problem  $\Pi$  is a subset of  $\Gamma^* \times \mathbb{N}$  for some finite alphabet  $\Gamma$ . An instance of a parameterized problem consists of  $(x, k)$ , where  $k$  is called the parameter. A central notion in parameterized complexity is *fixed parameter tractability (FPT)* which means, for a given instance  $(x, k)$ , solvability in time  $f(k) \cdot p(|x|)$ , where  $f$  is an arbitrary function of  $k$  and  $p$  is a polynomial in the input size. The notion of *kernelization* is formally defined as follows.

**DEFINITION 1. [Kernelization, Kernel]** [14] *A kernelization algorithm for a parameterized problem  $\Pi \subseteq \Sigma^* \times \mathbb{N}$  is an algorithm that, given  $(x, k) \in \Sigma^* \times \mathbb{N}$ , outputs, in time polynomial in  $(|x| + k)$ , a pair  $(x', k') \in \Sigma^* \times \mathbb{N}$  such that: (a)  $(x, k) \in \Pi$  if and only if  $(x', k') \in \Pi$  and (b)  $|x'|, k' \leq g(k)$ , where  $g$  is some computable function. The output instance  $x'$  is called the kernel, and the function  $g$  is referred to as the size of the kernel. If  $g(k) = k^{O(1)}$ , then we say that  $\Pi$  admits a polynomial kernel.*

It is important to mention here that the early definitions of kernelization required that  $k' \leq k$ . This makes intuitive sense, as the parameter  $k$  measures the complexity of the problem — thus the larger the  $k$ , the harder the problem. This requirement was subsequently relaxed, notably in the context of lower bounds. An advantage of the more liberal notion of kernelization is that it is robust with respect to polynomial transformations of the kernel. However, it limits the connection with practical preprocessing. All the kernels obtained in this paper respect the fact that the output parameter is at most the input parameter, that is,  $k' \leq k$ .

**2.3. Tree-width and Protrusions.** Let  $G$  be a graph. A *tree decomposition* of a graph  $G$  is a pair  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$ , where  $T$  is a tree, and for all  $t \in V(T)$ ,  $X_t$  is a subset of  $V(G)$ , such that:

- $\cup_{t \in V(T)} X_t = V(G)$ ,
- for every edge  $(x, y) \in E(G)$  there is a  $t \in V(T)$  such that  $\{x, y\} \subseteq X_t$ , and
- for every vertex  $v \in V(G)$  the subgraph of  $T$  induced by the set  $\{t \mid v \in X_t\}$  is connected.

The *width* of a tree decomposition is  $(\max_{t \in V(T)} |X_t|) - 1$  and the *treewidth* of  $G$  is the minimum width over all tree decompositions of  $G$ . A tree decomposition  $(T, \mathcal{X})$  is called a *nice tree decomposition* if  $T$  is a tree rooted at some node  $r$  where  $X_r = \emptyset$ , each node of  $T$  has at most two children, and each node is of one of the following kinds:

1. *Introduce node*: a node  $t$  that has only one child  $t'$  where  $X_t \supset X_{t'}$  and  $|X_t| = |X_{t'}| + 1$ .
2. *Forget node*: a node  $t$  that has only one child  $t'$  where  $X_t \subset X_{t'}$  and  $|X_t| = |X_{t'}| - 1$ .
3. *Join node*: a node  $t$  with two children  $t_1$  and  $t_2$  such that  $X_t = X_{t_1} = X_{t_2}$ .
4. *Base node*: a node  $t$  that is a leaf of  $t$ , is different than the root, and  $X_t = \emptyset$ .

Notice that, according to the above definition, the root  $r$  of  $T$  is either a forget node or a join node. It is well known that any tree decomposition of  $G$  can be transformed into a nice tree decomposition in time  $O(|V(G)| + |E(G)|)$  maintaining the same width [49]. We use  $G_t$  to denote the graph induced on the vertices  $\cup_{t' \in \text{desc}(t)} X_{t'}$ , where  $t'$  ranges over all descendants of  $t$ , including  $t$ . We use  $H_t$  to denote  $G_t \setminus X_t$ . We now define the notion of a *protrusion*.

**DEFINITION 2. [*r*-protrusion]** *Given a graph  $G$ , we say that a set  $X \subseteq V(G)$  is an *r*-protrusion of  $G$  if  $\text{tw}(G[X]) \leq r$  and  $|\partial(X)| \leq r$ .*

**2.4. *t*-Boundaried Graphs.** In this section we define *t*-boundaried graphs and various operations on them. Throughout this section,  $t$  is an arbitrary positive integer.

**DEFINITION 3. [*t*-Boundaried Graphs]** *A *t*-boundaried graph is a graph  $G$  with  $t$  distinguished vertices, uniquely labeled from 1 to  $t$ . The set  $\partial(G)$  of labeled vertices is called the boundary of  $G$ . The vertices in  $\partial(G)$  are referred to as boundary vertices or terminals.*

For a graph  $G$  and a vertex set  $S \subseteq V(G)$ , we will sometimes consider the graph  $G[S]$  as the  $|\partial(S)|$ -boundaried graph with  $\partial(S)$  being the boundary.

**DEFINITION 4. [Gluing by  $\oplus$ ]** *Let  $G_1$  and  $G_2$  be two *t*-boundaried graphs. We denote by  $G_1 \oplus G_2$  the *t*-boundaried graph obtained by taking the disjoint union of  $G_1$  and  $G_2$  and identifying each vertex of  $\partial(G_1)$  with the vertex of  $\partial(G_2)$  with the same label; that is, we glue them together on the boundaries. In  $G_1 \oplus G_2$  there is an edge between two labeled vertices if there is an edge between them in  $G_1$  or in  $G_2$ .*

In this paper, *t*-boundaried graphs often come coupled with a vertex set which represents a partial solution to some optimization problem. For ease of notation we define  $\mathcal{H}_t$  to be the set of pairs  $(G, S)$ , where  $G$  is a *t*-boundaried graph and  $S \subseteq V(G)$ .

**DEFINITION 5. [Replacement]** *Let  $G$  be a graph containing an *r*-protrusion  $X$ . Let  $G_1$  be an *r*-boundaried graph. The act of replacing  $G[X]$  with  $G_1$  corresponds to changing  $G$  into  $G[(V(G) \setminus X) \cup \partial(X)] \oplus G_1$ .*

**2.5. Finite Integer Index.** In this section we define the notion of finite integer index. This first appeared in the work of Bodlaender and Fluiter [13, 18, 32].

**DEFINITION 6. [Canonical Equivalence]** *For a parameterized problem  $\Pi$  and two *t*-boundaried graphs  $G_1$  and  $G_2$ , we say that  $G_1 \equiv_{\Pi} G_2$  if there exists a constant  $c$  such that for all *t*-boundaried graphs  $G_3$  and for all  $k$ ,*

$$(G_1 \oplus G_3, k) \in \Pi \text{ if and only if } (G_2 \oplus G_3, k + c) \in \Pi.$$

DEFINITION 7. [**Finite Integer Index**] We say that a parameterized problem  $\Pi$  has finite integer index if for every  $t$  there exists a finite set  $\mathcal{S}$  of  $t$ -boundaried graphs such that for any  $t$ -boundaried graph  $G_1$  there exists  $G_2 \in \mathcal{S}$  such that  $G_2 \equiv_{\Pi} G_1$ . Such a set  $\mathcal{S}$  is called a set of representatives for  $(\Pi, t)$ . Note that for every  $t$ , the relation  $\equiv_{\Pi}$  on  $t$ -boundaried graphs is an equivalence relation. A problem  $\Pi$  is finite integer index if and only if for every  $t$ ,  $\equiv_{\Pi}$  is of finite index, that is, has a finite number of equivalence classes. The notion of *strong monotonicity* is an easily checked sufficient condition for a  $p$ -MIN-MSO problem to have finite integer index.

DEFINITION 8. [**Signatures**] Let  $\Pi$  be a  $p$ -MIN-MSO problem. For a  $t$ -boundaried graph  $G$  we define the signature function  $\zeta_G^{\Pi} : \mathcal{H}_t \rightarrow \mathbb{N} \cup \{\infty\}$  as follows. For a pair  $(G', S') \in \mathcal{H}_t$ , if there is no set  $S \subseteq V(G)$  ( $S \subseteq E(G)$ ) such that  $P_{\Pi}(G \oplus G', S \cup S')$  holds, then  $\zeta_G^{\Pi}((G', S')) = \infty$ . Otherwise  $\zeta_G^{\Pi}((G', S'))$  is the size of the smallest  $S \subseteq V(G)$  ( $S \subseteq E(G)$ ) such that  $P_{\Pi}(G \oplus G', S \cup S')$  holds.

DEFINITION 9. [**Strong Monotonicity**] A  $p$ -MIN-MSO problem  $\Pi$  is said to be strongly monotone if there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that the following condition is satisfied. For every  $t$ -boundaried graph  $G$ , there is a subset  $S \subseteq V(G)$  such that for every  $(G', S') \in \mathcal{H}_t$  such that  $\zeta_G^{\Pi}((G', S'))$  is finite,  $P_{\Pi}(G \oplus G', S \cup S')$  holds and  $|S| \leq \zeta_G^{\Pi}((G', S')) + f(t)$ .

**2.6. MSO Formulations.** We now give MSO formulations for some properties involving  $\mathcal{F}$  or  $\theta_c$  that we use in our arguments. For a graph  $G$  and a vertex set  $S \subseteq V(G)$ , let  $Conn(G, S)$  denote the MSO formula which states that  $G[S]$  is connected, and let  $MaxConn(G, S)$  denote the MSO formula which states that  $G[S]$  is a maximal connected subgraph of  $G$ .

*H minor-models.* Let  $\mathcal{F}$  be the finite forbidden set. For a graph  $G$ , we use  $\phi_H(G)$  to denote an MSO formula which states that  $G$  contains  $H$  as a minor — equivalently, that  $G$  contains a minimal  $H$  minor model. Let  $V(H) = \{h_1, \dots, h_c\}$ . Then, for a simple graph  $H$ ,  $\phi_H(G)$  is given by:

$$\begin{aligned} \phi_H(G) \equiv & \exists X_1, \dots, X_c \subseteq V(G) [ \\ & \bigwedge_{i \neq j} (X_i \cap X_j = \emptyset) \wedge \bigwedge_{1 \leq i \leq c} Conn(G, X_i) \wedge \\ & \bigwedge_{(h_i, h_j) \in E(H)} \exists x \in X_i \wedge y \in X_j [(x, y) \in E(G)] \\ & ] \end{aligned}$$

More generally, if  $H$  has  $p_j$  parallel edges between vertices  $(h_i, h_j)$ , then the formula  $\phi_H(G)$  is given by:

$$\begin{aligned} \phi_H(G) \equiv & \exists X_1, \dots, X_c \subseteq V(G) [ \\ & \bigwedge_{i \neq j} (X_i \cap X_j = \emptyset) \wedge \bigwedge_{1 \leq i \leq c} Conn(G, X_i) \wedge \\ & \bigwedge_{(h_i, h_j) \in E(H)} \exists x_1, \dots, x_{p_{ij}} \in X_i \wedge y_1, \dots, y_{p_{ij}} \in X_j [\wedge_{\ell \in [p_{ij}]} (x_{\ell}, y_{\ell}) \in E(G) \\ & \wedge \bigwedge_{(r \neq s) \in [p_{ij}] \times [p_{ij}]} [x_r \neq x_s] \wedge \bigwedge_{(r \neq s) \in [p_{ij}] \times [p_{ij}]} y_r \neq y_s] \\ & ] \end{aligned}$$



*Minimum-size  $\mathcal{F}$ -hitting set.* A minimum-size  $\mathcal{F}$ -hitting set of graph  $G$  can be expressed as:

$$\text{Minimize } S \subseteq V(G) \left[ \bigwedge_{H \in \mathcal{F}} \neg \phi_H(G \setminus S) \right]$$

*Largest  $\theta_c$  “flower”.* Let  $v$  be a vertex in a graph  $G$ . A maximum-size set  $M$  of  $\theta_c$  minor-models in  $G$ , all of which contain  $v$  and no two of which share any vertex other than  $v$ , can be represented as:

$$\begin{aligned} \text{Maximize } S \subseteq V(G) [ \\ \exists F \subseteq E(G) [\forall x \in S [ \\ \exists X \subseteq V' [ \text{MaxConn}(G', X) \wedge x \in X \wedge \forall y \in S [y \neq x \implies y \notin X] \wedge \phi_{\theta_c}(X \cup \{v\}) ] \\ ] ] ] \end{aligned}$$

Here  $G'$  is the graph with vertex set  $V(G)$  and edge set  $F$ , and  $V' = V(G) \setminus \{v\}$ .  $S$  is a system of distinct representatives for the vertex sets that constitute the elements of  $M$ .

**3. Kernelization for  $p$ - $\mathcal{F}$ -Deletion on  $K_{1,t}$  free Graphs.** In this section we show that if the obstruction set  $\mathcal{F}$  contains a planar graph then the  $p$ - $\mathcal{F}$ -DELETION problem has a linear vertex kernel on graphs excluding  $K_{1,t}$  as an induced subgraph. We start with the following lemma, which is crucial to our kernelization algorithms.

**LEMMA 3.1.** *Let  $\mathcal{F}$  be an obstruction set containing a planar graph of size  $h$ . If  $G$  has an  $\mathcal{F}$ -hitting set  $S$  of size at most  $k$ , then  $\text{tw}(G \setminus S) \leq d$  and  $\text{tw}(G) \leq k + d$ , where  $d = 20^{2(14h-24)^5}$ .*

*Proof.* By assumption,  $\mathcal{F}$  contains at least one planar graph. Let  $h$  be the size of the smallest planar graph  $H$  contained in  $\mathcal{F}$ . By a result of Robertson et al. [60],  $H$  is a minor of the  $(\ell \times \ell)$ -grid, where  $\ell = 14h - 24$ . In the same paper Robertson et al. [60] have shown that any graph with treewidth greater than  $20^{2\ell^5}$  contains a  $(\ell \times \ell)$ -grid as a minor. Let  $S$  be a  $\mathcal{F}$ -hitting set of  $G$  of size at most  $k$ . Since the  $(\ell \times \ell)$ -grid contains  $H$  as a minor, we have that  $\text{tw}(G \setminus S) \leq 20^{2\ell^5}$ . Therefore,  $\text{tw}(G) \leq k + d$ , where  $d = 20^{2\ell^5}$  — indeed, a tree decomposition of width  $(k + d)$  can be obtained by adding the vertices of  $S$  to every bag in an optimal tree decomposition of  $G \setminus S$ . This completes the proof of the lemma.  $\square$

In a series of recent developments, Chekuri and Chuzhoy [21] and Chuzhoy [25] have demonstrated polynomial bounds on the treewidth of graphs that exclude a planar graph as a minor. In particular, they show that there is a universal constant, say  $b$ , such that if  $G$  excludes a planar graph  $H$  as a minor, then the treewidth of  $G$  is  $O(|V(H)|^b)$ . The most recent work [25] achieves a bound of  $O(g^{36}(\log g)^{O(1)})$  on the treewidth of graphs that contain a  $g \times g$  grid minor. This implies that the bound in Lemma 3.1 can in fact be improved to  $\text{tw}(G) \leq k + d$ , where  $d = O(h^b)$ .

### 3.1. The Protrusion Rule — Reductions Based on Finite Integer Index.

We obtain our kernelization algorithm for  $p$ - $\mathcal{F}$ -DELETION by applying a protrusion based reduction rule. That is, any large  $r$ -protrusion for a fixed constant  $r$  that depends only on  $\mathcal{F}$  (that is, only on the problem) is replaced with a smaller equivalent  $r$ -protrusion. For this, we utilize the following lemma of Bodlaender et al. [15].

LEMMA 3.2 ([15]). *Let  $\Pi$  be a parameterized problem that has finite integer index. Then there exists a computable function  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  and an algorithm that given an instance  $(G, k)$  and an  $r$ -protrusion  $X$  of  $G$  of size at least  $\gamma(r)$ , runs in  $O(|X|)$  time and outputs an instance  $(G^*, k^*)$  such that  $|V(G^*)| < |V(G)|$ ,  $k^* \leq k$ , and  $(G^*, k^*) \in \Pi$  if and only if  $(G, k) \in \Pi$ .*

REMARK 1. *Let us remark that if  $G$  does not have  $K_{1,t}$  as an induced subgraph then the proof of Lemma 3.2 also ensures that the graph  $G'$  does not contain  $K_{1,t}$  as an induced subgraph. This ensures that the reduced instance belongs to the same graph class as the original. The remark is not only true about the class of graphs excluding  $K_{1,t}$  as an induced subgraph, but also for any graph class  $\mathcal{G}$  that can be characterized by either finite set of forbidden subgraphs or induced subgraphs or minors. That is, if  $G$  is in  $\mathcal{G}$  then the graph  $G'$  returned by Lemma 3.2 is also in  $\mathcal{G}$ .*

In order to apply Lemma 3.2 we need to be able to efficiently find large  $r$ -protrusions whenever the instance considered is large enough. Also, we need to prove that  $p$ - $\mathcal{F}$ -DELETION has finite integer index. The next lemma yields a divide and conquer algorithm for efficiently finding large  $r$ -protrusions.

LEMMA 3.3. *There is a linear time algorithm that given an  $n$ -vertex graph  $G$  and a set  $X \subseteq V(G)$  such that  $\mathbf{tw}(G \setminus X) \leq d$ , outputs a  $2(d+1)$ -protrusion of  $G$  of size at least  $\frac{n-|X|}{4|N(X)|+1}$ . Here  $d$  is some constant.*

*Proof.* Let  $F = G \setminus X$ . The algorithm starts by computing a nice tree decomposition of  $F$  of width at most  $d$ . Notice that since  $d$  is a constant this can be done in linear time [10]. Let  $S$  be the vertices in  $V(F)$  that are neighbors of  $X$  in  $G$ , that is,  $S = N_G(X)$ .

The nice tree decomposition of  $F$  is a pair  $(T, \mathcal{B} = \{B_\ell\}_{\ell \in V(T)})$ , where  $T$  is a rooted binary tree. We will now *mark* some of the nodes of  $T$ . For every  $v \in S$ , we mark the topmost node  $\ell$  in  $T$  such that  $v \in B_\ell$ . In this manner, at most  $|S|$  nodes are marked. Now we mark more nodes of  $T$  by exhaustively applying the following rule: if  $u$  and  $v$  are marked, mark their least common ancestor in  $T$ . Let  $M$  be the set of all marked nodes of  $T$ . Standard counting arguments on trees give that  $|M| \leq 2|S|$ .

Since  $T$  is a binary tree, it follows that  $T \setminus M$  has at most  $2|M| + 1$  connected components. Let the vertex sets of these connected components be  $C_1, C_2, \dots, C_\eta$ ,  $\eta \leq 2|M| + 1$ . For every  $i \leq \eta$ , let  $C'_i = N_T(C_i) \cup C_i$  and let  $P_i = \bigcup_{u \in C'_i} B_u$ . By the construction of  $M$ , every component of  $T \setminus M$  has at most 2 neighbors in  $M$ . Also for every  $1 \leq i \leq \eta$  and  $v \in S$ , we have that if  $v \in P_i$ , then  $v$  should be contained in one of the bags of  $N_T(C_i)$ . In other words,  $S \cap P_i \subseteq \bigcup_{u \in C'_i \setminus C_i} B_u$ . Thus every  $P_i$  is a  $2(d+1)$ -protrusion of  $G$  with boundary  $\bigcup_{u \in C'_i \setminus C_i} B_u$ . Since  $\eta \leq 2|M| + 1 \leq 4|S| + 1$ , the pigeon-hole principle yields that there is a protrusion  $P_i$  with at least  $\frac{n-|X|}{4|S|+1}$  vertices. The algorithm constructs  $M$  and  $P_1, \dots, P_\eta$  and outputs the largest protrusion  $P_i$ . It is easy to implement this procedure to run in linear time. This concludes the proof.  $\square$

Now we show that  $p$ - $\mathcal{F}$ -DELETION has finite integer index. For this we need the following lemma.

LEMMA 3.4 ([15]). *Every strongly monotone  $p$ -MIN-MSO problem has finite integer index.*

LEMMA 3.5.  *$p$ - $\mathcal{F}$ -DELETION has finite integer index.*

*Proof.* One can easily formulate  $p$ - $\mathcal{F}$ -DELETION in MSO, which shows that it is a  $p$ -MIN-MSO problem (see Section 2.6). To complete the proof that  $p$ - $\mathcal{F}$ -DELETION has finite integer index we show that  $\Pi = p$ - $\mathcal{F}$ -DELETION is strongly monotone. Given

a  $t$ -boundaried graph  $G$ , with  $\partial(G)$  as its boundary, let  $S'' \subseteq V(G)$  be a minimum set of vertices in  $G$  such that  $G \setminus S''$  does not contain any graph in  $\mathcal{F}$  as a minor. Let  $S = S'' \cup \partial(G)$ .

Now for any  $(G', S') \in \mathcal{H}_t$  such that  $\zeta_G^\Pi((G', S'))$  is finite, we have that  $G \oplus G'[(V(G) \cup V(G')) \setminus (S \cup S')]$  does not contain any graph in  $\mathcal{F}$  as a minor and  $|S| \leq \zeta_G^\Pi((G', S')) + t$ . This proves that  $p$ - $\mathcal{F}$ -DELETION is strongly monotone. By Lemma 3.4,  $p$ - $\mathcal{F}$ -DELETION has finite integer index.  $\square$

**3.2. Analysis and Kernel Size – Proof of Theorem 1.1.** Now we give the desired kernel for  $p$ - $\mathcal{F}$ -DELETION. We first prove an useful combinatorial lemma.

**LEMMA 3.6.** *Let  $G$  be a graph excluding  $K_{1,t}$  as an induced subgraph and  $S$  be an  $\mathcal{F}$ -hitting set. If  $\mathcal{F}$  contains a planar graph of size  $h$ , then  $|N(S)| \leq g(h, t) \cdot |S|$  for some computable function  $g$  of  $h$  and  $t$ .*

*Proof.* By Lemma 3.1,  $\mathbf{tw}(G \setminus S) \leq d$  for  $d = 20^{2(14h-24)^5}$ . It is well known that a graph of treewidth  $d$  is  $d + 1$  colorable (see, for instance, [41]). Let  $v \in S$  and let  $S_v$  be its neighbors in  $G \setminus S$ . We first show that  $|S_v| \leq (t - 1)(d + 1)$ . Consider the graph  $G^* = G[S_v]$ . Since  $\mathbf{tw}(G \setminus S) \leq d$  we have that  $\mathbf{tw}(G^*) \leq d$  and hence  $G^*$  is  $d + 1$  colorable. Fix a coloring  $\kappa$  of  $G^*$  with  $d + 1$  colors and let  $\eta$  be the size of the largest color class. Clearly  $\eta \geq (|S_v|/d + 1)$ . Since each color class is an independent set, we have that  $\eta \leq (t - 1)$ , else we will get  $K_{1,t}$  as an induced subgraph in  $G$ . This implies that  $|S_v| \leq (t - 1)(d + 1)$ . Since  $v$  was an arbitrary vertex of  $S$ , we have that  $\sum_{v \in S} |S_v| \leq \sum_{v \in S} (t - 1)(d + 1) = |S| \cdot g(h, t)$ . Here  $g(h, t) = (t - 1)(20^{2(14h-24)^5} + 1)$ . Finally the observation that  $N(S) = \cup_{v \in S} S_v$ , yields the result.  $\square$

Now we are ready to prove Theorem 1.1.

*Proof.* [Proof of Theorem 1.1] Let  $(G, k)$  be an instance of  $p$ - $\mathcal{F}$ -DELETION and  $h$  be the size of a smallest planar graph in the obstruction set  $\mathcal{F}$ . We first apply Theorem 1.3 (to be proved in next section), an approximation algorithm for  $p$ - $\mathcal{F}$ -DELETION with factor  $O(\log^{3/2} OPT)$ , and obtain a set  $X \subseteq V(G)$  such that  $G \setminus X$  contains no graph in  $\mathcal{F}$  as a minor. If the size of the set  $X$  is more than  $O(k \log^{3/2} k)$  then we return that  $(G, k)$  is a NO-instance to  $p$ - $\mathcal{F}$ -DELETION. This is justified by the approximation guarantee provided by Theorem 1.3.

Let  $d$  denote the treewidth of the graph after the removal of  $X$ , that is,  $d := \mathbf{tw}(G \setminus X)$ . Now we obtain the kernel in two phases: we first apply the protrusion rule selectively (Lemma 3.2) and get a polynomial kernel. Then, we apply the protrusion rule exhaustively on the obtained kernel to get a smaller kernel. This is done in order to reduce the running time complexity of the kernelization algorithm. To obtain the kernel we follow the following steps.

*Applying the Protrusion Rule.* By Lemma 3.1,  $d \leq 20^{2(14h-24)^5}$ . We apply Lemma 3.3 and obtain a  $2(d + 1)$ -protrusion  $Y$  of  $G$  of size at least  $\frac{|V(G')| - |X|}{4|N(X)| + 1}$ . By Lemma 3.5,  $p$ - $\mathcal{F}$ -DELETION has finite integer index. Let  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  be the function defined in Lemma 3.2. If  $\frac{|V(G')| - |X|}{4|N(X)| + 1} \geq \gamma(2d + 2)$ , then using Lemma 3.2 we replace the  $2(d + 1)$ -protrusion  $Y$  in  $G$  and obtain an instance  $(G^*, k^*)$  such that  $|V(G^*)| < |V(G)|$ ,  $k^* \leq k$ , and  $(G^*, k^*)$  is a YES-instance of  $p$ - $\mathcal{F}$ -DELETION if and only if  $(G, k)$  is a YES-instance of  $p$ - $\mathcal{F}$ -DELETION. Recall that  $G^*$  also excludes  $K_{1,t}$  as an induced subgraph.

Let  $(G^*, k^*)$  be a reduced instance with hitting set  $X^*$  of size at most  $O(k^* \log^{3/2} k^*)$ . In other words, there is no  $(2d + 2)$ -protrusion of size  $\gamma(2d + 2)$  in  $G^* \setminus X^*$ , and Protrusion Rule no longer applies. We claim that the number of vertices in this graph is bounded by  $O(k \log^{3/2} k)$ . Indeed, since we cannot apply the Protrusion Rule, we

have that  $\frac{|V(G^*)| - |X^*|}{4|N(X^*)| + 1} \leq \gamma(2d + 2)$ , which can be rewritten as follows:

$$|V(G^*)| \leq \gamma(2d + 2)(4|N(X^*)| + 1) + |X^*|. \quad (3.1)$$

By Lemma 3.6,  $|N(X^*)| \leq g(h, d) \cdot |X^*|$ . Since  $k^* \leq k$ , we have that

$$|V(G^*)| = O(\gamma(2d + 2) \cdot k \log^{3/2} k) = O(k \log^{3/2} k). \quad (3.2)$$

This gives us a polynomial time algorithm that returns a vertex kernel of size  $O(k \log^{3/2} k)$ .

Now we give a kernel of smaller size. We would like to replace every large  $(2d + 2)$ -protrusion in graph by a smaller one. We find a  $(2d + 2)$ -protrusion  $Y$  of size at least  $\gamma(2d + 2)$  by guessing the boundary  $\partial(Y)$  of size at most  $2d + 2$ . This could be performed in time  $|V(G^*)|^{O(d)} = k^{O(d)}$ . Let  $(G^*, k^*)$  be the reduced instance on which we cannot apply the Protrusion Rule. If  $(G, k)$  is a YES-instance, then  $G^*$  admits a  $\mathcal{F}$ -hitting set  $S$  of size at most  $k^*$ . Note that  $\mathbf{tw}(G \setminus S) \leq d$ , and applying Equation (3.1) with  $S$  yields that  $|V(G^*)| = O(k)$ . Recall that this analysis applies when we begin with a YES-instance. Therefore, if the number of vertices in the reduced instance  $G^*$  exceeds the bound derived in Equation (3.2), then we return that  $G$  is a NO-instance. This concludes the proof of the theorem.  $\square$

**COROLLARY 3.7.**  *$p$ -FEEDBACK VERTEX SET,  $p$ -DIAMOND HITTING SET,  $p$ -PATHWIDTH ONE DELETION SET,  $p$ -OUTERPLANAR DELETION SET admit linear vertex kernel on graphs excluding  $K_{1,t}$  as an induced subgraph.*

The methodology used in proving Theorem 1.1 is not limited to  $p$ - $\mathcal{F}$ -DELETION. For example, it is possible to obtain an  $O(k \log k)$  vertex kernel on  $K_{1,t}$ -free graphs for  $p$ -DISJOINT CYCLE PACKING, which is for a given graph  $G$  and positive integer  $k$  to determine if there are  $k$  vertex disjoint cycles in  $G$ . It is interesting to note that  $p$ -DISJOINT CYCLE PACKING does not admit a polynomial kernel on general graphs [17]. For our kernelization algorithm, we use the following Erdős-Pósa property [36]: given a positive integer  $\ell$  every graph  $G$  either has  $\ell$  vertex disjoint cycles or there exists a set  $S \subseteq V(G)$  of size at most  $O(\ell \log \ell)$  such that  $G \setminus S$  is a forest. So given a graph  $G$  and positive integer  $k$ , we first apply the factor 2 approximation algorithm given in [7] and obtain a set  $S$  such that  $G \setminus S$  is a forest. If the size of  $S$  is more than  $O(k \log k)$  then we return that  $G$  has  $k$  vertex disjoint cycles. Else, we use the fact that  $p$ -DISJOINT CYCLE PACKING [15] has finite integer index and apply the protrusion reduction rule in  $G \setminus S$  to obtain an equivalent instance  $(G^*, k^*)$ , as in Theorem 1.1. The analysis for kernel size used in the proof of Theorem 1.1 together with the observation that  $\mathbf{tw}(G \setminus S) \leq 1$  shows that if  $(G, k)$  is a yes instance then the size of  $V(G^*)$  is at most  $O(k \log k)$ .

**COROLLARY 3.8.**  *$p$ -DISJOINT CYCLE PACKING has  $O(k \log k)$  vertex kernel on graphs excluding  $K_{1,t}$  as an induced graph.*

Next, we extend the methods used in this section for obtaining kernels for  $p$ - $\mathcal{F}$ -DELETION on graphs excluding  $K_{1,t}$  as an induced graph to all graphs, though for restricted  $\mathcal{F}$  — we consider the families  $\mathcal{F}$  that contain  $\theta_c$ .

**4. An Approximation Algorithm for Finding a  $\mathcal{F}$ -Hitting Set.** To extend our results to all graphs, we need a polynomial time approximation algorithm with a factor polynomial in optimum size and not depending on the input size. For example, an approximation algorithm with factor  $O(\log n)$  would not serve our purpose. Here, we obtain an approximation algorithm for  $p$ - $\mathcal{F}$ -DELETION with a factor  $O(\log^{3/2} OPT)$  whenever the finite obstruction set  $\mathcal{F}$  contains a planar graph. Here

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**Algorithm 1** HIT-SET-I-( $G$ )

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- 1: **if**  $\text{tw}(G) \leq d$  **then**
  - 2:   Find a minimum  $\mathcal{F}$ -hitting set  $Y$  of  $G$  and return  $Y$ .
  - 3: **end if**
  - 4: Compute an approximate tree decomposition  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  of width  $\ell$ .
  - 5: **if**  $\ell > (k + d)d' \sqrt{\log(k + d)}$ , where  $d$  is as in Lemma 3.1 **then**
  - 6:   Return that  $G$  does not have  $\mathcal{F}$ -hitting set of size at most  $k$ .
  - 7: **end if**
  - 8: Convert  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  to a nice tree decomposition of the same width.
  - 9: Find a partitioning of vertex set  $V(G)$  into  $V_1, V_2$  and  $X$  (a bag corresponding to a node in  $T$ ) such that  $\text{tw}(G[V_1]) = d$  as described in the proof.
  - 10: Return  $(X \cup \text{HIT-SET-I-}(G[V_1]) \cup \text{HIT-SET-I-}(G[V_2]))$ .
- 

$OPT$  is the size of a minimum  $\mathcal{F}$ -hitting set. This immediately implies a factor  $O(\log^{3/2} n)$  algorithm for all the problems that can be categorized by  $p$ - $\mathcal{F}$ -DELETION when  $\mathcal{F}$  contains a planar graph. We believe this result has its own significance and is of independent interest.

LEMMA 4.1. *There is a polynomial time algorithm that, given a graph  $G$  and a positive integer  $k$ , either reports that  $G$  has no  $\mathcal{F}$ -hitting set of size at most  $k$  or finds an  $\mathcal{F}$ -hitting set of size at most  $O(k \log^{3/2} k)$ .*

*Proof.* We begin by introducing some definitions that will be useful for describing our algorithms. First is the notion of a *good labeling function*. Given a nice tree decomposition  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  of a graph  $G$ , a function  $g : V(T) \rightarrow \mathbb{N}$  is called a *good labeling function* if it satisfies the following properties:

- if  $t$  is a base node then  $g(t) = 0$ ;
- if  $t$  is an introduce node, then  $g(t) = g(s)$ , where  $s$  is the child of  $t$ ;
- if  $t$  is a join node, then  $g(t) = g(s_1) + g(s_2)$ , where  $s_1$  and  $s_2$  are the children of  $t$ ; and
- if  $t$  is a forget node, then  $g(t) \in \{g(s), g(s) + 1\}$ , where  $s$  is the child of  $t$ .

A *max labeling function*  $g$  is defined analogously to a good labeling function, the only difference being that for a join node  $t$ , we have the condition  $g(t) = \max\{g(s_1), g(s_2)\}$ . We now turn to the approximation algorithm.

Our algorithm has two phases. In the first phase we obtain an  $\mathcal{F}$ -hitting set of size  $O(k^2 \sqrt{\log k})$  and in the second phase we use the hitting set obtained in the first phase to get an  $\mathcal{F}$ -hitting set of size  $O(k \log^{3/2} k)$ . The second phase could be thought of as “bootstrapping” where one uses an initial solution to a problem to obtain a better solution.

By assumption we know that  $\mathcal{F}$  contains at least one planar graph. Let  $h$  be the number of vertices in the smallest planar graph  $H$  contained in  $\mathcal{F}$ . By a result of Robertson et al. [60],  $H$  is a minor of the  $(\rho \times \rho)$ -grid, where  $\rho = 14h - 24$ . Robertson et al. [60] have also shown that any graph with treewidth greater than  $20^{2\rho^5}$  contains a  $\rho \times \rho$  grid as a minor. In the algorithm we set  $d = 20^{2\rho^5} + 1$ .

We begin by describing the first phase of the algorithm, see **Algorithm 1**. We start by checking whether a graph  $G$  has treewidth at most  $d$  (the first step of the algorithm) using the linear time algorithm of Bodlaender [10]. If  $\text{tw}(G) \leq d$  then we find an optimum  $\mathcal{F}$ -hitting set of  $G$  in linear time using a modification of Lemma 5.4. If the treewidth of the input graph is more than  $d$  then we find an approximate tree

decomposition of width  $\ell$  using an algorithm of Feige et al. [37] such that  $\mathbf{tw}(G) \leq \ell \leq d' \mathbf{tw}(G) \sqrt{\log \mathbf{tw}(G)}$ , where  $d'$  is a fixed constant.

If  $\ell > (k+d)d' \sqrt{\log(k+d)}$ , then by Lemma 3.1, we know that the size of a minimum  $\mathcal{F}$ -hitting set of  $G$  is at least  $(k+1)$ . Hence from now onwards we assume that  $\mathbf{tw}(G) \leq \ell \leq (k+d)d' \sqrt{\log(k+d)}$ . In the next step we convert the given tree decomposition to a nice tree decomposition of the same width in linear time [49]. Given a nice tree decomposition  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  of  $G$ , we compute a *partial* function  $\beta : V(T) \rightarrow \mathbb{N}$ , defined as  $\beta(t) = \mathbf{tw}(H_t)$  (recall that  $H_t$  refers to  $G_t[V(G_t) \setminus X_t]$ ). Observe that  $\beta$  is a max labeling function. We compute  $\beta$  in a bottom up fashion starting from base nodes and moving towards the root. We stop this computation the first time that we find a node  $t$  such that  $\beta(t) = \mathbf{tw}(H_t) = d$ . Let  $V_1 = V(H_t)$ ,  $V_2 = (V(G) \setminus V_1) \setminus X_t$  and  $X = X_t$ . After this we recursively solve the problem on the graphs induced on  $V_1$  and  $V_2$ .

Let us assume that  $G$  has an  $\mathcal{F}$ -hitting set of size at most  $k$ . We show that in this case the size of the hitting set returned by the algorithm can be bounded by  $O(k^2 \sqrt{\log k})$ . The above recursive procedure can be thought of as a rooted binary tree  $\mathcal{T}$  where at each non-leaf node of the tree the algorithm makes two recursive calls. We will assume that the left child of a node of  $\mathcal{T}$  corresponds to the graph induced on  $V_1$  such that the treewidth of  $G[V_1]$  is  $d$ . Assuming that the root is at depth 0 we show that the depth of  $\mathcal{T}$  is bounded by  $k$ . Let  $P = a_0 a_1 \cdots a_q$  be a longest path from the root to a leaf and let  $G_i$  be the graph associated with the node  $a_i$ . Observe that for every  $i \in \{0, \dots, q-1\}$ ,  $a_i$  has a left child, or else  $a_i$  cannot be a non-leaf node of  $\mathcal{T}$ . Let the graph associated with the left child of  $a_i$ ,  $i \in \{0, \dots, q-1\}$ , be denoted by  $H_i$ . Observe that for every  $0 \leq i < j \leq q-1$ ,  $V(H_i) \cap V(H_j) = \emptyset$  and  $\mathbf{tw}(H_i) = d$ . This implies that every  $H_i$  has at least one  $H$  minor model and all of these are vertex-disjoint. This implies that  $q \leq k$  and hence the depth of  $\mathcal{T}$  is bounded by  $k$ .

Let us look at all the subproblems at depth  $i$  in the recursion tree  $\mathcal{T}$ . Suppose at depth  $i$  the induced subgraphs associated with these subproblems are  $G[V_j]$ ,  $j \in [\tau]$ , where  $\tau$  is some positive integer. Then observe that for every  $j_1, j_2 \in [\tau]$  and  $j_1 \neq j_2$ , we have that  $V_{j_1} \cap V_{j_2} = \emptyset$ , there is no edge  $(u, v)$  such that  $u \in V_{j_1}$ ,  $v \in V_{j_2}$ , and hence  $\sum_{j=1}^{\tau} k_j \leq k$ , where  $k_j$  is the size of the minimum  $\mathcal{F}$ -hitting set of  $G[V_j]$ . Furthermore the number of instances at depth  $i$  such that it has at least one  $H$  minor model and hence contributes to the hitting set is at most  $k$ . Now Lemma 3.1 together with the factor  $d' \sqrt{\log \mathbf{tw}(G)}$  approximation algorithm of Feige et al. [37] implies that the treewidth of every instance is upper bounded by  $(k_j + d)d' \sqrt{\log(k_j + d)}$ , where  $k_j$  is the size of the minimum  $\mathcal{F}$ -hitting set of  $G[V_j]$ . Hence the total size of the union of sets added to our hitting set at depth  $i$  is at most

$$\sum_{j=1}^{\tau} \chi(j) (k_j + d) d' \sqrt{\log(k_j + d)} \leq d' (k + d) \sqrt{\log(k + d)}.$$

Here  $\chi(j)$  is 1 if  $G[V_j]$  contains at least one  $H$  minor model and is 0 otherwise. We have shown that for each  $i$  the size of the union of the sets added to the hitting set is at most  $d' (k + d) \sqrt{\log(k + d)}$ . This together with the fact that the depth is at most  $k$  implies that the size of the  $\mathcal{F}$ -hitting set is at most  $O(k^2 \sqrt{\log k})$ . Hence if the size of the hitting set returned by the algorithm is more than  $d' (k + d) k \sqrt{\log(k + d)}$  then we return that  $G$  has at no  $\mathcal{F}$ -hitting set of size at most  $k$ . Hence when we move to the second phase we assume that we have a hitting set of size  $O(k^2 \sqrt{\log k})$ . This concludes the description of the first phase of the algorithm.

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**Algorithm 2** HIT-SET-II- $(G, Z)$ 

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- 1: **if**  $\text{tw}(G) \leq d$  **then**
  - 2:   Find a minimum  $\mathcal{F}$ -hitting set  $Y$  of  $G$  and Return  $Y$ .
  - 3: **end if**
  - 4: Compute an approximate tree decomposition  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  of width  $\ell$ .
  - 5: Convert it to a nice tree decomposition of  $G$ . Now compute the function  $\mu : V(T) \rightarrow \mathbb{N}$ , defined as follows:  $\mu(t) = |V(H_t) \cap Z|$ .
  - 6: **if**  $(\mu(r) = 0)$  **then**
  - 7:   Return  $\phi$ .
  - 8: **else**
  - 9:   Find the partitioning of the vertex set  $V(G)$  into  $V_1, V_2$  and  $X$  (a bag corresponding to a node in  $T$ ) as described in Cases 1 and 2 of the proof of Lemma 4.1.
  - 10: **end if**
  - 11: Return  $(X \cup \text{HIT-SET-II-}(G[V_1], Z \cap V_1) \cup \text{HIT-SET-II-}(G[V_2], Z \cap V_2))$ .
- 

Now we describe the second phase of the algorithm. Here we are given the hitting set  $Z$  of size  $O(k^2 \sqrt{\log k})$  obtained from the first phase of the algorithm. The algorithm is depicted in **Algorithm 2**. The new algorithm essentially uses  $Z$  to define a good labeling function  $\mu$  which enables us to argue that the depth of recursion is upper bounded by  $O(\log |Z|)$ . In particular, consider the function  $\mu : V(T) \rightarrow \mathbb{N}$ , defined as follows:  $\mu(t) = |V(H_t) \cap Z|$ . Let  $k' := \mu(r)$ , where  $r$  is the node corresponding to the root of a fixed nice tree decomposition of  $G$ .

Let  $t \in V(T)$  be the node where  $\mu(t) > 2k'/3$  and for each child  $t'$  of  $t$ ,  $\mu(t') \leq 2k'/3$ . Since  $\mu$  is a good labeling function, it is easy to see that this node exists and is unique provided that  $k' > 0$ . Moreover, observe that  $t$  could either be a forget node or a join node. We distinguish these two cases.

- *Case 1.* If  $t$  is a forget node, we set  $V_1 = V(H_{t'})$  and  $V_2 = V(G) \setminus (V_1 \cup X_{t'})$  and observe that  $|V(G_i) \cap Z| \leq \lfloor 2k'/3 \rfloor, i = 1, 2$ . Also we set  $X = X_{t'}$ .
- *Case 2.* If  $t$  is a join node with children  $t_1$  and  $t_2$ , we have that  $\mu(t_i) \leq 2k'/3, i = 1, 2$ . However, as  $\mu(t_1) + \mu(t_2) > 2k'/3$ , we also have that either  $\mu(t_1) \geq k'/3$  or  $\mu(t_2) \geq k'/3$ . Without loss of generality we assume that  $\mu(t_1) \geq k'/3$  and we set  $V_1 = V(H_{t_1}), V_2 = V(G) \setminus (V_1 \cup X_{t_1})$  and  $X = X_{t_1}$ .

Now we argue that if  $G$  has an  $\mathcal{F}$ -hitting set of size at most  $k$  then then the size of the hitting set returned by the algorithm is upper bounded by  $O(k \log^{3/2} k)$ . As in the first phase we can argue that the size of the union of the sets added to the hitting set in the subproblems at depth  $i$  is at most  $d'(k+d)\sqrt{\log(k+d)}$ . Observe that the recursive procedure in Algorithm 2 is such that the value of the function  $\mu()$  drops by at least a constant fraction at every level of recursion. This implies that the depth of recursion is upper bounded by  $O(\log |Z|) = O(\log k)$ . Hence the size of the hitting set returned by the algorithm is upper bounded by  $O(k \log^{3/2} k)$  whenever  $G$  has an  $\mathcal{F}$ -hitting set of size at most  $k$ . Thus if the size of the hitting set returned by HIT-SET-II- $(G, Z)$  is more than  $d'(k+d)\sqrt{\log^{3/2}(k+d)}$ , we return that  $G$  does not have an  $\mathcal{F}$ -hitting set of size at most  $k$ . This concludes the proof.  $\square$

*Proof.* [Proof of Theorem 1.3] Given a graph  $G$  on  $n$  vertices, let  $k$  be the minimum positive integer in  $\{1, \dots, n\}$  such that Lemma 4.1 returns an  $\mathcal{F}$ -hitting set  $S$  when applied on  $(G, k)$ . We return this  $S$  as an approximate solution. By our choice of  $k$  we

know that  $G$  does not have  $\mathcal{F}$ -hitting set of size at most  $k - 1$  and hence  $OPT \geq k$ . This implies that the size of  $S$  returned by Lemma 4.1 is at most  $O(k \log^{3/2} k) = O(OPT \log^{3/2} OPT)$ . This concludes the proof.  $\square$

We now define a generic problem. Let  $\eta$  be a fixed constant. In the TREEWIDTH  $\eta$ -DELETION SET problem, we are given an input graph  $G$  and the objective is to delete minimum number of vertices from a graph such that the resulting graph has treewidth at most  $\eta$ . For an example TREEWIDTH 1-DELETION SET is simply the FEEDBACK VERTEX SET problem. We obtain the following corollary of Theorem 1.3.

**COROLLARY 4.2.** FEEDBACK VERTEX SET, DIAMOND HITTING SET, PATH-WIDTH ONE DELETION SET, OUTERPLANAR DELETION SET and TREEWIDTH  $\eta$ -DELETION SET admit a factor  $O(\log^{3/2} n)$  approximation algorithm on general undirected graphs.

**5. Kernelization for  $p$ - $\theta_c$ -Deletion.** In this section we obtain a polynomial kernel for  $p$ - $\theta_c$ -DELETION on general graphs, where  $c$  is a fixed constant. To obtain our kernelization algorithm we not only need the approximation algorithm presented in the last section but also a variation of classical Hall's theorem. We first present this combinatorial tool and other auxiliary results that we make use of.

**5.1. Combinatorial Lemma and Some Linear-Time Subroutines.** We need a variation of the celebrated Hall's Theorem, which we call the  $q$ -Expansion Lemma. The  $q$ -Expansion Lemma is a generalization of Hall's Theorem, and captures a certain property of neighborhood sets in graphs that has been used by several authors to obtain polynomial kernels for many graph problems. For  $q = 1$  this lemma is Hall's theorem and its usage is equivalent to the application of the Crown Decomposition technique [1, 23]. For  $q = 2$  it was known as double crown decomposition and was used by Prieto and Sloper for the first time to obtain a quadratic kernel for packing  $k$  disjoint copies of stars with  $s$  leaves [58]. Prieto [57] used  $q$ -Expansion Lemma in its most general form to obtain a quadratic kernel for finding a maximal matching with at most  $k$  edges. Prieto [57] referred to this technique as the  $q$ -spike crown decomposition.

*The Expansion Lemma.* Consider a bipartite graph  $G$  with vertex bipartition  $A \uplus B$ . Given subsets  $S \subseteq A$  and  $T \subseteq B$ , we say that  $S$  has  $|S|$   $q$ -stars in  $T$  if to every  $x \in S$  we can associate a subset  $F_x \subseteq N(x) \cap T$  such that (a) for all  $x \in S$ ,  $|F_x| = q$ ; (b) for any pair of vertices  $x, y \in S$ ,  $F_x \cap F_y = \emptyset$ . Observe that if  $S$  has  $|S|$   $q$ -stars in  $T$  then every vertex  $x$  in  $S$  could be thought of as the center of a star with its  $q$  leaves in  $T$ , with all these stars being vertex-disjoint. Further, a collection of  $|S|$   $q$ -stars is also a family of  $q$  edge-disjoint matchings, each saturating  $S$ . We use the following result in our kernelization algorithm to bound the degrees of vertices. The proof of next lemma is just a slight variation of the proof of  $q$ -spike crown decomposition given in [57] and is primarily given here for the sake of completeness. (See the remark following the lemma for an explanation of the differences.)

**LEMMA 5.1. [The  $q$ -Expansion Lemma]** *Let  $q$  be a positive integer, and let  $m$  be the size of the maximum matching in a bipartite graph  $G$  with vertex bipartition  $A \uplus B$ . If  $|B| > mq$ , and there are no isolated vertices in  $B$ , then there exist nonempty vertex sets  $S \subseteq A, T \subseteq B$  such that  $S$  has  $|S|$   $q$ -stars in  $T$  and no vertex in  $T$  has a neighbor outside  $S$ . Furthermore, the sets  $S, T$  can be found in time polynomial in the size of  $G$ .*

*Proof.* Consider the graph  $H$  obtained from  $G = (A \uplus B, E)$  by adding  $(q - 1)$  copies of all the vertices in  $A$ , and giving all copies of a vertex  $v$  the same neighborhood



in  $B$  as  $v$ . Formally, let  $\{u_1, u_2, \dots, u_p\}$  denote the vertices of  $A$ , and let  $A_1, \dots, A_q$  denote  $q$  vertex sets, with  $q$  vertices each:

$$A_i := \{u_1^{(i)}, \dots, u_p^{(i)}\}.$$

Further, we use  $X$  to denote  $A_1 \cup \dots \cup A_q$ . The graph  $H$  is the bipartite graph  $(X \uplus B, E^*)$ , where  $E^*$  is given by:

$$\bigcup_{1 \leq j \leq q} \{(u_i^{(j)}, v) \mid u_i^{(j)} \in A_j, v \in B \text{ such that } (u_i, v) \in E\}.$$

Let  $M$  be a maximum matching in  $H$ . For the rest of this discussion, vertices are saturated and unsaturated with respect to this fixed matching  $M$ .

Let  $U_X$  be the vertices in  $X$  that are unsaturated, and  $R_X$  be those that are reachable from  $U_X$  via non-trivial alternating paths, i.e, paths with at least two edges. We let  $S_A = X \setminus (U_X \cup R_X)$ . Let  $U_B$  be the set of unsaturated vertices in  $B$ , and let  $T$  denote the set of partners of  $S_A$  in the matching  $M$ , that is,  $T = \{x \in B \mid (u, x) \in M \text{ and } u \in S_A\}$  (see Figure 5.1).

Note that  $S_A$  is non-empty: since  $|B| > mq$ , the set  $U_B$  of unsaturated vertices of  $B$  in  $H$  is non-empty. Further, by the assumption that  $B$  admits no isolated vertices, the neighbors of  $U_B$  form a non-trivial subset of  $A$ . Now, notice that neighbors of  $U_B$  cannot lie in either  $U_X$  or  $R_X$  (in both cases we obtain augmenting paths, contradicting the fact that  $M$  is a maximum matching). Therefore, the neighbors of  $U_B$  must lie in  $S_A$ , and therefore  $S_A$  is non-empty.

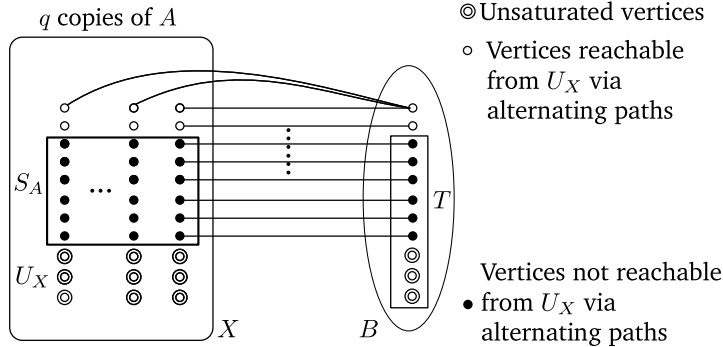


FIG. 5.1. The construction used in the proof of the  $q$ -Expansion Lemma

For every  $v \in A$ , let  $C(v)$  be the set of all copies of  $v$  (including  $v$ ). We claim that either  $C(v) \cap S_A = C(v)$ , or  $C(v) \cap S_A = \emptyset$ . Suppose that  $v \in S_A$  but a copy of  $v$ , say  $u$ , is in  $U_X$ . Let  $(v, w) \in M$ . Then  $v$  is reachable from  $u$  because  $(u, w) \in E(H)$ , contradicting the assumption that  $v \in S_A$ . In the case when  $v \in S_A$  but a copy of  $u$  is in  $R_X$ , let  $(w, u)$  be the last edge on some alternating path from  $U_X$  to  $u$ . Since  $(w, v) \in E(H)$ , we have that there is also an alternating path from  $U_X$  to  $v$ , contradicting the fact that  $v \in S_A$ . Now, let  $S = \{v \in A \mid C(v) \subseteq S_A\}$ . Then the subgraph  $G[S \cup T]$  contains  $q$  edge-disjoint matchings, each of which saturates  $S$  in  $G$  — this is because in  $H$ ,  $M$  saturates each copy of  $v \in S$  separately.

We now show that no vertex in  $T$  has a neighbor outside  $S$  in  $G$ . Notice that if no vertex in  $T$  has a neighbor outside  $S_A$  in  $H$ , then from the construction no vertex

in  $T$  has a neighbor outside  $S$  in  $G$ , thus it suffices to prove that no vertex in  $T$  has a neighbor outside  $S_A$  in  $H$ . For the purpose of contradiction, let us assume that for some  $v \in T$ ,  $u \in N(v)$ , but  $u \notin S_A$ . Suppose  $u \in R_X$ . We know that  $u \in R_X$  because there is some unsaturated vertex (say  $w$ ) that is connected by an alternating path to  $u$ . This path can be extended to a path to  $v$  using the edge  $(u, v)$ , and can be further extended to  $v'$ , where  $(v, v') \in M$ . However,  $v' \in S_A$ , and by construction, there is no path from  $w \in U_X$  to  $v'$ , a contradiction. If  $u \in U_X$ , then we arrive at a contradiction along the same lines (in fact, the paths from  $w$  to a vertex in  $S$  will be of length two in this case). This proves the claim that no vertex in  $T$  has a neighbor outside  $S_A$  in  $H$ . This concludes the proof.  $\square$

REMARK 2. Prieto [57] or Thomasse [61, Theorem 2.3] proved a slightly different version of Lemma 5.1. The statement in [57, 61] assumes that  $|B| \geq q|A|$  however Lemma 5.1 only assumes that  $|B| > qm$  where  $m$  is the size of maximum matching.

We will need the following proposition for the proof of next observation. Its proof follows from definitions.

PROPOSITION 5.2. For any  $c \in \mathbb{N}$ , a subgraph  $M$  of graph  $G$  is a minimal minor-model of  $\theta_c$  in  $G$  if and only if  $M$  consists of two trees, say  $T_1$  and  $T_2$ , and a set  $S$  of  $c$  edges, each of which has one end vertex in  $T_1$  and the other in  $T_2$ , and further, for each leaf vertex  $v$  of  $T_1$  and  $T_2$ , there is an edge in  $S$  incident on  $v$ .

OBSERVATION 1. For  $c \geq 2$ , any minimal  $\theta_c$  minor-model  $M$  of a graph  $G$  is a connected subgraph of  $G$ , and does not contain a vertex whose degree in  $M$  is less than 2, or a vertex whose deletion from  $M$  results in a disconnected graph (a cut vertex of  $M$ ).

*Proof.* From Proposition 5.2, whose terminology we use in this proof,  $M$  is connected and contains no isolated vertex. Suppose  $x$  is a vertex of degree exactly one in  $M$ . Then  $x$  is a leaf node in one of the two trees in  $M$ , say  $T_1$ , and no edge in  $S$  is incident on  $x$ . Removing  $x$  from  $T_1$  results in a smaller  $\theta_c$  minor-model, contradicting the minimality of  $M$ . It follows that every vertex of  $M$  has degree at least two.

Now suppose  $x$  is a cut vertex in  $M$  which belongs to, say, the tree  $T_1$ . Let  $T_1^1, T_1^2, \dots, T_1^l$  be the subtrees of  $T_1$  obtained when  $x$  is deleted from  $T_1$ . Let  $M'$  be the graph obtained by deleting  $x$  from  $M$ . If  $l > 0$ , then each  $T_1^i$  has a leaf node, which, by the above argument, has at least one neighbor in  $T_2$ . If  $l = 0$ , then  $M' = T_2$ . Thus  $M'$  is connected in all cases, and so  $x$  is not a cut vertex, a contradiction.  $\square$

The following well known result states that every optimization problem expressible in MSO has a linear time algorithm on graphs of bounded treewidth.

PROPOSITION 5.3 ([6, 10, 19, 27, 29]). Let  $\phi$  be a property that is expressible in Monadic Second Order Logic. For any fixed positive integer  $t$ , there is an algorithm that, given a graph  $G$  of treewidth at most  $t$  as input, finds a largest (alternatively, smallest) set  $S$  of vertices of  $G$  that satisfies  $\phi$  in time  $f(t, |\phi|)|V(G)|$ .

Proposition 5.3 together with MSO formulations ?? and ?? given in Section 2.6 implies the following lemma.

LEMMA 5.4. Let  $G$  be a graph on  $n$  vertices and  $v$  a vertex of  $G$ . Given a tree decomposition of width  $t = O(1)$  of  $G$ , we can, in  $O(n)$  time, find both (1) a smallest set  $S \subseteq V$  of vertices of  $G$  such that the graph  $G \setminus S$  does not contain  $\theta_c$  as a minor, and (2) a largest collection  $\{M_1, M_2, \dots, M_l\}$  of  $\theta_c$  minor models of  $G$  such that for  $1 \leq i < j \leq l$ ,  $(V(M_i) \cap V(M_j)) = \{v\}$ .

Now we describe the reduction rules used by the kernelization algorithm. In contrast to the reduction rules employed by most known kernelization algorithms, these rules cannot always be applied on general graphs in polynomial time. Hence

the algorithm does not proceed by applying these rules exhaustively, as is typical in kernelization programs. We describe how to arrive at situations where these rules can in fact be applied in polynomial time, and prove that even this selective application of rules results in a kernel of size polynomial in the parameter  $k$ .

**5.2. Bounding the Maximum Degree of a Graph.** Now we present a set of reduction rules which, given an input instance  $(G, k)$  of  $p$ - $\theta_c$ -DELETION, obtains an equivalent instance  $(G', k')$  where  $k' \leq k$  and the maximum degree of  $G'$  is at most a polynomial in  $k$ . In the sequel a vertex  $v$  is *irrelevant* if it is not a part of any  $\theta_c$  minor model, and is *relevant* otherwise. For each rule below, the input instance is  $(G, k)$ .

**REDUCTION RULE 1 (Irrelevant Vertex Rule).** *Delete all irrelevant vertices in  $G$ .*

Given a graph  $G$  and a vertex  $v \in V(G)$ , an  $\ell$ -flower passing through  $v$  is a set of  $\ell$  different  $\theta_c$  minor-models in  $G$ , each containing  $v$  and no two sharing any vertex other than  $v$ .

**REDUCTION RULE 2 (Flower Rule).** *If a  $(k + 1)$ -flower passes through a vertex  $v$  of  $G$ , then include  $v$  in the solution and remove it from  $G$  to obtain the equivalent instance  $(G \setminus \{v\}, (k - 1))$ .*

The argument for the soundness of these reduction rules is simple and is hence omitted. One can test whether a particular vertex  $v$  is part of any minimal minor-model corresponding to  $\theta_c$  using the rooted minor testing algorithm of Robertson and Seymour [59]. It is not clear, however, that one might check whether a vertex is a part of  $(k + 1)$ -flower in polynomial time. Hence we defer the application of these rules and apply them only when the vertices are “evidently” irrelevant or finding a flower can be solved in polynomial time. Now we state an auxiliary lemma which will be useful in bounding the maximum degree of the graph.

**LEMMA 5.5.** *Let  $G$  be an  $n$ -vertex graph containing  $\theta_c$  as a minor and  $v$  be a vertex such that  $G' = G \setminus \{v\}$  does not contain  $\theta_c$  as a minor and the maximum  $\ell$  such that  $G$  has an  $\ell$ -flower containing  $v$  is at most  $k$ . Then there exists a vertex subset  $T_v$  of size  $O(k)$  such that  $v \notin T_v$  and  $G \setminus T_v$  does not contain  $\theta_c$  as a minor. Moreover, we can find  $T_v$  in polynomial time.*

*Proof.* We first bound the treewidth of  $G'$ . Robertson, Seymour and Thomas [60] have shown that any graph with treewidth greater than  $20^{2c^5}$  contains a  $c \times c$  grid, and hence  $\theta_c$ , as a minor. This implies that for a fixed  $c$ ,  $\mathbf{tw}(G') \leq 20^{2c^5} = O(1)$ . Now we show the existence of a  $T_v$  of the desired kind. Recall the algorithm used to obtain the  $\mathcal{F}$ -hitting set for a graph described in **Algorithm 2**. We use the same algorithm to construct the desired  $T_v$ . Let  $F_{\theta_c}(G)$  denote the size of the maximum flower passing through  $v$  in  $G$ . Consider a nice tree decomposition  $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$  of  $G'$  of width at most  $\mathbf{tw}(G')$ . We define the function  $\mu(t) := F_{\theta_c}(G[V(H_t) \cup \{v\}])$ . It is easy to see that  $\mu$  is a good labeling function, and can be computed in polynomial time due to Lemma 5.4. Observe that  $\mu(r) \leq k$ , where  $r$  is the root node of the tree decomposition. Let  $\mathcal{S}(G', k)$  denote the size of the hitting set returned by the algorithm. Thus the size of the hitting set returned by the **Algorithm 2** is governed by the following recurrence:

$$\mathcal{S}(G', k) \leq \max_{1/3 \leq \alpha \leq 2/3} \left\{ \mathcal{S}(G[V_1], \alpha k) + \mathcal{S}(G[V_2], (1 - \alpha)k) + O(1) \right\}.$$

Using Akra-Bazzi [3] it follows that the above recurrence solves to  $O(k)$ . This implies that there exists a set  $T_v$  of size  $O(k)$  such that  $v \notin T_v$  and  $G \setminus T_v$  does not contain  $\theta_c$

as a minor. We now proceed to find an optimal hitting set in  $G$  avoiding  $v$ . To make **Algorithm 2** run in polynomial time we only need to find the tree decomposition and compute the function  $\mu()$  in polynomial time. Since  $\mathbf{tw}(G) = O(1)$ , we can find the desired tree decomposition of  $G$  or one of its subgraphs in linear time using the algorithm of Bodlaender [10]. Similarly we can compute a flower of the maximum size using Lemma 5.4 in linear time. Hence the function  $\mu()$  can also be computed in polynomial time. This concludes the proof of the lemma.  $\square$

*Flowers, Expansion and the Maximum Degree.* Now we are ready to prove the lemma which bounds the maximum degree of the instance.

**LEMMA 5.6.** *There exists a polynomial time algorithm that, given an instance  $(G, k)$  of  $p$ - $\theta_c$ -DELETION returns an equivalent instance  $(G', k')$  such that  $k' \leq k$  and that the maximum degree of  $G'$  is  $O(k \log^{3/2} k)$ . Moreover it also returns a  $\theta_c$ -hitting set of  $G'$  of size  $O(k \log^{3/2} k)$ .*

*Proof.* Given an instance  $(G, k)$  of  $p$ - $\theta_c$ -DELETION, we first apply Lemma 4.1 on  $(G, k)$ . The polynomial time algorithm described in Lemma 4.1, given a graph  $G$  and a positive integer  $k$  either reports that  $G$  has no  $\theta_c$ -hitting set of size at most  $k$ , or finds a  $\theta_c$ -hitting set of size at most  $k^* = O(k \log^{3/2} k)$ . If the algorithm reports that  $G$  has no  $\theta_c$ -hitting set of size at most  $k$ , then we return that  $(G, k)$  is a NO-instance to  $p$ - $\theta_c$ -DELETION. So we assume that we have a hitting set  $\mathcal{S}$  of size  $k^*$ . Now we proceed with the following two rules.

(1) *Selective Flower Rule.* To apply the Flower Rule selectively we use  $\mathcal{S}$ , the  $\theta_c$ -hitting set. For a vertex  $v \in \mathcal{S}$  let  $\mathcal{S}_v := \mathcal{S} \setminus \{v\}$  and let  $G_v := G \setminus \mathcal{S}_v$ . By a result of Robertson et. al. [60] we know that any graph of treewidth greater than  $20^{2c^5}$  contains a  $c \times c$  grid, and hence  $\theta_c$ , as a minor. Since deleting  $v$  from  $G_v$  makes it  $\theta_c$ -minor-free,  $\mathbf{tw}(G_v) \leq 20^{2c^5} + 1 = O(1)$ . Now by Lemma 5.4, we find in linear time the size of the largest flower centered at  $v$ , in  $G_v$ . If for any vertex  $v \in \mathcal{S}$  the size of the flower in  $G_v$  is at least  $k + 1$ , we apply the Flower Rule and get an equivalent instance  $(G \leftarrow G \setminus \{v\}, k \leftarrow k - 1)$ . Furthermore, we set  $\mathcal{S} := \mathcal{S} \setminus \{v\}$ . We apply the Flower Rule selectively until no longer possible. We abuse notation and continue to use  $(G, k)$  to refer to the instance that is reduced with respect to exhaustive application of the Selective Flower Rule. Thus, for every vertex  $v \in \mathcal{S}$  the size of any flower passing through  $v$  in  $G_v$  is at most  $k$ .

Now we describe how to find, for a given  $v \in V(G)$ , a hitting set  $H_v \subseteq V(G) \setminus \{v\}$  for all minor-models of  $\theta_c$  that contain  $v$ . Notice that this hitting set is required to *exclude*  $v$ , so  $H_v$  cannot be the trivial hitting set  $\{v\}$ . If  $v \notin \mathcal{S}$ , then  $H_v = \mathcal{S}$ . On the other hand, suppose  $v \in \mathcal{S}$ . Since the maximum size of a flower containing  $v$  in the graph  $G_v$  is at most  $k$  by Lemma 5.5, we can find a set  $T_v$  of size  $O(k)$  that does not contain  $v$  and hits all the  $\theta_c$  minor-models passing through  $v$  in  $G_v$ . Hence in this case we set  $H_v = \mathcal{S}_v \cup T_v$  (See Figure 5.2.). We denote  $|H_v|$  by  $h_v$ . Notice that  $H_v$  is defined algorithmically, that is, there could be many small hitting sets in  $V(G) \setminus \{v\}$  hitting all minor-models containing  $v$ , and  $H_v$  is one of them.

(2) *q-expansion Rule with  $q = c$ .* Given an instance  $(G, k)$ ,  $\mathcal{S}$ , and a family of sets  $H_v$ , we show that if there is a vertex  $v$  with degree more than  $ch_v + c(c - 1)h_v$ , then we can reduce its degree to at most  $ch_v + c(c - 1)h_v$  by repeatedly applying the  $q$ -Expansion Lemma with  $q = c$ . Assume, without loss of generality, that the instance is reduced with respect to the Irrelevant Vertex Rule. Observe that for every vertex  $v$  the set  $H_v$  is also a  $\theta_c$  hitting set for  $G$ , that is,  $H_v$  hits *all* minor-models of  $\theta_c$  in  $G$ . Consider the graph  $G \setminus (H_v \cup \{v\})$ . Let the components of this graph that contain a neighbor of  $v$  be  $C_1, C_2, \dots, C_r$ . Note that  $v$  cannot have more than  $(c - 1)$  neighbors

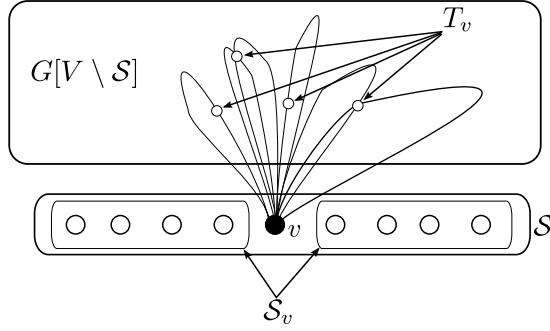


FIG. 5.2. *The hitting set in Selective Flower Rule*

into any component, else the component together with  $v$  will form a  $\theta_c$  minor and will contradict the fact that  $H_v$  hits all the  $\theta_c$  minors. Also note that none of the  $C_i$ 's can contain a minor model of  $\theta_c$ . Since  $v$  has at most  $(c-1)$  edges to each  $C_i$ , it follows that if  $d(v) > ch_v + c(c-1)h_v$ , then the number of components  $|C|$  is more than  $ch_v$ .

We say that a component  $C_i$  is adjacent to  $H_v$  if there exists a vertex  $u \in C_i$  and  $w \in H_v$  such that  $(u, w) \in E(G)$ . Next we show that vertices in components that are not adjacent to  $H_v$  are irrelevant in  $G$ . Recall a vertex is irrelevant if there is no minimal minor model of  $\theta_c$  that contains it. Consider a vertex  $u$  in a component  $C$  that is not adjacent to  $H_v$ . Since  $G[V(C) \cup \{v\}]$  does not contain any  $\theta_c$  minor we have that if  $u$  is a part of a minimal minor model  $M \subseteq G$ , then  $v \in M$  and also there exists a vertex  $u' \in M$  such that  $u' \notin C \cup \{v\}$ . Then the removal of  $v$  disconnects  $u$  from  $u'$  in  $M$ , a contradiction to Observation 1 that for  $c \geq 2$ , any minimal  $\theta_c$  minor model  $M$  of a graph  $G$  does not contain a cut vertex. Therefore, any vertex in a component not adjacent to  $H_v$  is irrelevant. Since the instance is reduced with respect to the Irrelevant Vertex Rule, we conclude that in all the components  $C_i$ , there is at least one vertex that is adjacent to a vertex in  $H_v$ .

Now, consider a bipartite graph  $\mathcal{G}$  with vertex bipartitions  $H_v$  and  $C$ . Here  $C = \{c_1, \dots, c_s\}$  contains a vertex  $c_i$  corresponding to each component  $C_i$ . For every  $u \in H_v$ , we add the edge  $(u, c_i)$  if there is a vertex  $w \in C_i$  such that  $\{u, w\} \in E(G)$ . Even though we start with a simple graph (graphs without parallel edges) it is possible that after applying reduction rules parallel edges may appear. However, throughout the algorithm, we ensure that the number of parallel edges between any pair of vertices is at most  $c$ . Now,  $v$  has at most  $ch_v$  edges to vertices in  $H_v$ . By applying  $q$ -Expansion Lemma with  $q = c$ ,  $A = H_v$ , and  $B = D$ , we find a subset  $S \subseteq H_v$  and  $T \subseteq D$  such that  $S$  has  $|S|$   $c$ -stars in  $T$  and  $N(T) = S$ .

The reduction rule involves deleting edges of the form  $(v, u)$  for all  $u \in C_i$ , such that  $c_i \in T$ , and adding  $c$  edges between  $v$  and  $w$  for all  $w \in S$ . We add these edges only if they were not present before so that the number of edges between any pair of vertices remains at most  $c$ . This completes the description of the  $q$ -expansion reduction rule with  $q = c$ . Let  $G_R$  be the graph obtained after applying the reduction rule. The following lemma shows the correctness of the rule.

**LEMMA 5.7.** *Let  $G$ ,  $S$  and  $v$  be as above and  $G_R$  be the graph obtained after applying the  $c$ -expansion rule. Then  $(G, k)$  is a yes instance of  $p$ - $\theta_c$ -DELETION if and only if  $(G_R, k)$  is a yes instance of  $p$ - $\theta_c$ -DELETION.*

*Proof.* We first show that if  $G_R$  has hitting set  $Z$  of size at most  $k$ , then the

same hitting set  $Z$  hits all the minor-models of  $\theta_c$  in  $G$ . Observe that either  $v \in Z$  or  $S \subseteq Z$ . Suppose  $v \in Z$ , then observe that  $G_R \setminus \{v\}$  is the same as  $G \setminus \{v\}$ . Therefore  $Z \setminus \{v\}$ , a hitting set of  $G_R \setminus \{v\}$  is also a hitting set of  $G \setminus \{v\}$ . This shows that  $Z$  is a hitting set of size at most  $k$  of  $G$ . The case when  $S \subseteq Z$  is similar.

To prove that a hitting set of size at most  $k$  in  $G$  implies a hitting set of size at most  $k$  in  $G_R$ , it suffices to prove that whenever there is a hitting set of size at most  $k$ , there also exists a hitting set of size at most  $k$  that contains either  $v$  or all of  $S$ . Consider a hitting set  $W$  that does not contain  $v$ , and omits at least one vertex from  $S$ . Note the  $|S|$   $c$ -stars in  $\mathcal{G}[S \cup T]$ , along with  $v$ , correspond to minor-models of  $\theta_c$  centered at  $v$  in  $G$ , vertex-disjoint except for  $v$ . Thus, such a hitting set must pick at least one vertex from one of the components. Let  $\mathcal{C}$  be the collection of components  $C_i$  such that the (corresponding) vertex  $c_i \in T$ . Let  $X$  denote the set of all vertices of  $W$  that appeared in any  $C_i \in \mathcal{C}$ . Consider the hitting set  $W'$  obtained from  $W$  by removing  $X$  and adding  $S$ , that is,  $W' := (W \setminus X) \cup S$ .

We now argue that  $W'$  is also a hitting set of size at most  $k$ . Indeed, let  $S'$  be the set of vertices in  $S$  that do not already belong to  $W$ . Clearly, for *every* such vertex that  $W$  omitted,  $W$  must have had to pick distinct vertices from  $\mathcal{C}$  to hit the  $\theta_c$  minor-models formed by the corresponding  $c$ -stars. Formally, there exists a  $X' \subseteq X$  such that there is a bijection between  $S'$  and  $X'$ , implying that  $|W'| \leq |W| \leq k$ .

Finally, observe that  $W'$  must also hit all minor-models of  $\theta_c$  in  $G$ . If not, there exists a minor-model  $M$  that contains some vertex  $u \in X$ . Hence,  $u \in C_i$  for some  $i$ , and  $M$  contains some vertex in  $H_v \setminus S$ . However,  $v$  separates  $u$  from  $H_v \setminus S$  in  $G \setminus S$ , contradicting Observation 1 that  $M$  does not contain a cut vertex. This concludes the proof.  $\square$

Observe that all edges that are added during the application of the  $q$ -expansion reduction rule have at least one end point in  $\mathcal{S}$ , and hence  $\mathcal{S}$  remains a hitting set of  $G_R$ . We are now ready to summarize the algorithm that bounds the degree of the graph (see Algorithm 3).

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**Algorithm 3** BOUND-DEGREE( $G, k, \mathcal{S}$ )

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- 1: Apply the Selective Flower Rule
  - 2: **if**  $\exists v \in V(G)$  such that  $d(v) > ch_v + c(c-1)h_v$  **then**
  - 3:   Apply the  $q$ -expansion reduction rule with  $q = c$ .
  - 4: **else**
  - 5:   Return  $(G, k, \mathcal{S})$ .
  - 6: **end if**
  - 7: Return BOUND-DEGREE( $G, k, \mathcal{S}$ ).
- 

Let the instance output by Algorithm 3 be  $(G', k', \mathcal{S})$ . Clearly, in  $G'$ , the degree of every vertex is at most  $ch_v + c(c-1)h_v \leq O(k \log^{3/2} k)$ . The routine also returns  $\mathcal{S}$  — a  $\theta_c$ -hitting set of  $G'$  of size at most  $O(k \log^{3/2} k)$ .

We now show that the algorithm runs in polynomial time. For  $x \in V(G)$ , let  $\nu(x)$  be the number of neighbors of  $x$  to which  $x$  has fewer than  $c$  parallel edges. Observe that the application of  $q$ -expansion reduction rule never increases  $\nu(x)$  for any vertex. For the vertex  $v$  that is considered in Line 2 of Algorithm 3, the application of the  $q$ -expansion lemma with  $q = c$  reduces the degree of  $v$  to at most  $ch_v + c(c-1)h_v$ , therefore reducing the degree of  $v$ . Further, the edges that are removed by the reduction rule are from  $v$  to its neighbors in  $T$ , which have fewer than  $c$  parallel edges with  $v$ . It follows that  $\nu(v)$  strictly decreases by one. The other rules delete vertices,

and can never increase  $\nu(x)$  for any vertex. This concludes the proof of Lemma 5.6.  $\square$

**5.3. Analysis and Kernel Size – Proof of Theorem 1.2.** In this section we give the desired kernel for  $p$ - $\theta_c$ -DELETION.

*Proof.* [Proof of Theorem 1.2] Let  $(G, k)$  be an instance to  $p$ - $\theta_c$ -DELETION. We first bound the maximum degree of the graph by applying Lemma 5.6 on  $(G, k)$ . If Lemma 5.6 returns that  $(G, k)$  is a NO-instance to  $p$ - $\theta_c$ -DELETION then we return the same. Else we obtain an equivalent instance  $(G', k')$  such that  $k' \leq k$  and the maximum degree of  $G'$  is bounded by  $O(k \log^{3/2} k)$ . Moreover it also returns a  $\theta_c$ -hitting set,  $X'$ , of  $G'$  of size at most  $O(k \log^{3/2} k)$ . Let  $d$  denote the treewidth of the graph after the removal of  $X'$ , that is,  $d := \mathbf{tw}(G' \setminus X')$ .

Now, we obtain our kernel in two phases: we first apply the protrusion rule selectively (Lemma 3.2) and get a polynomial kernel. Then, we apply the protrusion rule exhaustively on the obtained kernel to get a smaller kernel. To obtain the kernel we follow the following steps.

*Applying the Protrusion Rule.* By a result of Robertson et. al. [60] we know that any graph of treewidth greater than  $20^{2c^5}$  contains a  $c \times c$  grid, and hence  $\theta_c$ , as a minor. Hence  $d \leq 20^{2c^5}$ . Now we apply Lemma 3.3 and get a  $2(d+1)$ -protrusion  $Y$  of  $G'$  of size at least  $\frac{|V(G')| - |X'|}{4|N(X')| + 1}$ . By Lemma 3.5,  $p$ - $\theta_c$ -DELETION has finite integer index.

Let  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  be the function defined in Lemma 3.2. Hence if  $\frac{|V(G')| - |X'|}{4|N(X')| + 1} \geq \gamma(2d+2)$  then using Lemma 3.2 we replace the  $2(d+1)$ -protrusion  $Y$  of  $G'$  and obtain an instance  $G^*$  such that  $|V(G^*)| < |V(G')|$ ,  $k^* \leq k'$ , and  $(G^*, k^*)$  is a YES-instance of  $p$ - $\theta_c$ -DELETION if and only if  $(G', k')$  is a YES-instance of  $p$ - $\theta_c$ -DELETION.

Before applying the Protrusion Rule again, if necessary, we bound the maximum degree of the graph by reapplying Lemma 5.6. This is done because the application of the protrusion rule could potentially increase the maximum degree of the graph. We alternately apply the protrusion rule and Lemma 5.6 in this fashion, until either Lemma 5.6 returns that  $G$  is a NO instance, or the protrusion rule ceases to apply. Observe that this process will always terminate as the procedure that bounds the maximum degree never increases the number of vertices and the protrusion rule always reduces the number of vertices.

Let  $(G^*, k^*)$  be a reduced instance with hitting set  $X$ . In other words, there is no  $(2d+2)$ -protrusion of size  $\gamma(2d+2)$  in  $G^* \setminus X$ , and the protrusion rule no longer applies. Also, since Lemma 5.6 has also been applied exhaustively, we have that  $\Delta(G^*) = O(k \log^{3/2} k)$ . Now we show that the number of vertices and edges of this graph is bounded by  $O(k^2 \log^3 k)$ . We first bound the number of vertices. Since we cannot apply the Protrusion Rule,  $\frac{|V(G^*)| - |X|}{4|N(X)| + 1} \leq \gamma(2d+2)$ . Since  $k^* \leq k$  this implies that

$$\begin{aligned} |V(G^*)| &\leq \gamma(2d+2)(4|N(X)| + 1) + |X| \\ &\leq \gamma(2d+2)(4|X|\Delta(G^*) + 1) + |X| \\ &\leq \gamma(2d+2)(O(k \log^{3/2} k) \times O(k \log^{3/2} k) + 1) + O(k \log^{3/2} k) \\ &\leq O(k^2 \log^3 k). \end{aligned}$$

To get the desired bound on the number of edges we first observe that since  $\mathbf{tw}(G^* \setminus X) \leq 20^{2c^5} = d$ , we have that the number of edges in  $G^* \setminus X \leq d|V(G^*) \setminus X| = O(k^2 \log^3 k)$ . Also the number of edges incident on the vertices in  $X$  is at most  $|X| \cdot$

$\Delta(G^*) \leq O(k^2(\log k)^3)$ . This gives us a polynomial time algorithm that returns a kernel of size  $O(k^2 \log^3 k)$ .

To obtain a kernel of smaller size, we apply combination of rules to bound the degree and the protrusion rule as before. The only difference is that we would like to replace any large  $(2d + 2)$ -protrusion in the graph by a smaller one. We find a  $2d + 2$ -protrusion  $Y$  of size at least  $\gamma(2d + 2)$  by guessing the boundary  $\partial(Y)$  of size at most  $2d + 2$ . This could be performed in time  $k^{O(d)}$ . So let  $(G^*, k^*)$  be the reduced instance on which we can not apply either the Protrusion Rule or Lemma 5.6. Then we know that  $\Delta(G^*) = O(k \log^{3/2} k)$ . If  $G$  is a YES-instance then there exists a  $\theta_c$ -hitting set  $X$  of size at most  $k$  such that  $\mathbf{tw}(G \setminus X) \leq 20^{2c^5} = d$ . Now applying the analysis above with this  $X$  yields that  $|V(G^*)| = O(k^2 \log^{3/2} k)$  and  $|E(G^*)| \leq O(k^2 \log^{3/2} k)$ . Hence if the number of vertices or edges in the reduced instance  $G^*$ , to which we can not apply the Protrusion Rule, is more than  $O(k^2 \log^{3/2} k)$  then we return that  $G$  is a NO-instance. This concludes the proof of the theorem.  $\square$

Theorem 1.2 has following immediate corollary.

**COROLLARY 5.8.**  *$p$ -VERTEX COVER,  $p$ -FEEDBACK VERTEX SET and  $p$ -DIAMOND HITTING SET have kernel of size  $O(k^2 \log^{3/2} k)$ .*

**6. Conclusion.** In this paper we gave the first kernelization algorithms for a subset of  $p$ - $\mathcal{F}$ -DELETION problems and a generic approximation algorithm for the  $p$ - $\mathcal{F}$ -DELETION problem when the set of excluded minors  $\mathcal{F}$  contains at least one planar graph. Our approach generalizes and unifies known kernelization algorithms for  $p$ -VERTEX COVER and  $p$ -FEEDBACK VERTEX SET. By the celebrated result of Robertson and Seymour, every  $p$ - $\mathcal{F}$ -DELETION problem is FPT and our work naturally leads to the following question: does every  $p$ - $\mathcal{F}$ -DELETION problem have a polynomial kernel? Can it be that for some finite sets of minor obstructions  $\mathcal{F} = \{O_1, \dots, O_p\}$  the answer to this question is NO? Even the case  $\mathcal{F} = \{K_5, K_{3,3}\}$ , vertex deletion to planar graphs, is an interesting challenge.

In the follow-up work of the subset of the authors [42], it was shown that  $p$ - $\mathcal{F}$ -DELETION admits a polynomial kernel and randomized constant factor approximation when  $\mathcal{F}$  contains a planar graph. However our algorithm is still the best known deterministic approximation algorithm for this problem.

The kernelization results from [42] are also of different nature than ours. Our kernel is uniform, i.e. its size is bounded by  $f(\theta_c) \cdot k^2 \log^{3/2} k$  for some function  $f$ . The polynomial kernel for  $p$ - $\mathcal{F}$ -DELETION given in [42] is non-uniform, in a sense that the size of the kernel is  $k^{f(\mathcal{F})}$ , where  $f$  is a non-elementary function. Moreover, as it was shown very recently the improvement of general case is highly unlikely—unless some complexity assumption fails, there is no kernel of size  $f(\mathcal{F}) \cdot k^{O(1)}$  for any function of  $\mathcal{F}$ , when  $\mathcal{F}$  contains a planar graph [45].

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