Towards Practical Physical-Optics Rendering — Supplemental and Derivations

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S1 THE JONES CALCULUS AND GENERALIZED MUELLER CALCULUS

Given a matrix $A \in \mathbb{C}^{n\times n}$ with rows $\vec{a}_j$, let the *stack operator* acting on A be denoted as $A^{\mathcal{S}}$. That is, $A^{\mathcal{S}} = [\vec{a}_1^{\top}, \vec{a}_2^{\top}, \ldots, \vec{a}_n^{\top}]^{\top}$ is the vectorized from of the matrix A: the column vector that arises by stacking the rows of A.

We mention a well-known property of the Kronecker product:

Lemma S1.1. $ADB^{\top} = C$ *if and only if* $(B \otimes A)D^{\mathcal{S}} = C^{\mathcal{S}}$.

Proof. See [Loan 2000]. □

Jones calculus. The Jones calculus [Savenkov 2009] is a simple and useful formal method, where the transverse components of light are written as a 2-element complex vector—the *Jones vector*—and the actions of linear optical elements are quantified via 2 × 2 matrices—the *Jones matrices*—that act upon these vectors. Let $\vec{\psi}(\vec{k}, t)$ be a plane wave, as in the paper, with wavevector $\vec{k}$. Under a given transverse basis $\hat{x}, \hat{y}$, the transverse components of this plane wave are written as the Jones vector $\vec{j}(\vec{k}, t) = [\hat{x} \cdot \vec{\psi},\ \hat{y} \cdot \vec{\psi}]^{\top}$. Let that plane wave be incident upon a particle, whose scattering characteristics are quantified by the Jones matrix J. The scattered plane wave is then $\vec{j}' = J\vec{j}$. Jones matrices can be complex, quantifying the phase-shifts induced by conductive particles. This calculus deals directly with the underlying electric field, and may only quantify interactions with perfectly-coherent, perfectly-polarized light.

Generalized Mueller calculus. In order to derive a coherence-aware linear calculus, Korotkova [2017] generalize the classical *Mueller-Stokes calculus* into a two-point formalism.

We start with the *cross-spectral density matrix*, which is simply the CSDs $C_{\alpha\beta}$ written in matrix form:

$$\mathcal{C}\,(\vec{r}_1, \vec{r}_2\,;\omega) \triangleq \left\langle \vec{E}(\vec{r}_1)\vec{E}(\vec{r}_2)^{\dagger} \right\rangle_{\omega} = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix} . \tag{S1.1}$$

Note that, compared with the paper, we re-parametrize the CSDs and the generalized radiance and irradiance such that $\vec{r}_{1,2} = \vec{p} \pm \frac{1}{2}\vec{\xi}$, where $\vec{p}.\vec{\xi}$ are as defined in Eq. (7), are the two points at which these quantities are evaluated. This notation is more common in optical literature. The generalized Stokes parameters (Eq. (8) in the paper) and the CSD matrix above describe the same information, in different analytic forms. Observe that we may trivially write the gSP as follows:

$$\overleftrightarrow{\mathcal{S}}^{[\mu]} = A_{\text{mül}}\, \mathcal{C}^{\mathcal{S}} , \tag{S1.2}$$

where the $A_{\text{mül}}$, defined by Savenkov [2009, Eq. 3.12], is dictated by Eq. (8):

$$A_{\text{mül}} \triangleq \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \end{pmatrix} \tag{S1.3}$$

Let the local, spatially-varying scattering characteristics of a medium be quantified by the Jones matrix $J(\vec{p})$, with $\vec{p}$ being a point in the medium. Apply J to the electric field $\vec{E}$, and substitute into Eq. (S1.1), yielding the CSD matrix of the radiation after interaction, viz.

$$\begin{aligned} \mathcal{C}_{(\mathrm{o})}\,(\vec{r}_1, \vec{r}_2\,;\omega) &= \left\langle \left[J(\vec{r}_1)\vec{E}(\vec{r}_1)\right]\left[J(\vec{r}_2)\vec{E}(\vec{r}_2)\right]^{\dagger} \right\rangle_{\omega} \\ &= J(\vec{r}_1)\,\mathcal{C}\,(\vec{r}_1, \vec{r}_2\,;\omega) J(\vec{r}_2)^{\dagger} . \end{aligned} \tag{S1.4}$$

The above relates the CSD matrices of the incident radiation and radiation immediately after interaction (but before propagation), $\mathcal{C}$ and $\mathcal{C}'$, respectively. Using Lemma S1.1 we deduce

$$\mathcal{C}_{(\mathrm{o})}\,(\vec{r}_1, \vec{r}_2\,;\omega)^{\mathcal{S}} = \left[J(\vec{r}_2)^{\star} \otimes J(\vec{r}_1)\right] \mathcal{C}\,(\vec{r}_1, \vec{r}_2\,;\omega)^{\mathcal{S}} , \tag{S1.5}$$

and, finally, apply the equivalence relation between the CSD matrix and the generalized Stokes parameters, Eq. (S1.2), yielding,

$$\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathrm{o})} = A_{\text{mül}} \left[J(\vec{r}_2)^{\star} \otimes J(\vec{r}_1)\right] A_{\text{mül}}^{-1}\, \overleftrightarrow{\mathcal{S}}^{[\mu]} , \tag{S1.6}$$

which is a relation between the generalized Stokes parameters of the radiation before and after interaction. The quantity

$$\mathcal{M}(\vec{p}_1, \vec{p}_2) \triangleq A_{\text{mül}}[J(\vec{p}_2)^{\star} \otimes J(\vec{p}_1)] A_{\text{mül}}^{-1} \tag{S1.7}$$

is the *generalized Mueller matrix*. That is, $\mathcal{M}(\vec{p}_1, \vec{p}_2)$, acting upon the generalized Stokes parameters, quantifies the same action as the pair of Jones matrices $J(\vec{p}_1)$ and $J(\vec{p}_2)$ acting upon the CSD matrix. Hence, $\mathcal{M}$ is the equivalent, matricial form of Eq. (S1.4), giving rise to the generalized Mueller calculus [Korotkova 2017].

Discussion. Note that the scattered CSD matrix $\mathcal{C}'$ and generalized Stokes parameters vector $\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathrm{o})}$, which arise in Eqs. (S1.5) and (S1.6), are local quantities before propagation. These quantities carry units of intensity (spectral irradiance). Both the CSD matrix and generalized Stokes parameters describe the same information. We choose to employ a formalism based on the generalized Stokes parameters and generalized Mueller calculus due to: (a) The equivalent relation written in terms of the CSD matrix and Jones matrices, Eq. (S1.4), frustrates analytic progress because we may not take advantage of the properties of locally-stationary matter (Eq. (S2.8)). This is because these matrices, $\mathcal{C}$ and J, generally do not commute. (b) The classical Mueller matrix $\bar{m}$ that arises in Eq. (S2.8) is a simple, classical quantity, that is easy to understand and has seen wide usage in computer graphics.

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S2 THE LIGHT-MATTER INTERACTION THEOREM

Propagation. The far-field Smythe diffraction formula can be written as [Zangwill 2013]

$$\vec{E}'(\vec{r}) = \frac{\mathrm{i}e^{\mathrm{i}kr}}{\lambda r}\hat{\boldsymbol{r}} \times \int_{\mathbf{P}} \mathrm{d}^2\vec{\boldsymbol{p}}_\perp \left[\hat{\boldsymbol{n}} \times \vec{E}(\vec{\boldsymbol{p}}_\perp)\right] e^{-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\vec{\boldsymbol{p}}_\perp}, \tag{S2.1}$$

where $\mathbf{P}$ is a planar aperture (centred at the origin) with normal $\hat{\boldsymbol{n}}$, $\vec{E}$ is the monochromatic electric field that is incident upon the aperture and $\vec{E}'$ is the scattered field. Using the far-field Smythe diffraction formula, Steinberg and Yan [2021a, supplemental Sec. 7.2] derive a propagation formula for the CSD matrix. We trivially generalize the above to diffraction by a volumetric region (under the Born first-order approximation) by extending $\mathbf{P}$ to an arbitrary three-dimensional volume. The far-field propagation formula for the CSD matrix becomes:

$$\mathcal{C}_{(\mathrm{o})}(\vec{r}_1,\vec{r}_2\,;\omega) = \frac{\cos^2\vartheta_o e^{\mathrm{i}k\hat{\boldsymbol{r}}\cdot(\vec{r}_1-\vec{r}_2)}}{\lambda^2 r^2}\int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_1\int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_2 \times e^{-\mathrm{i}k(\hat{\boldsymbol{r}}_1\cdot\vec{\boldsymbol{p}}_1-\hat{\boldsymbol{r}}_2\cdot\vec{\boldsymbol{p}}_2)}\boldsymbol{\tau}\,\mathcal{C}_{(\mathbf{P})}(\vec{r}_1',\vec{r}_2'\,;\omega)\boldsymbol{\tau}^\top, \tag{S2.2}$$

which is a double three-dimensional Fourier transform with $\mathcal{C}_{(\mathbf{P})}$ being the CSD matrix of the radiation at the region $\mathbf{P}$ and ϑ_o the inclination angle. $\hat{\boldsymbol{r}}$ and r are the mean propagation direction and distance. In the far field, $\hat{\boldsymbol{r}} \approx \hat{\boldsymbol{r}}_1 \approx \hat{\boldsymbol{r}}_2$ and likewise $r \approx r_1 \approx r_2$. The transformation matrix $\boldsymbol{\tau}$ transforms the incident CSD matrix from its chosen transverse basis to the transverse basis of the scattered radiation, viz. (simplified version of Steinberg and Yan [2021a, supplemental Eq. 7.17])

$$\boldsymbol{\tau} \triangleq \begin{pmatrix} \hat{\boldsymbol{x}}'_\perp\cdot\hat{\boldsymbol{x}} & \hat{\boldsymbol{x}}'_\perp\cdot\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{y}}'_\perp\cdot\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}}'_\perp\cdot\hat{\boldsymbol{y}} \end{pmatrix}, \tag{S2.3}$$

where $\{\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}\}$ and $\{\hat{\boldsymbol{x}}',\hat{\boldsymbol{y}}'\}$ are the given transverse bases of the incident and scattered radiation, respectively, and $\hat{\boldsymbol{x}}'_\perp,\hat{\boldsymbol{y}}'_\perp$ are the normalized projections of the scattered transverse basis onto the incident transverse plane. The above is a simple rotation matrix with angle φ, which is the angle between $\hat{\boldsymbol{x}}'_\perp$ and $\hat{\boldsymbol{x}}$ (in the singular case where $\hat{\boldsymbol{x}}'$ is perpendicular to the incident transverse plane, we take the angle between $\hat{\boldsymbol{y}}'_\perp$ and $\hat{\boldsymbol{y}}$). The corresponding Mueller matrix arises immediately:

$$\boldsymbol{T} \triangleq A_{\text{mül}}[\boldsymbol{\tau}\otimes\boldsymbol{\tau}]A_{\text{mül}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & -\sin 2\varphi & 0 \\ 0 & \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{S2.4}$$

We may now write the general diffraction formulae Eq. (S2.2) in an equivalent Stokes parameters form:

$$\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathrm{o})} = \frac{\cos^2\vartheta_o e^{\mathrm{i}k\hat{\boldsymbol{r}}\cdot(\vec{r}_1-\vec{r}_2)}}{\lambda^2 r^2}\int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_1\int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_2\, e^{-\mathrm{i}k(\hat{\boldsymbol{r}}_1\cdot\vec{\boldsymbol{p}}_1-\hat{\boldsymbol{r}}_2\cdot\vec{\boldsymbol{p}}_2)}\boldsymbol{T}\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathbf{P})}. \tag{S2.5}$$

with $\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathrm{o})}$ and $\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathbf{P})}$ being the Stokes parameters vectors after propagation and at the scattering volume, respectively. Applying Eq. (10) to the above, we may write an equivalent propagation formula for the radiance-carrying gSP $\overleftrightarrow{\mathcal{L}}^{[\mu]}$

$$\overleftrightarrow{\mathcal{L}}^{[\mu]}_{(\mathrm{o})} = \frac{\cos^2\vartheta_o e^{\mathrm{i}k\hat{\boldsymbol{r}}\cdot(\vec{r}_1-\vec{r}_2)}}{\lambda^2 A} \times \int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_1\int_{\mathbf{P}}\mathrm{d}^3\vec{\boldsymbol{p}}_2\, e^{-\mathrm{i}k(\hat{\boldsymbol{r}}_1\cdot\vec{\boldsymbol{p}}_1-\hat{\boldsymbol{r}}_2\cdot\vec{\boldsymbol{p}}_2)}\boldsymbol{T}\overleftrightarrow{\mathcal{S}}^{[\mu]}_{(\mathbf{P})}, \tag{S2.6}$$

where A is the projected area of P in direction $\hat{\boldsymbol{r}}$.

S2.1 Interaction With Matter

Locally-stationary matter. Let the scattering characteristics of the scattering region $\mathbf{P}$ be quantified by a Jones matrix $\boldsymbol{J}(\vec{\boldsymbol{p}})$, which we now be regarded as a spatial stochastic process, describing the spatially-varying polarimetric scattering potential of the medium. The respective Mueller matrix $\boldsymbol{\mathcal{M}}$ is defined via Eq. (S1.7). An autocorrelation function of that process can defined as the generalized Mueller matrix

$$\boldsymbol{\rho}_{JJ}(\vec{\boldsymbol{p}}_1,\vec{\boldsymbol{p}}_2) \triangleq \left\langle \boldsymbol{\mathcal{M}}(\vec{\boldsymbol{p}}_1,\vec{\boldsymbol{p}}_2)\right\rangle, \tag{S2.7}$$

where the operator $\langle\cdot\rangle$ averages over the statistical ensemble of all matter realizations that conform to the statistics of the process $\boldsymbol{J}$ (not to be confused with $\langle\cdot\rangle_\omega$). The quantity $\langle\boldsymbol{\mathcal{M}}\rangle$ qualifies as an autocorrelation function, as it quantifies the ensemble-averaged mutual scattering characteristics at a pair of points.

We now restrict the stochastic process $\boldsymbol{J}$ to the class of locally-stationary processes (a less restrictive class than weak stationarity) [Silverman 1957]. This gives rise to *locally-stationary matter* [Steinberg and Yan 2021b]. Under this description of matter, the single-point function $\bar{\boldsymbol{m}}(\vec{r}) \triangleq \langle\boldsymbol{\mathcal{M}}(\vec{r},\vec{r})\rangle$ is the ensemble-averaged, local Mueller matrix that describes the deterministic features of the matter (e.g., density and polarimetric properties of scattering particles in a medium). The autocorrelation function of such a process takes the form of a product of real, classical Mueller matrices:

$$\boldsymbol{\rho}_{JJ}(\vec{\boldsymbol{p}}_1,\vec{\boldsymbol{p}}_2) = \bar{\boldsymbol{m}}\left(\tfrac{\vec{\boldsymbol{p}}_1+\vec{\boldsymbol{p}}_2}{2}\right)\boldsymbol{R}_{JJ}(\vec{\boldsymbol{p}}_1-\vec{\boldsymbol{p}}_2). \tag{S2.8}$$

The Mueller matrix $\boldsymbol{R}_{JJ}$ is the *stationary autocorrelation* (a function of the difference vector $\vec{\boldsymbol{p}}_1-\vec{\boldsymbol{p}}_2$ only) of the locally-stationary process, which describes statistical perturbations across the matter. These quantities are typically wavelength dependent.

Interaction formulae. The light-matter interaction is then quantified by the matter's stochastic generalized Mueller matrix $\boldsymbol{\rho}_{JJ}$. Set

$$\overleftrightarrow{\mathcal{S}}^{[\mu_\mathrm{o}]}_{(\mathbf{P})}(\vec{r}_1-\vec{r}_2\,;\omega) \triangleq \boldsymbol{\rho}_{JJ}(\vec{r}_1,\vec{r}_2\,;\omega)\overleftrightarrow{\mathcal{S}}^{[\mu_\mathrm{i}]}_{(\mathrm{i})}(\vec{r}_1-\vec{r}_2\,;\omega), \tag{S2.9}$$

where $\overleftrightarrow{\mathcal{L}}^{[\mu]}_{(\mathrm{i})}$ and $\overleftrightarrow{\mathcal{L}}^{[\mu]}_{(\mathbf{P})}$ are the incident canonical wave packet (Definition 3.1) before interaction with the matter and immediately after interaction, but before propagation, respectively. The propagation is then done via the wave packet propagation formula, Eq. (S2.6). Substituting Eq. (S2.8) into Eq. (S2.6) yields

$$\overleftrightarrow{\mathcal{L}}^{[\mu_\mathrm{o}]}_{(\mathrm{o})} = \frac{\cos^2\vartheta_o e^{\mathrm{i}k\hat{\boldsymbol{r}}\cdot(\vec{r}_1-\vec{r}_2)}}{\lambda^2 A}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{\boldsymbol{p}}_1\int_{\mathbb{R}^3}\mathrm{d}^3\vec{\boldsymbol{p}}_2\, e^{-\mathrm{i}k(\hat{\boldsymbol{r}}_1\cdot\vec{\boldsymbol{p}}_1-\hat{\boldsymbol{r}}_2\cdot\vec{\boldsymbol{p}}_2)} \times \bar{\boldsymbol{m}}\left(\tfrac{\vec{\boldsymbol{p}}_1+\vec{\boldsymbol{p}}_2}{2}\right)\boldsymbol{R}_{JJ}(\vec{\boldsymbol{p}}_1-\vec{\boldsymbol{p}}_2)\boldsymbol{T}\overleftrightarrow{\mathcal{S}}^{[\mu_\mathrm{i}]}_{(\mathrm{i})}, \tag{S2.10}$$

with $\bar{\boldsymbol{m}}$ and $\boldsymbol{R}_{JJ}$ as in the paper and we denote $A = A_{(\mathbf{P})}/A_{(\mathrm{i})}$. Note, we are now integrating over $\mathbb{R}^3\times\mathbb{R}^3$, the restriction to the scattering region is assumed to be performed by $\bar{\boldsymbol{m}}$, i.e. set $\bar{\boldsymbol{m}}(\vec{\boldsymbol{p}}) \equiv 0$

when $\vec{\boldsymbol{p}} \notin \mathbf{P}$. Perform the variable changes $\vec{\xi} = \vec{\boldsymbol{p}}_1 - \vec{\boldsymbol{p}}_2$ and $\vec{\zeta} = \frac{1}{2}(\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2)$. Then, the quantities that appear in the integrand above can be rewritten as follows:

$$e^{-ik\left(\hat{\boldsymbol{r}}_1\cdot\vec{\boldsymbol{p}}_1 - \hat{\boldsymbol{r}}_2\cdot\vec{\boldsymbol{p}}_2\right)} = e^{-ik\frac{1}{2}\vec{\xi}\cdot(\hat{\boldsymbol{r}}_1+\hat{\boldsymbol{r}}_2)} e^{-ik\vec{\zeta}\cdot(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2)}, \tag{S2.11}$$

$$\bar{\boldsymbol{m}}\left(\frac{\vec{\boldsymbol{p}}_1+\vec{\boldsymbol{p}}_2}{2}\right) = \bar{\boldsymbol{m}}\left(\vec{\zeta}\right), \tag{S2.12}$$

$$\boldsymbol{R}_{JJ}\left(\vec{\boldsymbol{p}}_1 - \vec{\boldsymbol{p}}_2\right) = \boldsymbol{R}_{JJ}\left(\vec{\xi}\right), \tag{S2.13}$$

and recall that a generalized irradiance is a function of $\vec{\xi}$ only. This serves to decouple the double integral in Eq. (S2.10) into separate spatial integrals for each integration variable, $\vec{\zeta}$ and $\vec{\xi}$.

The first integral becomes the following Fourier transform, which can be understood as the *angular spectrum function*:

$$\begin{aligned}\widetilde{\boldsymbol{M}}(k\hat{\boldsymbol{r}}_1 - k\hat{\boldsymbol{r}}_2) &\triangleq \int_{\mathbb{R}^3} d^3\vec{\zeta}\, e^{-ik\vec{\zeta}\cdot(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2)}\bar{\boldsymbol{m}}\left(\vec{\zeta}\right) \\ &=(2\pi)^{\frac{3}{2}}\mathscr{F}\left\{\bar{\boldsymbol{m}}\right\}(k\hat{\boldsymbol{r}}_1 - k\hat{\boldsymbol{r}}_2)\ .\end{aligned} \tag{S2.14}$$

As $\hat{\boldsymbol{r}}_1 \approx \hat{\boldsymbol{r}}_2$ and by the far-field assumption $\bar{\boldsymbol{m}}$ changes slowly as a function of $\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}$, $\widetilde{\boldsymbol{M}}(0)$ is a very good approximation to that integral. The second integral is a Fourier transform of the product of the matter's stationary autocorrelation function with the incident wave packet, viz.

$$\vec{\boldsymbol{f}}\left(k\frac{\hat{\boldsymbol{r}}_1+\hat{\boldsymbol{r}}_2}{2}\right) \triangleq \int_{\mathbb{R}^3} d^3\vec{\xi}\, e^{-i\frac{k}{2}\vec{\xi}\cdot(\hat{\boldsymbol{r}}_1+\hat{\boldsymbol{r}}_2)}\boldsymbol{R}_{JJ}\left(\vec{\xi}\right)\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}\left(\vec{\xi}\right). \tag{S2.15}$$

We first prove a pair of simple Lemmas.

Lemma S2.1. *Let $\boldsymbol{\Sigma} \succ 0$ and $\boldsymbol{B}$ be non-singular real matrices. Then,*

$$\mathscr{F}\left\{g^{\boldsymbol{\Sigma}}\left(\boldsymbol{B}\vec{\xi}\right)\right\}(\vec{\boldsymbol{q}}) = \frac{|\boldsymbol{\Sigma}|^{1/2}}{|\boldsymbol{B}|}g^{\boldsymbol{\Sigma}^{-1}}\left(\boldsymbol{B}^{-1}\vec{\boldsymbol{q}}\right).$$

Proof. Denote $\boldsymbol{\Xi} = \boldsymbol{B}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{B}$, a positive-definite matrix, and $\vec{\xi}' = \boldsymbol{\Xi}^{1/2}\vec{\xi}$. As $\boldsymbol{\Xi} \succ 0, \boldsymbol{\Xi}^{1/2}$ exists, is unique, real and positive-definite. Then, $d^3\vec{\xi} = |\boldsymbol{\Xi}|^{-1/2} d^3\vec{\xi}'$, as $|\boldsymbol{\Xi}|^{-1/2}$ is the Jacobian, and

$$\begin{aligned}\mathscr{F}\left\{g^{\boldsymbol{\Sigma}}\left(\boldsymbol{B}\vec{\xi}\right)\right\}(\vec{\boldsymbol{q}}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} d^3\vec{\xi}\, e^{-i\vec{\xi}\cdot\vec{\boldsymbol{q}}}e^{-\frac{1}{2}\vec{\xi}^\intercal\boldsymbol{B}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\vec{\xi}} \\ &=\frac{1}{(2\pi)^{\frac{3}{2}}}|\boldsymbol{\Xi}|^{-1/2}\int_{\mathbb{R}^3} d^3\vec{\xi}'\, e^{-i\vec{\xi}'\cdot\boldsymbol{\Xi}^{-1/2}\vec{\boldsymbol{q}}}e^{-\frac{1}{2}(\xi')^2} \\ &=|\boldsymbol{\Xi}|^{-1/2}\mathscr{F}\left\{g^{\boldsymbol{I}}\right\}\left(\boldsymbol{\Xi}^{-1/2}\vec{\boldsymbol{q}}\right) = |\boldsymbol{\Xi}|^{-1/2}g^{\boldsymbol{\Sigma}^{-1}}\left(\boldsymbol{B}^{-1}\vec{\boldsymbol{q}}\right),\end{aligned}$$

where the Fourier transform of a Gaussian identity was used. □

Let $\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}$ be an incident generalized irradiance, with shape matrices $\boldsymbol{\Theta}_{x,y,1/2}$. Let $S_{x,y}$ be the irradiance carried by each transverse component of $\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}$ and χ, ς as in Definition 3.1. Define the shorthands

$$\vec{\boldsymbol{S}}^{(i)}_{x,y} \triangleq \begin{pmatrix} S_{x,y} \\ \pm S_{x,y} \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{\boldsymbol{S}}^{(i)}_{1/2} \triangleq \sqrt{S_x S_y}\begin{pmatrix} 0 \\ 0 \\ \chi \\ \varsigma \end{pmatrix}. \tag{S2.16}$$

Corollary S2.2. *Given a generalized irradiance $\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}$ with mean propagation distance s,*

$$\mathscr{F}\left\{\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}\right\}(\vec{\boldsymbol{q}}) = \sum_{\alpha\in\{x,y,1/2\}}|\boldsymbol{\Theta}_\alpha|^{1/2}g^{\boldsymbol{\Theta}_\alpha^{-1}}\left(\boldsymbol{Q}^\intercal_{[\mu]}(\vec{\boldsymbol{q}} + k\hat{\boldsymbol{s}})\right)\vec{\boldsymbol{S}}^{(i)}_\alpha\ .$$

Proof. Immediately via the shift property of the Fourier transform and Lemma S2.1. □

Define $p \triangleq \mathscr{F}\left\{\boldsymbol{R}_{JJ}\right\}$, the Fourier transform of the stationary autocorrelation function, i.e. the stationary power spectral density (Definition 3.3.(i)). Applying Corollary S2.2, the convolution theorem (Eq. (3)) and the approximation $\frac{1}{2}(\hat{\boldsymbol{r}}_1 + \hat{\boldsymbol{r}}_2) \approx \hat{\boldsymbol{r}}$ to Eq. (S2.15), and simplifying, yields:

$$\begin{aligned}\vec{\boldsymbol{f}}(k\hat{\boldsymbol{r}}) &=(2\pi)^{\frac{3}{2}}\mathscr{F}\left\{\boldsymbol{R}_{JJ}\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}\right\}(k\hat{\boldsymbol{r}}) = 8\pi^3\left[p * \mathscr{F}\left\{\overleftrightarrow{\boldsymbol{\mathcal{S}}}^{[\mu_i]}_{(i)}\right\}\right](k\hat{\boldsymbol{r}}) \\ &=8\pi^3\sum_{\alpha\in\{x,y,1/2\}}\left|\boldsymbol{\Theta}^{(i)}_\alpha\right|^{\frac{1}{2}}\left[p * g^{\boldsymbol{\Xi}^{(i)}_\alpha}\right]\left(\vec{\boldsymbol{h}}\right)\vec{\boldsymbol{S}}^{(i)}_\alpha,\end{aligned} \tag{S2.17}$$

where the convolution is with respect to the integration variable $\vec{\boldsymbol{q}}'$, we define the shorthands

$$\vec{\boldsymbol{h}} \triangleq k(\hat{\boldsymbol{r}} + \hat{\boldsymbol{s}}) \qquad \text{and} \qquad \boldsymbol{\Xi}^{(i)}_\alpha \triangleq \boldsymbol{Q}_{[\mu_i]}\left(\boldsymbol{\Theta}^{(i)}_\alpha\right)^{-1}\boldsymbol{Q}^\intercal_{[\mu_i]}\ . \tag{S2.18}$$

We also define the classical pBSDF as the averaged (over the entire scattering region) Mueller matrix, viz.

$$\boldsymbol{M} \triangleq \frac{1}{|P|}\int_P \bar{\boldsymbol{m}} = \frac{1}{|P|}\widetilde{\boldsymbol{M}}(0)\ . \tag{S2.19}$$

Substituting Eqs. (S2.14) and (S2.17) into Eq. (S2.10) finally results in:

$$\overleftrightarrow{\boldsymbol{\mathcal{L}}}^{[\mu_o]}_{(o)} = \frac{\cos\vartheta_o e^{ik\hat{\boldsymbol{r}}\cdot(\vec{\boldsymbol{r}}_1-\vec{\boldsymbol{r}}_2)}}{\lambda^2(2\pi)^{\frac{3}{2}}}\boldsymbol{M}\sum_{\alpha\in\{x,y,1/2\}}\left|\boldsymbol{\Theta}^{(i)}_\alpha\right|^{\frac{1}{2}}\left(p * g^{\boldsymbol{\Xi}^{(i)}_\alpha}\right)\left(\vec{\boldsymbol{h}}\right)\boldsymbol{T}\vec{\boldsymbol{S}}^{(i)}_\alpha \tag{S2.20}$$

and up to the coherence functions. We used the fact that $|P|/A = 1/\cos\vartheta_o$. Note, in the paper, Eq. (21), absorbs the $1/\lambda^2$ into the pBSDF, for succinctness.

The stationary power spectral density (PSD) function p and the classical pBSDF fully quantify the matter's scattering properties. As the PSD p is real, the convolution that appears in Eq. (S2.20) is real as well, and this convolution quantifies the wave-interference effects. The classical Mueller matrix $\boldsymbol{M}$ acts upon the classical incident Stokes parameters vector (incident irradiance). The resulting $\overleftrightarrow{\boldsymbol{\mathcal{L}}}^{[\mu_o]}_{(o)}$ is a (radiance-carrying) classical Stokes parameters vector, that is constant up to the propagator term.

While Eq. (S2.20) quantifies the radiometric and polarimetric properties well, the coherence information was lost when we discarded the small perturbations of the vectors $\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_2$. We discuss the transformation of the shape matrices on interaction with matter in the paper.

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