Supplemental document: Fractional Gaussian Fields for Modeling and Rendering of Spatially-Correlated Media

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This document is supplemental to the paper entitled *Fractional Gaussian Fields for Modeling and Rendering of Spatially-Correlated Media*. In the following sections, we provide derivations of some important formulas and detailed discussions of some conclusions, as well as additional results.

1 DERIVATION OF $C(H, 1)$

The scaling term $C(H, d)$ defined on $\mathbb{R}^d$ is given by

$$C(H,d) = \frac{2^{-2H-d}\Gamma(-H)}{\pi^{d/2}\Gamma(H+d/2)}. \tag{1}$$

When $d = 1$, we have

$$\begin{aligned} C(H,1) &= \frac{2^{-2H-1}\Gamma(-H)}{\pi^{1/2}\Gamma(H+1/2)} \\ &= \frac{2^{-2H-1}\Gamma(-H)\Gamma(H)}{\pi^{1/2}\Gamma(H+1/2)\Gamma(H)} \\ &= -\frac{2^{-2H-1}\frac{\pi}{H\sin(\pi H)}}{\pi^{1/2}2^{1-2H}\pi^{1/2}\Gamma(2H)} \\ &= -\frac{1}{4H\Gamma(2H)\sin(\pi H)} \\ &= -\frac{1}{2\Gamma(2H+1)\sin(\pi H)}. \end{aligned} \tag{2}$$

The third equation is based on the Euler's reflection formula

$$\Gamma(1-H)\Gamma(H) = \frac{\pi}{\sin(\pi H)} \tag{3}$$

and the duplication formula

$$\Gamma(H+1/2)\Gamma(H) = 2^{1-2H}\pi^{1/2}\Gamma(2H). \tag{4}$$

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Note that $C(H, 1)$ satisfy

$$C(H+1,1) = \frac{C(H,1)}{(2H+1)(2H+2)}. \tag{5}$$

2 DERIVATION OF $\mathrm{var}_p[\bar{\sigma}_t]$

Defining $\mathbf{x}' = \mathbf{x} + t'\boldsymbol{\omega}$ and $\mathbf{x}'' = \mathbf{x} + t''\boldsymbol{\omega}$, the variance of $\bar{\sigma}_t(\mathbf{x}) = [\int_0^t \sigma_t(\mathbf{x} + t'\boldsymbol{\omega})\mathrm{d}t']/t$ is calculated as

$$\mathrm{var}[\bar{\sigma}_t] = \frac{1}{t^2}\int_0^t\int_0^t \mathrm{cov}(\mathbf{x}', \mathbf{x}'')\mathrm{d}t'\mathrm{d}t''. \tag{6}$$

Substituting the autocovariance function of 1D pink noise into the above formula, we have

$$\begin{aligned} \mathrm{var}_p[\bar{\sigma}_t] &= \frac{1}{t^2}\int_0^t\int_0^t C(H)S_w|t'-t''|^{2H}\mathrm{d}t'\mathrm{d}t'' \\ &= \frac{1}{t^2}C(H)S_w\int_0^t\int_0^t |t'-t''|^{2H}\mathrm{d}t'\mathrm{d}t'' \\ &= \frac{C(H)S_w}{(2H+1)(H+1)}t^{2H} \\ &= \frac{-S_w t^{2H}}{2(2H+1)(H+1)\Gamma(2H+1)\sin(\pi H)}. \end{aligned} \tag{7}$$

Using the fact that

$$2(2H+1)(H+1)\Gamma(2H+1) = \Gamma(2H+3) \tag{8}$$

we can simplify the above expression to

$$\begin{aligned} \mathrm{var}_p[\bar{\sigma}_t] &= \frac{-S_w t^{2H}}{\Gamma(2H+3)\sin(\pi H)} \\ &= -2C(H+1)S_w t^{2H}. \end{aligned} \tag{9}$$

3 DERIVATION OF $\mathrm{var}_f[\bar{\sigma}_t]$

For 1D fBm, we have

$$\begin{aligned} &\mathrm{var}_f[\bar{\sigma}_t(x)] \\ &= \frac{1}{t^2}\int_0^t\int_0^t C(H)(|x'-x''|^{2H} - |x'|^{2H} - |x''|^{2H})\mathrm{d}t'\mathrm{d}t'' \\ &= \frac{2C(H)}{2H+1}S_w\left\{\frac{t^{2H}}{2H+2} - \frac{(x+t)^{2H+1} - x^{2H+1}}{t}\right\} \end{aligned} \tag{10}$$

in which $x' = x + t'$ and $x'' = x + t''$. Note that this expression depends on the spatial position x and has a never-ending growth with respect to x as we claimed in the paper.

To use fBm in practice, it is required to define fBm on a limited scale from 0 to an outer-scale L. Now, performing spatial averaging on the above expression, we get

$$\begin{aligned}
&\mathrm{var}_f[\bar{\sigma}_t] \\
&= \frac{1}{L}\int_0^L \frac{2C(H)}{2H+1} S_w \left\{ \frac{t^{2H}}{2H+2} - \frac{(x+t)^{2H+1} - x^{2H+1}}{t} \right\} \\
&= -\frac{2C(H)}{2H+1} S_w \frac{(L+t)^{2H+2} - L^{2H+2} - t^{2H+2} - t^{2H+1}L}{(2H+2)tL}.
\end{aligned} \tag{11}$$

Using the expansion $(L+t)^{2H+2} = L^{2H+2} + (2H+2)tL^{2H+1} + \ldots + t^{2H+2}$, we further arrive at

$$\begin{aligned}
\mathrm{var}_f[\bar{\sigma}_t] &= -\frac{2C(H)}{2H+1} S_w \frac{(2H+2)tL^{2H+1} + \ldots - t^{2H+1}L}{(2H+2)tL} \\
&= -\frac{2C(H)}{2H+1} S_w \frac{(2H+2)L^{2H+1} + \ldots - t^{2H}L}{(2H+2)L}.
\end{aligned} \tag{12}$$

If we assume $L \gg t$, we will obtain

$$\mathrm{var}_f[\bar{\sigma}_t] \approx -\frac{2C(H)}{2H+1} S_w L^{2H} \tag{13}$$

considering that terms in "…" all contain t. This expression is consistent with the one derived in the paper using the one-point scale-independence property of fBm.

4 ONE-POINT SCALE-INDEPENDENCE OF FBM

In this section, we prove the one-point scale-independence property of fBm [Davis and Marshak 2004]. In the context of random extinction field σ_t of a fBm type, the property of one-point scale-independence requires:

(1) The ensemble average of the line-averaged extinction is the same as the ensemble average of the extinction itself, i.e.,

$$\langle \bar{\sigma}_t \rangle = \langle \sigma_t \rangle. \tag{14}$$

(2) The variance of the line-averaged field and the variance of the field itself differ at most by a small amount on the order of a very small ratio, i.e.,

$$\frac{\mathrm{var}[\bar{\sigma}_t]}{\mathrm{var}[\sigma_t]} - 1 = O\left(\left(\frac{t}{L}\right)^{2H}\right). \tag{15}$$

It is easy to prove the first requirement. For the second requirement, we know that the variance of the line-averaged field is given by

$$\begin{aligned}
&\mathrm{var}_f[\bar{\sigma}_t] \\
&= -\frac{2C(H)}{2H+1} S_w \frac{(L+t)^{2H+2} - L^{2H+2} - t^{2H+2} - t^{2H+1}L}{(2H+2)tL}
\end{aligned} \tag{16}$$

while the variance of the field itself is

$$\begin{aligned}
\mathrm{var}_f[\sigma_t] &= \frac{1}{L}\int_0^L -2C(H)S_w |x|^{2H} \mathrm{d}x \\
&= -\frac{2C(H)S_w}{2H+1} L^{2H}.
\end{aligned} \tag{17}$$

Their ratio is

$$\begin{aligned}
\frac{\mathrm{var}[\bar{\sigma}_t]}{\mathrm{var}[\sigma_t]} &= \frac{(L+t)^{2H+2} - L^{2H+2} - t^{2H+2} - t^{2H+1}L}{(2H+2)tL^{2H+1}} \\
&= \frac{1}{2H+2}\left\{ \frac{L}{t}\left[\left(1+\frac{t}{L}\right)^{2H+2} - 1\right] \right. \\
&\quad \left. - \left(\frac{t}{L}\right)^{2H+1} - \left(\frac{t}{L}\right)^{2H} \right\}.
\end{aligned} \tag{18}$$

When $L \gg t$, $\frac{t}{L} \to 0$, we have

$$\lim_{\frac{t}{L}\to 0} \frac{L}{t}\left[\left(1+\frac{t}{L}\right)^{2H+2} - 1\right] = 2H+2. \tag{19}$$

Then,

$$\begin{aligned}
\frac{\mathrm{var}[\bar{\sigma}_t]}{\mathrm{var}[\sigma_t]} - 1 &= -\frac{1}{2H+2}\left[\left(\frac{t}{L}\right)^{2H+1} + \left(\frac{t}{L}\right)^{2H}\right] \\
&= O\left(\left(\frac{t}{L}\right)^{2H}\right).
\end{aligned} \tag{20}$$

5 DERIVATION OF $\mathrm{var}_{kf}[\bar{\sigma}_t]$

Based on the property of one-point scale-independence of k-fBm, we can obtain $\mathrm{var}_{kf}[\bar{\sigma}_t]$ using the variance of the field σ_t itself. In the case of k-fBm, the variance of σ_t is given by

$$\begin{aligned}
\mathrm{var}_{kf}[\sigma_t] &= \frac{1}{L}\int_0^L -2C(H)S_w \sum_{j=0}^{k-1} (-1)^j \binom{2H}{j} x^{2H} \mathrm{d}x \\
&= -\frac{2C(H)}{2H+1} S_w \sum_{j=0}^{k-1} (-1)^j \binom{2H}{j} L^{2H} \\
&= \frac{2C(H)S_w(-1)^k}{2H+1} \binom{2H-1}{k-1} L^{2H}
\end{aligned} \tag{21}$$

using the fact that

$$\sum_{j=0}^{k-1} (-1)^j \binom{2H}{j} = (-1)^{k-1} \binom{2H-1}{k-1}. \tag{22}$$

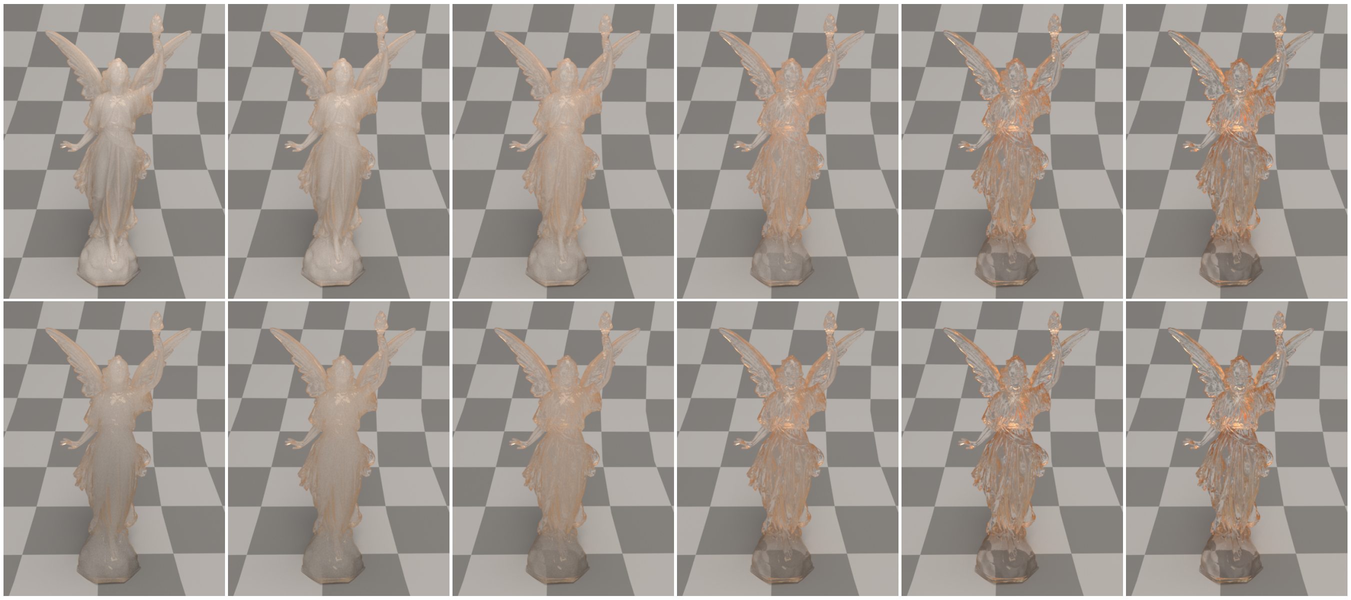
This is also the expression of $\mathrm{var}_{kf}[\bar{\sigma}_t]$.

6 ADDITIONAL RESULTS WITH ANISOTROPIC PHASE FUNCTIONS

In this section, we show the influence of phase functions on the appearance of spatially-correlated media. In Fig. 1, we choose the Henyey-Greenstein (HG) phase function parameterized by g (the asymmetry parameter) and render a homogeneous medium under different settings. Recall that negative values of g correspond to back-scattering while positive values correspond to forward-scattering. As seen, the phase function has a remarkable effect on the appearance when the H parameter is small. The influence weakens as H increases.

7 DISCUSSION ON NEGATIVE CORRELATIONS

If we use the pink-noise type transmittance function in our model and simply let $H \in (-1, -1/2)$, we will get faster-than-exponential attenuations as shown in Fig. 2. This is a typical characteristic of



(a) $H = -0.4$ (b) $H = -0.1$ (c) $H = 0.5$ (d) $H = 0.8$ (e) $H = 1.5$ (f) $H = 1.8$

Fig. 1. Effects of spatial correlations in random media with anisotropic HG phase functions. The symmetry parameter g is set to -0.5 (top row) and 0.5 (bottom row), respectively.

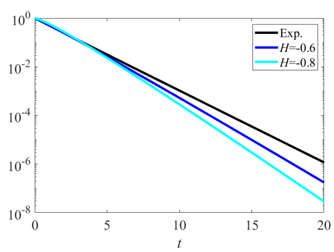


Fig. 2. Negative correlations can be achieved empirically by set $H \in (-1, -1/2)$ in the pink-noise type transmittance function.

Fig. 3. Comparison between negative (left) and positive (right) correlations.

the negatively-correlated media, implying that our model can also support negative correlations to a certain extent. See the visual comparison in Fig. 3. However, we should emphasis that this simple extension is not physically-based because when $H < -1/2$ we no longer have a straightforward Wiener-Khinchin connection between the PSD and the autocovariance function [Davis and Mineev-Weinstein 2011].

8 DISCUSSION ON WHITE NOISE

Our method converges to the classical transport with exponential falloff when the FGF is white noise. For white noise ($H = -1/2$), the variance of $\bar{\sigma}_t$ reduces to

$$\mathrm{var}[\bar{\sigma}_t] = S_w t^{-1} \tag{23}$$

and the transmittance is given by

$$\mathrm{Tr}(t) = \left(1 + \frac{S_w}{\sigma_m}\right)^{-\frac{\sigma_m^2}{S_w} t}. \tag{24}$$

Rearranging the above equation, we get

$$\mathrm{Tr}(t) = e^{-\ln\left(1+\frac{S_w}{\sigma_m}\right)\frac{\sigma_m^2}{S_w} t} \tag{25}$$

which means the transmittance is exponential in this case and the effective extinction is given by $\ln\left(1 + \frac{S_w}{\sigma_m}\right)\frac{\sigma_m^2}{S_w}$. Fig. 4 verifies that our model converges to the classical exponential transmittance with extinction $\ln\left(1 + \frac{S_w}{\sigma_m}\right)\frac{\sigma_m^2}{S_w}$ when H approaches $-1/2$.

(a) Ours $H = -0.49$ (b) Exp.

Fig. 4. Our method with a Hurst parameter close to $-1/2$ (a) converges to the classical transport with exponential falloff (b).

If we further assume that S_w is very small (much smaller than σ_m), the above expression of transmittance simplified to

$$\mathrm{Tr}(t) = -e^{\sigma_m t} \tag{26}$$

based on the fact $\ln\left(1 + \frac{S_w}{\sigma_m}\right) \approx \frac{S_w}{\sigma_m}$. This gives the classical transmittance without micro-scale fluctuations.

REFERENCES

Anthony B. Davis and Alexander Marshak. 2004. Photon propagation in heterogeneous optical media with spatial correlations: enhanced mean-free-paths and wider-than-exponential free-path distributions. *Journal of Quantitative Spectroscopy and Radiative Transfer* 84, 1 (2004), 3 – 34.

Anthony B. Davis and Mark Mineev-Weinstein. 2011. Radiation propagation in random media: From positive to negative correlations in high-frequency fluctuations. *Journal of Quantitative Spectroscopy and Radiative Transfer* 112 (2011), 632–645.