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Alternate Lucas Cubes

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We introduce alternate Lucas cubes, a new family of graphs designed as an alternative for the well known Lucas cubes. These interconnection networks are subgraphs of Fibonacci cubes and have a useful fundamental decomposition similar to the one for Fibonacci cubes. The vertices of alternate Lucas cubes are constructed from binary strings that are encodings of Lucas representation of integers. As well as ordinary hypercubes, Fibonacci cubes and Lucas cubes, alternate Lucas cubes have several interesting structural and enumerative properties. In this paper we study some of these properties. Specifically, we give the fundamental decomposition giving the recursive structure, determine the number of edges, number of vertices by weight, the distribution of the degrees; as well as the properties of induced hypercubes, q-cube polynomials and maximal hypercube polynomials. We also obtain the irregularity polynomials of this family of graphs, determine the conditions for Hamiltonicity, and calculate metric properties such as the radius, diameter, and the center.

Keywords: Hypercube; Fibonacci cube; Lucas cube; Alternate Lucas cube.

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1. Introduction

The hypercube graph $Q_n = (V_n, E_n)$ of dimension n is one of the basic models for interconnection networks. The vertex set V_n denotes the processors and the edge set E_n corresponds to the communication links between processors in an ideal interconnection network. The vertices of Q_n are represented by binary strings of length n with an edge between two vertices if and only if they differ in exactly one position. Q_n is a regular graph but the number of vertices grows very rapidly, as n increases.

Fibonacci cubes and Lucas cubes were introduced as new models of computation for interconnection networks [10, 16]. The growth rate of the number of vertices of these graphs are slower than the one of Q_n. Both families are subgraphs of Q_n and admit decompositions that allow recursive constructions. Fibonacci cubes decompose into two smaller Fibonacci cubes and a perfect matching, whereas in the analogous decomposition of Lucas cubes, instead of two lower dimensional Lucas cubes and a perfect matching, the decomposition is again in terms of a pair of lower dimensional Fibonacci cubes. Both families have interesting structural, metric, combinatorial and enumerative properties [10, 13, 16]. Similar results are also obtained for the graphs derived from hypercubes: generalized Fibonacci cubes [11], generalized Lucas cubes [12] and k-Fibonacci cubes [6].

We introduce *alternate Lucas cubes*, a new family of graphs whose number of vertices and the number of edges are equinumerous with those of Lucas cubes. As is the case with Lucas cubes, they are also induced subgraphs of Fibonacci cubes. Alternate Lucas cubes have a useful fundamental (canonical) decomposition similar to that of the Fibonacci cubes; they are constructed from two smaller alternate Lucas cubes and a perfect matching.

Alternate Lucas cubes have many interesting structural and enumerative properties. In this paper we describe the canonical decomposition which parallels the decomposition of Fibonacci cubes and then make use of this recursive structure. After the preliminaries in the next section we consider the canonical decomposition and the properties of the vertex labels in Sec. 3. In Sec. 4, we present enumerative properties such as the number of edges, number of vertices by weight, the distribution of the degrees, properties of induced hypercubes, q-cube and maximal hypercube polynomials. We obtain the irregularity polynomials of alternate Lucas cubes in Sec. 5. Hamiltonicity is considered in Sec. 6 where we give the conditions for the existence of Hamiltonian cycles and Hamiltonian paths, and give a construction for Hamiltonian paths. Finally we give a number of metric properties such as the radius, diameter and the center in Sec. 7 and show that the diameter of alternate Lucas cubes is less than or equal to the diameter of Lucas cubes.

2. Preliminaries

First we present some notation and preliminary results. We start with the description of a hypercube. The n-dimensional hypercube (or n-cube) Q_n is the simple graph with vertex set consisting of the 2^n binary strings

$$V_n = \{b_n \ldots b_2 b_1 \mid b_i \in \{0,1\}, 1 \leq i \leq n\}.$$

The edges of Q_n are between pairs of vertices differing in exactly one bit. The Fibonacci cube Γ_n is the induced subgraph of Q_n, obtained from Q_n by removing all vertices containing consecutive 1s. The number of vertices of Γ_n is F_{n+2}, where $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ are the Fibonacci numbers. If we remove the vertices with $b_1 = b_n = 1$ from Γ_n, then we obtain the Lucas cube Λ_n. For $n \geq 1$, Λ_n has L_n vertices, where $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ are the Lucas numbers.

Here we note that every positive integer can be represented uniquely as the sum of distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This representation is called the *Zeckendorf representation*, sometimes also called the *canonical representation*. By convention we assume that the integer 0 is represented by the n-bit string $(0 \ldots 0)$ when we are considering n-dimensional graphs with binary labels.

A similar representation of integers using Lucas numbers is considered in [4] where it is shown that every positive integer n can be expressed uniquely as a sum of distinct Lucas numbers in the form

$$n = \sum_{i \geq 0} b_{i+1} L_i,$$

where $b_i \cdot b_{i+1} = 0$ for $i \geq 1$ and $b_1 \cdot b_3 = 0$. We call this representation the *Lucas representation* of integers, or sometimes the Lucas basis. We will also refer to the binary encoding of an integer via its coefficients b_i in this representation as its *binary alternate Lucas string*. Lucas representation of the integers $n = 0, 1, \ldots, 6$ and their corresponding binary encodings are given in Table 1.

The n-dimensional alternate Lucas cube $\mathcal{L}_n$ is defined as the induced subgraph of Q_n obtained by removing vertices from Q_n that do not correspond to binary alternate Lucas strings. More precisely,

$$V(\mathcal{L}_n) = \{b_n \ldots b_2 b_1 \mid b_i \cdot b_{i+1} = 0 \text{ for } 1 \leq i < n \text{ and } b_1 \cdot b_3 = 0\} \subseteq V_n.$$

Example 1. At the top of Fig. 1, the first four Lucas cubes are presented with their vertices labeled with the corresponding binary strings in the hypercube graph.

Table 1. Lucas representation of $n = 0, 1, \ldots, 6$ and their binary encoding as used as vertex labels in the construction of the alternate Lucas cubes.

n	Lucas representation	Binary encoding $b_4b_3b_2b_1$
0	0	0000
1	L_1	0010
2	L_0	0001
3	L_2	0100
4	L_3	1000
5	$L_3 + L_1$	1010
6	$L_3 + L_0$	1001

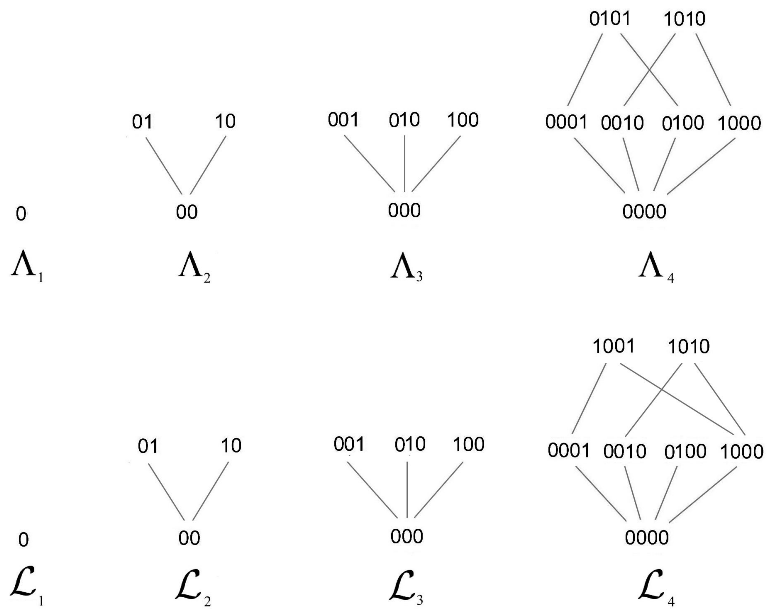

Fig. 1. Lucas cubes $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ and the alternate Lucas cubes $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$.

At the bottom of Fig. 1, the first four alternate Lucas cubes are presented with their labels that are their digits in the Lucas representation. The first three cubes $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are identical to the first three Lucas cubes Λ_1, Λ_2, Λ_3. However $\mathcal{L}_4$ is not isomorphic to Λ_4 because of the existence of a vertex of degree 3 in $\mathcal{L}_4$. This vertex has label 1000 as shown in Fig. 1.

In fact we can show via our calculation of the degree sequences of the family $\mathcal{L}_n$ and comparing with the known degree sequence of the Lucas cubes Λ_n that $\mathcal{L}_n$ is not isomorphic to Λ_n for $n \geq 4$.

The following decompositions of Γ_n and Λ_n can be obtained easily from the definitions.

For the Fibonacci cubes, one can classify the binary strings defining the vertices of Γ_n according to whether $b_n = 0$ or $b_n = 1$. In this way Γ_n decomposes into a subgraph Γ_{n-1}, whose vertices are given by the strings that start with 0, and a subgraph Γ_{n-2} whose vertices are given by the strings that start with 10. This decomposition can be denoted as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}. \tag{1}$$

Furthermore, Γ_{n-1} in turn has a subgraph Γ'_{n-2} (whose vertices start with 00 in Γ_n) isomorphic to Γ_{n-2}. Each vertex of this Γ'_{n-2} is connected by an edge to its twin in Γ_{n-2}. In other words, there is a perfect matching between these two copies of Γ_{n-2}. This is the *fundamental decomposition* of Γ_n, also referred to as its canonical decomposition.

Λ_n has a decomposition that comes from the classification of the binary strings defining its vertices. Λ_n has a subgraph Γ_{n-1} whose vertices are denoted by the corresponding strings starting with 0 and a subgraph Γ_{n-3} whose vertices are given by the strings that start with 10 and end with 0 in Λ_n. Furthermore, in this decomposition Γ_{n-1} has a subgraph Γ'_{n-3} (whose vertices start with 00 and end with 0) isomorphic to Γ_{n-3}, and each vertex of Γ'_{n-3} is connected by an edge to its twin in Γ_{n-3}. This decomposition is denoted by

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0. \tag{2}$$

However note that the lower dimensional graphs that appear in the decomposition of Λ_n are not Lucas cubes, but Fibonacci cubes.

In the fundamental decomposition for Fibonacci cubes, there are F_n edges between Γ'_{n-2} and Γ_{n-2}. For Lucas cubes, there are F_{n-1} edges between Γ'_{n-3} and Γ_{n-3} that arise in the decomposition.

In Fig. 2, we present Fibonacci cubes and their subgraphs Lucas and alternate Lucas cubes for $n = 1, 2, 3, 4$.

In Fig. 3, the labels of Λ_4 are given using the hypercube's binary digits as the Zeckendorf expansion of integers. The corresponding labeling of the vertices of $\mathcal{L}_4$ are the binary strings interpreted as the expansion of the labels in the Lucas basis. The newly added material, i.e. the subgraph induced by the vertices $\{4, 5, 6\}$ which

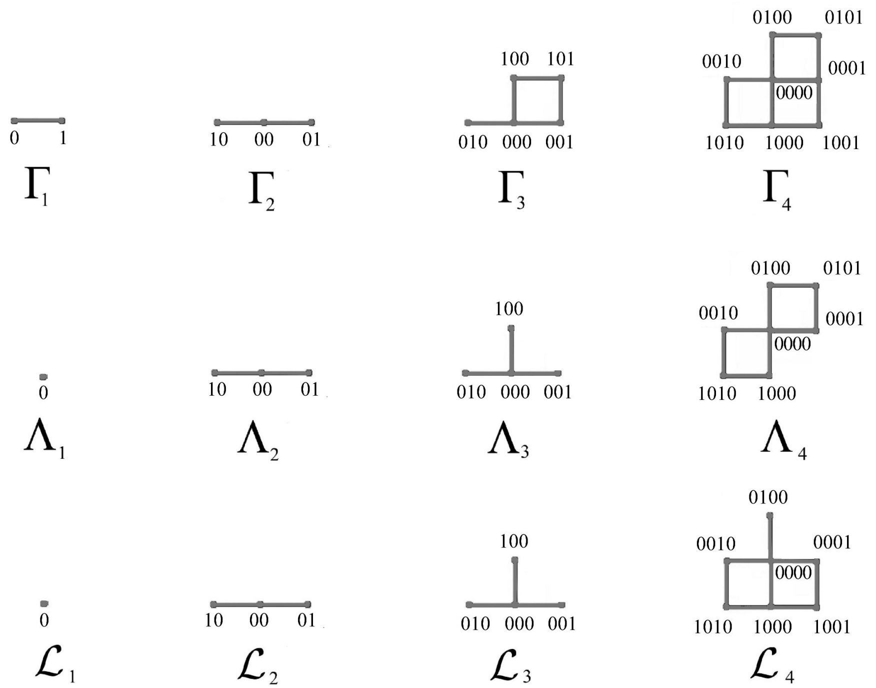

Fig. 2. Fibonacci, Lucas and alternate Lucas cubes Γ_n, Λ_n and $\mathcal{L}_n$ for $n = 1, 2, 3, 4$.

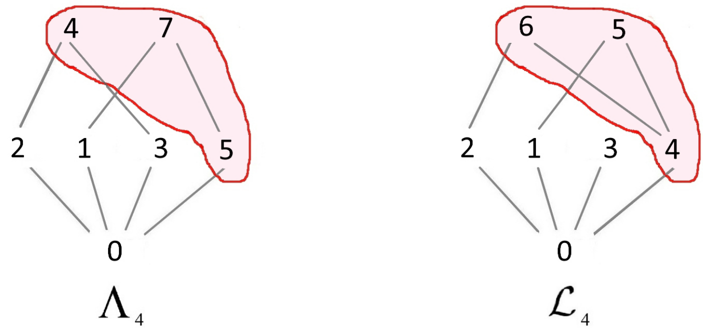

Fig. 3. Λ_4 and $\mathcal{L}_4$ with the labels of the newly added vertices indicated in decimal.

we view as having been added to the graph on the vertices induced by $\{0, 1, 2, 3\}$ in $\mathcal{L}_3$ is easily seen to be isomorphic to $\mathcal{L}_2$ as shown in Fig. 3. On the other hand if we consider the new vertices in Λ_4 that were added to Λ_3 on integer labels $\{0, 1, 2, 3\}$, this property fails to hold. Here 6 does not appear as a label since its Zeckendorf expansion starts and ends with a 1. We also note that the subgraph corresponding to the new labels $\{4, 5, 7\}$ has an isolated vertex, and therefore not isomorphic to Λ_2.

It is worth emphasizing that the n-bit binary representations of the integer labels $0, 1, \ldots, 2^n - 1$ correspond to the binary labels of the vertices of the hypercube graph Q_n. Similarly, the n-bit Zeckendorf representations of the integer labels $0, 1, \ldots, F_{n+2} - 1$ correspond to the binary labels of the vertices of the Fibonacci cube Γ_n. This property is carried over to alternate Lucas cubes; the n-bit Lucas representations of the integer labels $0, 1, \ldots, L_n - 1$ correspond to the binary labels of the vertices of the alternate Lucas cube $\mathcal{L}_n$. This pleasing property is missing in the classical Lucas cubes.

3. Vertex Labels and the Canonical Decomposition

In the definition of the Lucas cubes, the binary strings $b_n \ldots b_2 b_1$ that are the labels of the vertices satisfy $b_i \cdot b_{i+1} = 0$ for $i = 1, 2, \ldots, n-1$ as in the case of the Fibonacci graphs, and in addition also $b_1 \cdot b_n = 0$. There are some obvious reasons why this definition has been in use for defining Lucas cubes. For one thing, the number of such strings is the Lucas number L_n, so the number of vertices of Λ_n is L_n. For another, there is the pleasing symmetry afforded by viewing $b_n \ldots b_2 b_1$ circularly as a necklace, making b_1 adjacent to b_n, and then requiring the product of adjacent bits to be zero in this setting, generalizing the requirement of the case of Fibonacci cubes. However other than the notion of circularity and forbidden adjacent pairs of 1s, the mapping of the binary strings to the vertex names in Λ_n (as integer labels) is not satisfactory with this definition.

The vertices of Γ_n may be assigned consecutive integers from 0 to $F_{n+2} - 1$ where binary strings corresponding to these numbers are given by their Zeckendorf representations. Let us first start with Fibonacci cubes Γ_n to see this labeling property. The number of vertices in Γ_n is F_{n+2}, the number of n-bit Fibonacci strings. There is a one to one correspondence between these strings and integers

$$0, 1, \ldots, F_{n+2} - 1$$

with which the vertices of Γ_n are labeled. The correspondence is simply viewing the Fibonacci strings as the Zeckendorf expansion digits, where the rightmost digit corresponds to F_2, the second from the right to F_3 and so on. This labeling behaves extremely well in the case of Γ_n with respect to the fundamental recursion, as in the building of Γ_n from the two lower dimensional graphs Γ_{n-1} and Γ_{n-2}, the vertices of the form $0\Gamma_{n-1}$ retain their old labels as the integers

$$0, 1, \ldots, F_{n+1} - 1,$$

and the newly added vertices in $10\Gamma_{n-2}$ now have labels immediately following these sequentially as

$$F_{n+1}, F_{n+1} + 1, \ldots, F_{n+2} - 1,$$

making up the totality of the vertex labels $0, 1, \ldots, F_{n+2} - 1$ of Γ_n.

Lucas cubes Λ_n do not enjoy this property. In fact it can be argued that the symmetry of the idea of a circular string seems to work against the labeling of the vertices of Lucas cubes (see Fig. 3).

The alternate Lucas cube $\mathcal{L}_n$ can be decomposed into two subgraphs induced by the vertices that start with 0 and 10 respectively. The vertices that start with 0 constitute a graph isomorphic to $\mathcal{L}_{n-1}$ and the vertices that start with 10 constitute a graph isomorphic to $\mathcal{L}_{n-2}$. Additionally, there is a perfect matching between these two subgraphs, analogous to the decomposition of Fibonacci cubes. For $n \geq 3$ we denote this decomposition of $\mathcal{L}_n$ symbolically as

$$\mathcal{L}_n = 0\mathcal{L}_{n-1} + 10\mathcal{L}_{n-2} \tag{3}$$

just as we did for the case of the Fibonacci cubes themselves. In (3), there are L_{n-2} edges in the perfect matching between the vertices in $10\mathcal{L}_{n-2}$ and the corresponding vertices in $00\mathcal{L}_{n-2} \subset 0\mathcal{L}_{n-1}$, in complete analogy with the Fibonacci decomposition (see Fig. 4). Of course in the case of Γ_n, the corresponding perfect matching is enumerated by Fibonacci numbers.

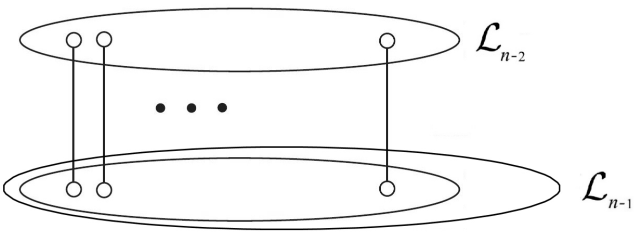

Fig. 4. Recursive decomposition of the alternate Lucas cube $\mathcal{L}_n$ in terms of $\mathcal{L}_{n-1}$ and $\mathcal{L}_{n-2}$, cf. [13, Fig. 3].

Remark 2. The canonical decomposition of Γ_n in (1) reflects the recursion $F_n = F_{n-1} + F_{n-2}$ whereas the corresponding decomposition of Λ_n in (2) reflects the well known identity $L_n = F_{n+1} + F_{n-1}$. The decomposition of alternate Lucas cubes as defined here in (3) corresponds directly to the numerical recursion $L_n = L_{n-1} + L_{n-2}$.

The decomposition (3) also has the welcome property that the labeling of the vertices behaves just as well as the case of Γ_n. In the construction of $\mathcal{L}_n$ from the two lower dimensional graphs $\mathcal{L}_{n-1}$ and $\mathcal{L}_{n-2}$, the vertices of the form $0\mathcal{L}_{n-1}$ retain their old labels as the integers

$$0, 1, \ldots, L_{n-1} - 1,$$

and the newly added vertices in $10\mathcal{L}_{n-2}$ now have labels immediately following these sequentially as

$$L_{n-1}, L_{n-1} + 1, \ldots, L_n - 1,$$

making up the totality of the vertex labels $0, 1, \ldots, L_n - 1$ of $\mathcal{L}_n$.

In the example given in Fig. 3, $n = 4$ and these vertex labels are $0, 1, 2, 3$ for $\mathcal{L}_3$ followed by $4, 5, 6$ for $\mathcal{L}_2$, with the corresponding vertex labels in binary that come from the Lucas basis expansions shown in Table 1.

4. Enumerative Properties of Alternate Lucas Cubes

By the definition of alternate Lucas cubes, we know that the number of vertices of $\mathcal{L}_n$ is equal to the number of vertices of Λ_n, that is, $|V(\mathcal{L}_n)| = |V(\Lambda_n)| = L_n$.

4.1. *The number of edges*

Denote the number of edges of $\mathcal{L}_n$ by e_n. First few values for $n \geq 2$ are

$$2, 3, 8, 15, 30, 56, 104, 189, 340, 605, \ldots$$

By the fundamental decomposition (3), the edges of $\mathcal{L}_n$ are of three types: those that are from $\mathcal{L}_{n-1}$, those that are from $\mathcal{L}_{n-2}$, and the L_{n-2} link edges that are added between the twin nodes in the two copies of $\mathcal{L}_{n-2}$. This gives the recursion

$$e_n = e_{n-1} + e_{n-2} + L_{n-2} \tag{4}$$

for $n \geq 3$. By using the well known relation

$$L_{n-2} = F_{n-1} + F_{n-3}$$

relating the Lucas and the Fibonacci numbers and the initial values, we find that the solution to the recursion (4) is given by $e_n = nF_{n-1}$. Therefore, the number of edges of $\mathcal{L}_n$ is identical to the number of edges of the Lucas cube Λ_n itself (see [16]).

Proposition 3. *The number of edges of alternate Lucas cube $\mathcal{L}_n$ is nF_{n-1}.*

4.2. *Number of vertices by weight*

The weight of a vertex v in $\mathcal{L}_n$ is its Hamming weight, in other words the number of 1s in its binary alternate Lucas string.

We can consider $\mathcal{L}_n$ as the Hasse diagram of the ranked poset of binary alternate Lucas strings of length n, where the covering relation is flipping a 0 to a 1. In this setting the weight of a vertex is its rank in the poset.

Let $r_{n,w}$ denote the number of elements of weight w in $\mathcal{L}_n$. Similar to the case of Fibonacci cubes [17, Remark 3.3] and Lucas cubes [14, Corollary 5.3.] we obtain

Proposition 4. *The number of vertices of weight w in $\mathcal{L}_n$ is given by*

$$r_{n,w} = \frac{n}{n-w}\binom{n-w}{w}. \tag{5}$$

Proof. From the fundamental decomposition (3) of alternate Lucas cubes, we obtain the recurrence relation

$$r_{n,w} = r_{n-1,w} + r_{n-2,w-1} \tag{6}$$

for $n \geq w \geq 1$ with $r_{n,0} = 1$ for $n \geq 1$ and $r_{n,1} = n$ for $n \geq 2$. The solution to (6) is given by (5). $\square$

We note that it follows from (5) that as a ranked poset, the rank generating function of $\mathcal{L}_n$ is given in closed form as

$$F(\mathcal{L}_n, q) = \sum_{w=0}^{\lfloor n/2 \rfloor} r_{n,w} q^w = 2^{-n}((1-\sqrt{1+4q})^n + (1+\sqrt{1+4q})^n).$$

4.3. *Degree sequences*

By using the fundamental decomposition (3) of alternate Lucas cubes we can find the degree sequence of $\mathcal{L}_n$ by considering the contribution of the vertices in $0\mathcal{L}_{n-1}$ and $10\mathcal{L}_{n-2}$ separately. For $n \geq 1$ and $0 \leq k \leq n$, let $a_{n,k}$ and $b_{n,k}$ denote the number of vertices of $10\mathcal{L}_{n-2}$ and $0\mathcal{L}_{n-1}$ of degree k respectively. Let

$$a(x,y) = \sum_{n,k\geq 0} a_{n,k} x^n y^k \quad \text{and} \quad b(x,y) = \sum_{n,k\geq 0} b_{n,k} x^n y^k$$

be their generating functions. Comparing with the linear system for the Fibonacci cubes that is derived in [14] and using the same notation, we see that these generating functions satisfy the similar linear system of equations below:

$$\begin{aligned} a(x,y) - 2x^2y &= x^2ya(x,y) + x^2yb(x,y) \\ b(x,y) - x + x^2y - x^2y^2 &= xyb(x,y) + xa(x,y). \end{aligned}$$

The solutions are

$$a(x,y) = \frac{x^2y(2+x-2xy-x^2y+x^2y^2)}{(1-xy)(1-x^2y)-x^3y}$$

$$b(x,y) = \frac{x(1-xy+x^2y+xy^2+x^3y^2-x^3y^3)}{(1-xy)(1-x^2y)-x^3y}.$$

In particular by adding the two, the degree enumerator polynomial for the alternate Lucas cube $\mathcal{L}_n$ is calculated as

$$\begin{aligned} &\frac{x(1+y(1+y)x+2y(1-y)x^2)}{(1-yx)(1-yx^2)-yx^3} \\ &\quad = \frac{x+yx^2+y^2x^2+2yx^3-2y^2x^3}{(1-yx)(1-yx^2)-yx^3} \\ &\quad = x+(2y+y^2)x^2+(3y+y^3)x^3+(y+4y^2+y^3+y^4)x^4 \\ &\qquad +(6y^2+3y^3+y^4+y^5)x^5+(4y^2+7y^3+5y^4+y^5+y^6)x^6+\cdots. \end{aligned} \tag{7}$$

Using technical computations which are standard, we obtain the following result along the lines of the proof of [14, Theorem 1.1].

Theorem 5. *Let $\ell_{n,k}$ denote the number of vertices of $\mathcal{L}_n$ of degree k for $n \geq k \geq 1$. Then*

$$\begin{aligned} \ell_{n,k} = \sum_{j=0}^{k} \Bigg[& 2\binom{n-2j-3}{k-j-1}\binom{j+1}{n-k-j-1} + \binom{n-2j-2}{k-j-2}\binom{j}{n-k-j} \\ & + \binom{n-2j-2}{k-j}\binom{j}{n-k-j-1} \Bigg]. \end{aligned}$$

Proof. By using Newton's expansion formula $\frac{z^k}{(1-z)^{k+1}} = \sum_{i \geq k} \binom{i}{k} z^i$, it was shown in [14] that

$$\frac{1}{(1-xy)(1-x^2y)-x^3y} = \sum_{n,k,j \geq 0} \binom{n-2j}{k-j}\binom{j}{n-k-j} x^n y^k. \tag{8}$$

To find the number of vertices of $\mathcal{L}_n$ of degree k we need to find the coefficient of the monomial $x^n y^k$ in the expansion of (7). Let $c_{n,k}$ be the coefficient of the monomial $x^n y^k$ in (8). Then using (7) we can write

$$\ell_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-2,k-2} + 2c_{n-3,k-1} - 2c_{n-3,k-2}. \tag{9}$$

Substituting $c_{n,k} = \sum_{j \geq 0} \binom{n-2j}{k-j}\binom{j}{n-k-j}$ in (9) and using the properties of binomial coefficients we get the desired result. $\square$

Next we consider a refinement of this result by keeping track of the weight of the vertex in the computation as well. Let $\ell_{n,k,w}$ be the number of vertices of $\mathcal{L}_n$ of degree k and weight w. Again, using techniques similar to the ones in [14, Sec. 4] we can determine $\ell_{n,k,w}$. From

$$\mathcal{L}_n = 0\mathcal{L}_{n-1} + 10\mathcal{L}_{n-2} = (00\mathcal{L}_{n-2} + 010\mathcal{L}_{n-3}) + 10\mathcal{L}_{n-2}$$

we obtain the recurrence relation

$$\ell_{n,k,w} = \ell_{n-1,k-1,w} + \ell_{n-2,k-1,w-1} + \ell_{n-3,k-1,w-1} - \ell_{n-3,k-2,w-1}.$$

Denote by $\ell(x, y, z)$ the generating function of this sequence so that $\ell_{n,k,w}$ is the coefficient of the term $x^n y^k z^w$ in the series expansion. Then

$$\ell(x,y,z) = \frac{2x^3yz + x^2y^2 + 2x^2yz + x - 2x^3y^2z - x^2y}{(1-xy)(1-x^2yz) - x^3yz}.$$

Our proof of the following proposition makes use of the results in [14, Sec. 4].

Proposition 6. *For $0 \le w \le k \le n$, the number of vertices of $\mathcal{L}_n$ having degree k and weight w is*

$$\begin{aligned}\ell_{n,k,w} = &\binom{w}{n-k-w}\left[2\binom{n-2w-1}{k-w} + \binom{n-2w-2}{k-w-2}\right] \\ &+ \binom{w}{n-k-w-1}\binom{n-2w-2}{k-w}.\end{aligned}$$

Proof. By direct inspection the claim holds for $n \le 5$. Let $f_{n,k,w}$ be the number of vertices of Γ_n having degree k and weight w and $s^0_{n,k,w}$ be the number of such vertices of Γ_n whose last bit is 0. Any vertex of $\mathcal{L}_n$ can be written as αabc where α is a Fibonacci string of length $n-3$ and abc is a Fibonacci string not equal to 101. Assume that αabc corresponds to a vertex of $\mathcal{L}_n$ having degree k and weight w. Then for $n \ge 5$ we have the following cases:

(1) If $abc = 001$ or $abc = 010$, then α corresponds to a vertex of Γ_{n-3} having weight $w-1$ and degree $k-1$. This is because αabc has neighbors of the form βabc and $\alpha 000$ in $\mathcal{L}_n$, where α and β correspond to two adjacent vertices of Γ_{n-3}. It follows that the number of such vertices is $2f_{n-3,k-1,w-1}$.

(2) If $abc = 100$, then α must end with a 0, that is, $\alpha = \alpha_1 0$ and α_1 corresponds to a vertex of Γ_{n-4} having weight $w-1$ and degree $k-1$. This is because $\alpha_1 0100$ has neighbors of the form $\beta_1 0100$ and $\alpha 000$ in $\mathcal{L}_n$, where α_1 and β_1 correspond to two adjacent vertices of Γ_{n-4}. Then the number of such vertices is $f_{n-4,k-1,w-1}$.

(3) If $abc = 000$, then we have two subcases:

   (a) Assume that α ends with a 1, that is, $\alpha = \alpha_2 01$. Then α_2 corresponds to a vertex of Γ_{n-5} having weight $w-1$ and degree $k-3$, since $\alpha_2 01000$ has neighbors of the form $\beta_2 01000$, $\alpha_2 00000$, $\alpha 010$ and $\alpha 001$ in $\mathcal{L}_n$, where α_2 and β_2 correspond to two adjacent vertices of Γ_{n-5}. The number of such vertices is $f_{n-5,k-3,w-1}$.

(b) Assume that α ends with a 0, that is, $\alpha = \alpha_1 0$. Then α corresponds to a vertex of Γ_{n-3} having weight w and degree $k-3$, since $\alpha000$ has neighbors of the form $\beta000$, $\alpha100$, $\alpha010$ and $\alpha001$ in $\mathcal{L}_n$, where α and β correspond to two adjacent vertices of Γ_{n-3}. Since α corresponds to a vertex of Γ_{n-3} ending with a 0, the number of such vertices is $s^0_{n-3,k-3,w}$.

It was shown in [14, Theorem 4.6] that

$$f_{n,k,w} = \binom{w+1}{n-k-w+1}\binom{n-2w}{k-w}.$$

Using [14, Lemma 4.5] we have

$$s^0_{n,k,w} = \binom{w}{n-k-w}\binom{n-2w-3}{k-w-3}.$$

Considering the cases above and using the binomial identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

we get the desired result. $\square$

Remark 7. Summing the formula in Proposition 6 over $w \geq 0$, we obtain an alternate expression for the $\ell_{n,k}$ given in Theorem 5:

$$\ell_{n,k} = \sum_{w\geq 0}\left[\binom{w}{n-k-w}\left[2\binom{n-2w-1}{k-w} + \binom{n-2w-2}{k-w-2}\right]\right.$$
$$\left. + \binom{w}{n-k-w-1}\binom{n-2w-2}{k-w}\right].$$

Remark 8. Similarly, the expression for $r_{n,w}$ in Proposition 4 can be written as the sum of $\ell_{n,k,w}$ over $k \geq 0$, giving the binomial identity

$$\frac{n}{n-w}\binom{n-w}{w} = \sum_{k\geq 0}\left[\binom{w}{n-k-w}\left[2\binom{n-2w-1}{k-w} + \binom{n-2w-2}{k-w-2}\right]\right.$$
$$\left. + \binom{w}{n-k-w-1}\binom{n-2w-2}{k-w}\right].$$

From the expression of the generating function (7), we immediately obtain the following recursion for the degree generating polynomial for $\mathcal{L}_n$. Set

$$g_n(y) = \sum_{k\geq 0} c_k y^k$$

where c_k is the number of vertices of degree k in $\mathcal{L}_n$ given in Theorem 5. Then

$$g_1(y) = 1, \quad g_2(y) = 2y + y^2, \quad g_3(y) = 3y + y^3$$

and for $n \geq 4$,

$$g_n(y) = yg_{n-1}(y) + yg_{n-2}(y) + y(1-y)g_{n-3}(y). \tag{10}$$

It is interesting that the denominators of the bivariate generating functions for the number of vertices of degree k in each of the families Γ_n, Λ_n and $\mathcal{L}_n$ are identical (see [14, Sec. 6]). Therefore, the corresponding degree generating polynomials for these families all satisfy the recursion in (10), except of course, they have different initial values.

The first four degree generating polynomials for the alternate Lucas cubes are

$$1, \quad 2y + y^2, \quad 3y + y^3, \quad y + 4y^2 + y^3 + y^4$$

whereas the corresponding polynomials for the Lucas cubes themselves are

$$1, \quad 2y + y^2, \quad 3y + y^3, \quad 6y^2 + y^4.$$

Clearly $\mathcal{L}_1$ through $\mathcal{L}_3$ are isomorphic to Λ_1 through Λ_3 but $\mathcal{L}_4$ is not isomorphic to Λ_4. In fact $\mathcal{L}_n$ and Λ_n are never isomorphic for $n \geq 4$. We record this as a proposition.

Proposition 9. *The alternate Lucas cube $\mathcal{L}_n$ and the Lucas cube Λ_n are not isomorphic for $n \geq 4$.*

Proof. For $n \geq 4$, $\mathcal{L}_n$ has a unique vertex of degree $n-1$, namely 10^{n-1}. Because of the additional requirement that $b_1 \cdot b_n = 0$ on the Fibonacci strings that constitute the vertices of the Lucas cubes, for the same values of n, Λ_n does not have vertex of degree $n-1$. □

The coefficient of the smallest degree term in $g_n(y)$ for the first few values of $n \geq 1$ are found to be

$$1, 2, 3, 1, 6, 4, 1, 11, 5, 1, 17, 6, 1, 24, 7, 1, 32, 8, 1, \ldots$$

The following result characterizes the smallest degree vertices in $\mathcal{L}_n$ as well as their number.

Proposition 10. *The smallest degree vertex in the alternate Lucas cube $\mathcal{L}_n$ has degree $\left\lfloor \frac{n+1}{3} \right\rfloor$. The number of such minimum degree vertices in $\mathcal{L}_n$ is*

$$\begin{array}{ll} m+2 & \textit{if } n = 3m, \\ 1 & \textit{if } n = 3m+1, \\ \dfrac{1}{2}(m^2+7m+4) & \textit{if } n = 3m+2. \end{array}$$

Proof. Let s_n be the smallest degree of a vertex in $\mathcal{L}_n$ and $a_{n,k}$ be the number of vertices of degree k in $\mathcal{L}_n$. Then write

$$g_n(y) = a_{n,s_n} y^{s_n} + \sum_{k > s_n} a_{n,k} y^k.$$

From the recurrence (10), s_n satisfies the recurrence relation

$$s_n = \min\{s_{n-1}+1, s_{n-2}+1, s_{n-3}+1\}$$

with $s_1 = 0, s_2 = s_3 = 1$. The result follows by induction on n by considering separately the three cases $n = 3m$, $n = 3m+1$ and $n = 3m+2$. The second part follows from Theorem 5. □

With a little more effort, we can actually obtain a more general version of the degree polynomial $g_n(y)$ whose recursive structure is given in (10). This generalization keeps track of the number of down-neighbors of a node v (i.e., the number of vertices obtained by changing a 1 to a 0), and the up-neighbors, which are the vertices obtained by changing a 0 to a 1 in v. These two quantities are denoted by $\deg_{down}(v)$ and $\deg_{up}(v)$, respectively. Clearly $\deg_{down}(v)$ is the Hamming weight of v and the sum of these two types of degrees is $\deg(v)$. Set

$$G_n(u,z) = \sum_{v\in\mathcal{L}_n} u^{\deg_{\mathrm{up}}(v)} z^{\deg_{\mathrm{down}}(v)}.$$

Clearly, $G_n(y,y) = g_n(y)$. Define the generating function of the polynomials $G_n(u,z)$ by

$$G(x) = G(u,z,x) = \sum_{n\geq 1} G_n(u,z)x^n.$$

Using standard technical computations that we omit here, the following closed form expression for $G(x)$ is obtained.

Theorem 11. *The generating function of the bivariate polynomials $G_n(u,z)$ is given by*

$$G(x) = \frac{x(1+(2z-u(1-u))x+2z(1-u)x^2)}{(1-ux)(1-zx^2)-zx^3}. \tag{11}$$

Rewriting (11) as a recurrence relation, we have

$$G_1(u,z) = 1, \quad G_2(u,z) = 2z+u^2, \quad G_3(u,z) = 3z+u^3$$

and for $n \geq 4$,

$$G_n(u,z) = uG_{n-1}(u,z) + zG_{n-2}(u,z) + z(1-u)G_{n-3}(u,z).$$

4.3.1. *Special cases of Theorem 11*

(1) Taking $u = z = 1$, in (11), we have the generating function of the Lucas numbers $L_1, L_2, \ldots$:

$$\frac{x+2x^2}{1-x-x^2}.$$

(2) Taking $u = z = y$, we have the generating function of the degree polynomials as given in (7).

(3) Taking $u = 1$ in $G_n(u, z)$, we obtain the weight enumerator polynomial $G_n(1, z)$ of $\mathcal{L}_n$. This specialization gives

$$\frac{x(1+2zx)}{1-x-zx^2} = x + (1+2z)x^2 + (1+3z)x^3 + (1+4z+2z^2)x^4 + (1+5z+5z^2)x^5 + (1+6z+9z^2+2z^3)x^6 + \cdots \tag{12}$$

(4) Similarly, the up-degree enumerators $G_n(u, 1)$ have the generating function

$$\begin{aligned} &\frac{x(1+(2-u(1-u))x+2(1-u)x^2)}{(1-ux)(1-x^2)-x^3} \\ &= x + (2+u^2)x^2 + (3+u^3)x^3 + (3+2u+u^2+u^4)x^4 \\ &\quad + (5+u+3u^2+u^3+u^5)x^5 + \cdots . \end{aligned} \tag{13}$$

(5) The number of edges between ranks w and $w-1$ in $\mathcal{L}_n$ is $wr_{n,w}$ where $r_{n,w}$ is the number of vertices of weight w. On the other hand, these edges can be counted as the sum of the up-degrees of all the vertices of weight $w-1$, which is the coefficient of z^{w-1} in $\frac{\partial}{\partial u}G_n(u, z)$ evaluated at $u = 1$. From (11), the generating function of $\frac{\partial}{\partial u}G_n(u, z)$ evaluated at $u = 1$ is

$$\frac{(2-x)x^2}{(1-x-zx^2)^2} = \sum_{w\geq 1} \frac{w(2-x)x^{2w}}{(1-x)^{w+1}} z^{w-1}$$

and the coefficient of z^{w-1} here is

$$\begin{aligned} \frac{w(2-x)x^{2w}}{(1-x)^{w+1}} &= w(2-x)x^{2w} \sum_{k\geq 0} \binom{k+w}{k} x^k \\ &= 2wx^{2w} + \sum_{k\geq 1} \left[2w\binom{k+w}{k} - w\binom{k+w-1}{k-1} \right] x^{k+2w}. \end{aligned} \tag{14}$$

Since the coefficient of x^n in (14) is equal to $wr_{n,w}$, we find

$$r_{n,w} = 2\binom{n-w}{w} - \binom{n-w-1}{w} = \frac{n}{n-w}\binom{n-w}{w}.$$

This is another derivation of the number of alternate Lucas strings of length n with weight w given in Proposition 4.

(6) In the generating function (11) the exponent of z is the weight of the vertex. If we replace z by zu, then the exponent of u now becomes the total degree of the vertex. Using the variable z for the weight and y for the total degree we obtain the generating function of the monomials that keep track of the length (as the exponent of x), weight (as the exponent of z) and the degree (as the exponent

of y) over all binary alternate Lucas strings as

$$\frac{x(1+(2zy-y(1-y))x+2zy(1-y)x^2)}{(1-yx)(1-zyx^2)-zyx^3}$$
$$=\frac{x(1-y(1-y)x)}{1-yx}+(2+(1-2y)x-y(1-y)x^2)$$
$$\times\sum_{k\geq 1}\left(\frac{y^k x^{2k}(1+(1-y)x)^k}{(1-yx)^{k+1}}\right)z^k.$$

This provides an alternate albeit more complicated expression for the number $\ell_{n,k,w}$ of vertices of degree k and weight w in $\mathcal{L}_n$ that was given in Proposition 6.

From Proposition 6, we obtain the distribution of the up-degrees in $\mathcal{L}_n$.

Proposition 12. *The number of vertices in $\mathcal{L}_n$ with up-degree k is given by*

$$\sum_{w\geq 0}\left[\binom{w}{n-k-2w}\left[2\binom{n-2w-1}{k}+\binom{n-2w-2}{k-2}\right]\right.$$
$$\left.+\binom{w}{n-k-2w-1}\binom{n-2w-2}{k}\right]. \tag{15}$$

Proof. In $\mathcal{L}_n$, we know that the number of vertices with up-degree k and weight w is equal to the number of vertices having degree $k+w$ and weight w. Using Proposition 6 we obtain formula (15) by summing $\ell_{n,k+w,w}$ over all $w\geq 0$. □

We note that we can obtain an alternate expression for (15) by using the generating function of the up-degree enumerators in (13). Following the idea in [14], we expand

$$\frac{1}{(1-ux)(1-x^2)-x^3}=\sum_{h,i,j\geq 0}\binom{i}{k}\binom{j}{k}u^{i-h}x^{i+2j}. \tag{16}$$

The coefficient of $u^k x^n$ in this expansion is given by

$$c_{n,k}=\sum_{i\geq 0}\binom{n-2i}{k}\binom{i}{n-2i-k}.$$

Multiplying (16) by the numerator in (13), we obtain an alternate expression for (15) as

$$c_{n-1,k}+2c_{n-2,k}+2c_{n-3,k}-c_{n-2,k-1}-2c_{n-3,k-1}+c_{n-2,k-2}.$$

4.4. *q-cube polynomials*

Let $h_{n,k}$ denote the number of k-dimensional hypercubes in $\mathcal{L}_n$. The cube polynomial [3], or the cube enumerator polynomial of $\mathcal{L}_n$ is defined as

$$c(\mathcal{L}_n,x)=\sum_{k\geq 0}h_{n,k}x^k.$$

Its q-analogue $c(\mathcal{L}_n, x; q)$ is defined as follows. Let $h_{n,k,d}$ be the number of k-dimensional hypercubes in $\mathcal{L}_n$ whose distance to the all 0 vertex in $\mathcal{L}_n$ is d. Then we set

$$c(\mathcal{L}_n) = c(\mathcal{L}_n, x; q) = \sum_{d,k\geq 0} h_{n,k,d} q^d x^k.$$

This definition is analogous to the q-cube polynomial of Γ_n introduced in [19]. Note that by taking $q = 1$ one obtains the cube polynomial of $\mathcal{L}_n$. Furthermore, the constant term $c(\mathcal{L}_n, 0, q)$ is the rank generating function of $\mathcal{L}_n$ as a polynomial in q, which for $q = 1$ gives the number of vertices in $\mathcal{L}_n$. The coefficient of x in $c(\mathcal{L}_n)$ evaluated at $q = 1$ gives the number of edges in $\mathcal{L}_n$. We define $c(\mathcal{L}_1) = 1$. First few of these q-cube polynomials of $\mathcal{L}_n$ are as follows:

$$\begin{aligned}
c(\mathcal{L}_1) &= 1,\\
c(\mathcal{L}_2) &= 1 + 2q + 2x,\\
c(\mathcal{L}_3) &= 1 + 3q + 3x,\\
c(\mathcal{L}_4) &= 1 + 4q + 2q^2 + (4 + 4q)x + 2x^2, \quad \text{(see Fig. 5)}\\
c(\mathcal{L}_5) &= 1 + 5q + 5q^2 + (5 + 10q)x + 5x^2.
\end{aligned} \tag{17}$$

Proposition 13. *For $n \geq 3$ the q-cube polynomial $c(\mathcal{L}_n)$ satisfies*

$$c(\mathcal{L}_n) = c(\mathcal{L}_{n-1}) + (q + x)c(\mathcal{L}_{n-1})$$

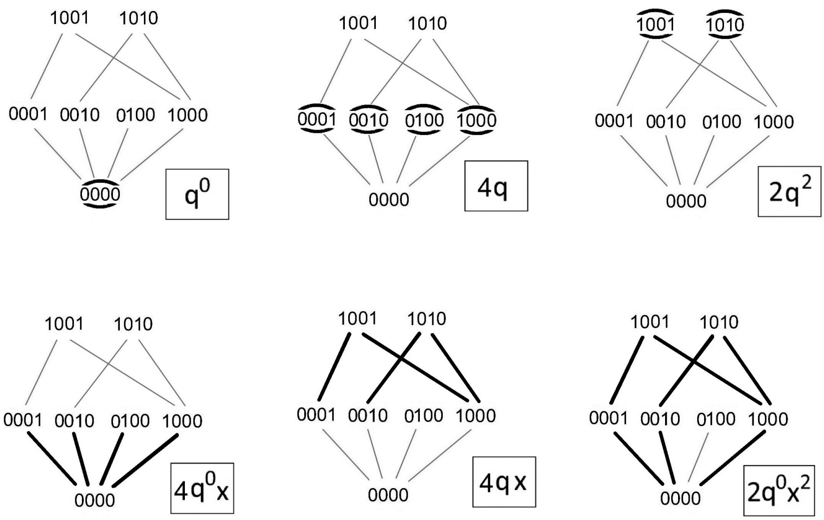

Fig. 5. The calculation of the q-cube polynomial $c(\mathcal{L}_4, x; q) = 1 + 4q + 2q^2 + (4 + 4q)x + 2x^2$ indicating the contribution of each hypercube in $\mathcal{L}_4$.

with $c(\mathcal{L}_1) = 1$ *and* $c(\mathcal{L}_2) = 1 + 2q + 2x$*. The generating function of* $\{c(\mathcal{L}_n)\}_{n\geq 1}$ *is given by*

$$\sum_{n\geq 1} c(\mathcal{L}_n)t^n = \frac{t + t^2(2q + 2x)}{1 - t - t^2(q + x)}. \tag{18}$$

Proof. The recurrence relation and the resulting generating function for the q-cube polynomials of $\mathcal{L}_n$ can be proved along the lines of the proofs of the analogous statements for the q-cube polynomials of Γ_n [19, Lemma 1, Proposition 1]. The recurrence relation is a consequence of the fact that the fundamental decompositions of Γ_n and $\mathcal{L}_n$ follow the same pattern. □

Remark 14. The q-cube polynomials of the Λ_n were determined in [20]. From the generating function given therein, we see that for $n \geq 1$, the q-cube polynomials of Λ_n and $\mathcal{L}_n$ are identical. This is a curious fact as Λ_n and $\mathcal{L}_n$ are nonisomorphic for $n \geq 4$ by Proposition 9. So not only does $\mathcal{L}_n$ have the same number of vertices and the same number of edges as Λ_n, but the number of induced hypercubes of every dimension is also the same for both, even when we take into account their distance to the all zero vertex in each.

Using the generating functions (18) and (12) we obtain

Corollary 15. *Let* $h_{n,k,d}$ *denote the number of* k*-dimensional hypercubes in* $\mathcal{L}_n$ *whose distance to the all* 0 *vertex is* d*. Then*

$$\sum_{k=0}^{w} h_{n,k,w-k} = \frac{2^w n}{n-w}\binom{n-w}{w} = 2^w r_{n,w}.$$

Proof. The generating function of the weight enumerator polynomial $G_n(1,z)$ of $\mathcal{L}_n$ given in (12) is identical to the generating function of the q-cube polynomials in (18) evaluated at $q = x = \frac{1}{2}z$. Comparing the coefficients of z^w in the resulting identity we obtain the formula in the corollary. □

Corollary 15 has the following combinatorial interpretation. For a fixed w, the number of vertices of weight w in $\mathcal{L}_n$ is $r_{n,w}$ by Proposition 5. For any such vertex, select k 1s in its string representation. By flipping these 1s to 0 all possible ways, we obtain the vertices of a copy of Q_k. The distance of this hypercube to all zero vertex is then $w - k$. So from any vertex with weight w we obtain $\binom{w}{k}$ different copies of Q_k. In total, we have $\sum_{k=0}^{w}\binom{w}{k} = 2^w$ hypercubes of dimension $k = 0, 1, \ldots, w$. Therefore, the total number of these is $2^w r_{n,w}$.

4.5. *Maximal hypercube polynomials*

The number of maximal hypercubes isomorphic to Q_k in Λ_n, which are not contained in any $H \subseteq \Lambda_n$ isomorphic to Q_{k+1} is studied in [15]. By generalizing this idea, the maximum number of disjoint subgraphs isomorphic to Q_k in Γ_n is presented in [9] and also studied in [18]. Now we consider the maximal hypercubes of dimension k in $\mathcal{L}_n$ as they appear in [15]. These are induced subgraphs H of $\mathcal{L}_n$ that are isomorphic to Q_k, and such that there exists no induced subgraph H' of $\mathcal{L}_n$ isomorphic to Q_{k+1} with $H \subset H'$. Let $m_{n,k}$ be the number of maximal hypercubes of dimension k of $\mathcal{L}_n$ with enumerator polynomial $m(\mathcal{L}_n, x) = \sum_{k=0}^{\infty} m_{n,k} x^k$. By direct inspection, first few $m(\mathcal{L}_n, x)$ are as follows:

$$m(\mathcal{L}_1, x) = 1,$$

$$m(\mathcal{L}_2, x) = 2x,$$

$$m(\mathcal{L}_3, x) = 3x,$$

$$m(\mathcal{L}_4, x) = x + 2x^2,$$

$$m(\mathcal{L}_5, x) = 5x^2.$$

Since the decomposition of Γ_n and $\mathcal{L}_n$ follow the same pattern, we have the recurrence relation in Proposition 16. This recurrence relation can be proved along the same lines as the proof of [15, Corollary 2.11].

Proposition 16. *For $n \geq 4$, $m(\mathcal{L}_n, x)$ satisfies*

$$m(\mathcal{L}_n, x) = x(m(\mathcal{L}_{n-2}, x) + m(\mathcal{L}_{n-3}, x))$$

with $m(\mathcal{L}_1, x) = 1$, $m(\mathcal{L}_2, x) = 2x$ and $m(\mathcal{L}_3, x) = 3x$. Furthermore, the generating function of $m(\mathcal{L}_n, x)$ is

$$\sum_{n \geq 1} m(\mathcal{L}_n, x) t^n = \frac{t + 2xt^2(1+t)}{1 - xt^2(1+t)}.$$

Remark 17. Note that although the recursive relations for $\mathcal{L}_n$, Λ_n and Γ_n are the same for the maximal hypercubes, the initial conditions are different, and consequently the enumerator polynomials are not the same.

5. Irregularity Polynomial of Alternate Lucas Cubes

A local measure of irregularity called imbalance $imb_G(e)$ of an edge $e = uv \in E(G)$ is defined as

$$imb_G(e) = |\deg_G(u) - \deg_G(v)|.$$

This quantity was transferred to a global irregularity measure by Albertson [1], who considered

$$irr(G) = \sum_{uv \in E(G)} |\deg_G(u) - \deg_G(v)|.$$

We define the irregularity polynomial $I_G(x)$ of G by

$$I_G(x) = \sum_{uv \in E(G)} x^{|\deg_G(u) - \deg_G(v)|}.$$

With this definition $|E(G)| = I_G(1)$, $irr(G) = \frac{d}{dx} I_G(x)$ evaluated at $x = 1$, and the coefficient of x^r in $I_G(x)$ is the number of edges $e \in G$ with $imb_G(e) = r$. In the rest of this section we determine the irregularity of alternate Lucas cubes using this polynomial.

Using (3) we can write

$$\mathcal{L}_n = 0\mathcal{L}_{n-1} + 10\mathcal{L}_{n-2} \tag{19}$$

$$= (00\mathcal{L}_{n-2} + 010\mathcal{L}_{n-3}) + 10\mathcal{L}_{n-2} \tag{20}$$

$$= ((000\mathcal{L}_{n-3} + 0010\mathcal{L}_{n-4}) + 010\mathcal{L}_{n-3}) + (100\mathcal{L}_{n-3} + 1010\mathcal{L}_{n-4}) \tag{21}$$

where there are perfect matchings (see Fig. 6) between

- $10\mathcal{L}_{n-2}$ and $00\mathcal{L}_{n-2} \subset 0\mathcal{L}_{n-1}$ in (19),
- $10\mathcal{L}_{n-2}$ and $00\mathcal{L}_{n-2}$; $010\mathcal{L}_{n-3}$ and $000\mathcal{L}_{n-3} \subset 00\mathcal{L}_{n-2}$ in (20),
- $010\mathcal{L}_{n-3}$ and $000\mathcal{L}_{n-3}$; $100\mathcal{L}_{n-3}$ and $000\mathcal{L}_{n-3}$; $1010\mathcal{L}_{n-4}$ and $0010\mathcal{L}_{n-4}$; $0010\mathcal{L}_{n-4}$ and $0000\mathcal{L}_{n-4} \subset 000\mathcal{L}_{n-3}$; $1010\mathcal{L}_{n-4}$ and $1000\mathcal{L}_{n-4} \subset 100\mathcal{L}_{n-3}$ in (21).

Let $I_n(x) = I_{\mathcal{L}_n}(x)$ denote the irregularity polynomial of $\mathcal{L}_n$. Then we have the following result.

Theorem 18. *The irregularity polynomial of $\mathcal{L}_n$ satisfies*

$$I_n(x) = 2I_{n-1}(x) + I_{n-2}(x) - 2I_{n-3}(x) - I_{n-4}(x) \tag{22}$$

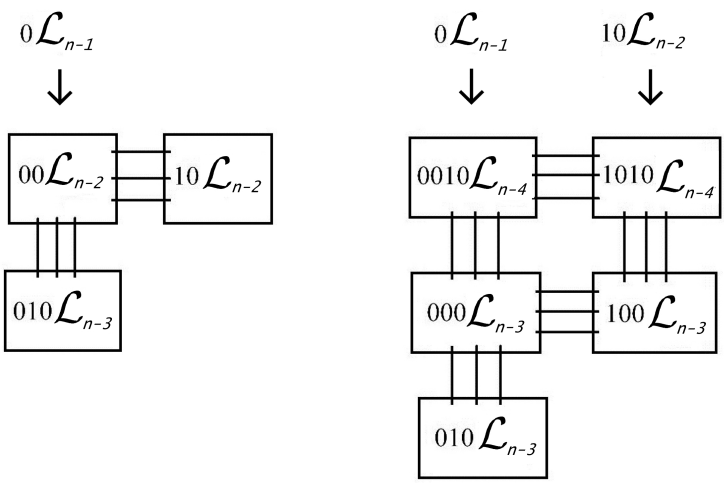

Fig. 6. Fundamental decomposition and perfect matchings in the alternate Lucas cube $\mathcal{L}_n$, $n \geq 4$.

for $n \geq 6$, *where* $I_1(x) = 0$, $I_2(x) = 2x$, $I_3(x) = 3x^2$, $I_4(x) = x^3 + 2x^2 + 3x + 2$ *and* $I_5(x) = x^3 + 6x^2 + 7x + 1$.

Proof. The values of $I_n(x)$ for $n \leq 5$ can be directly obtained from the definition of $\mathcal{L}_n$. Using the fundamental decomposition (19) of $\mathcal{L}_n$ we need to consider the following three cases:

(1) Assume that $e \in 10\mathcal{L}_{n-2}$. The irregularity polynomial of $\mathcal{L}_{n-2}$ is $I_{n-2}(x)$ and the degrees of vertices of all $e \in 10\mathcal{L}_{n-2}$ increase by one in $\mathcal{L}_n$. Consequently, there will be no change in the imbalance of such edges. Therefore, these edges contribute $I_{n-2}(x)$ to $I_n(x)$.

(2) Assume that $e = uv \in \mathcal{L}_n$ such that $u \in 10\mathcal{L}_{n-2}$ and $v \in 0\mathcal{L}_{n-1}$ (in particular, $v \in 00\mathcal{L}_{n-2}$). From (20) we know that there is perfect matching between $00\mathcal{L}_{n-2}$ and $10\mathcal{L}_{n-2}$, which means that the number of neighbors of u and v in $00\mathcal{L}_{n-2}$ and $10\mathcal{L}_{n-2}$ are the same. The only difference for the degrees of such vertices happens if there exists a neighbor of v in $010\mathcal{L}_{n-3}$ due to the perfect matching between $010\mathcal{L}_{n-3}$ and $000\mathcal{L}_{n-3} \subset 00\mathcal{L}_{n-2}$. In total, we have L_{n-3} edges each of which contributes x to $I_n(x)$ for a total of $L_{n-3}x$, and there are $L_{n-2} - L_{n-3} = L_{n-4}$ edges each of which contributes x^0 to $I_n(x)$, for a total contribution of $L_{n-4}x^0$. Therefore, these edges together contribute $L_{n-3}x + L_{n-4}$ to $I_n(x)$.

(3) Assume that $e \in 0\mathcal{L}_{n-1}$. Since $0\mathcal{L}_{n-1} = 00\mathcal{L}_{n-2} + 010\mathcal{L}_{n-3}$ we have three subcases to consider here.

    (a) Assume that $e \in 010\mathcal{L}_{n-3}$. The degrees of vertices of all of these edges increase by one in $\mathcal{L}_n$, and therefore they contribute $I_{n-3}(x)$ to $I_n(x)$.

    (b) Assume that $e = uv \in 0\mathcal{L}_{n-1}$ such that $u \in 010\mathcal{L}_{n-3}$ and $v \in 00\mathcal{L}_{n-2}$. As in Case (2) above, the contribution of these edges to $I_{n-1}(x)$ is $L_{n-4}x + L_{n-5}$. Since there is a perfect matching between $00\mathcal{L}_{n-2}$ and $10\mathcal{L}_{n-2}$, the degrees of all such vertices v must increase by 1 in $\mathcal{L}_n$. Therefore, the total contribution of these edges to $I_n(x)$ is $x \cdot (L_{n-4}x + L_{n-5}) = L_{n-4}x^2 + L_{n-5}x$.

    (c) Assume that $e \in 00\mathcal{L}_{n-2}$. These edges are the ones in $0\mathcal{L}_{n-1}$ that are not in $010\mathcal{L}_{n-3}$ and that are not created during the connection of $00\mathcal{L}_{n-2}$ and $010\mathcal{L}_{n-3}$. Furthermore, the degree of the vertices of each of these edges increases by 1 due to the perfect matching between $00\mathcal{L}_{n-2}$ and $10\mathcal{L}_{n-2}$, which does not change the contribution of these edges to $I_n(x)$. Therefore, their contribution to $I_n(x)$ is $I_{n-1}(x) - I_{n-3}(x) - (L_{n-4}x + L_{n-5})$.

Summing up all the above contributions we obtain that

$$I_n(x) = I_{n-1}(x) + I_{n-2}(x) + L_{n-4}x^2 + 2L_{n-5}x + L_{n-6}. \tag{23}$$

By using the recursion for Lucas numbers in (23) we can eliminate the terms which involve Lucas numbers and obtain the desired result. $\square$

Let E_n denote the number of edges in $E(\Gamma_n)$. It is shown in [16] that

$$E_n = \frac{1}{5}(nF_{n+1} + 2(n+1)F_n) \tag{24}$$

with generating function

$$\sum_{n\geq 0} E_n y^n = \frac{y}{(1-y-y^2)^2}. \tag{25}$$

$\mathcal{L}_n$ has the same number of edges as Λ_n. Putting $e_n = |E(\Lambda_n)| = |E(\mathcal{L}_n)|$, we have

$$e_n = nF_{n-1}. \tag{26}$$

Define the generating function of the irregularity polynomials $I_n(x)$ of $\mathcal{L}_n$ by

$$I(x,y) = \sum_{n\geq 1} I_n(x)y^n = 2xy^2 + 3x^2y^3 + (x^3 + 2x^2 + 3x + 2)y^4 + \cdots .$$

Using Theorem 18, we obtain a closed form for $I(x,y)$ and consequently for the polynomials $I_n(x)$ themselves. We then use the relationship between this function and the generating function of the number of edges in Γ_n to obtain further results.

Corollary 19. *The generating function of the irregularity polynomials $I_n(x)$ of $\mathcal{L}_n$ is*

$$I(x,y) = \sum_{n\geq 1} I_n(x)y^n = \frac{y(c_1(x)y + c_2(x)y^2 + c_3(x)y^3 + c_4(x)y^4)}{(1-y-y^2)^2} \tag{27}$$

where

$$\begin{aligned} c_1(x) &= 2x \\ c_2(x) &= x(3x-4) \\ c_3(x) &= x^3 - 4x^2 + x + 2 \\ c_4(x) &= -x^3 - x^2 + 5x - 3. \end{aligned}$$

Proof. We multiply identity (22) of Theorem 18 by y^n and sum for $n \geq 6$. Using the first few polynomials as given in Theorem 18 and with some algebra, we obtain an identity satisfied by $I(x,y)$ which is then solved and simplified to obtain the expression in (27). We omit the details. □

Corollary 20. *The irregularity polynomial and the irregularity of the alternate Lucas cube $\mathcal{L}_n$ for $n \geq 2$ are given by*

$$\begin{aligned} I_n(x) = {} & nF_{n-1} + 2(F_{n-1} + (n-1)F_{n-2})(x-1) \\ & + ((n+1)F_{n-3} + 3F_{n-4})(x-1)^2 \\ & + \frac{1}{5}(nF_{n-4} + (2n-3)F_{n-5})(x-1)^3, \end{aligned} \tag{28}$$

$$irr(\mathcal{L}_n) = 2(F_{n-1} + (n-1)F_{n-2}) = 2e_{n-1} + 2F_{n-1}. \tag{29}$$

Proof. Combining the generating functions in (27) and (25), we have

$$I(x,y) = \left(\sum_{n\geq 0} E_n y^n\right)(c_1(x)y + c_2(x)y^2 + c_3(x)y^3 + c_4(x)y^4)$$

where the coefficient polynomials are as they appear in Corollary 19. Comparing coefficients of y^n on both sides gives

$$\begin{aligned} I_n(x) &= c_1(x)E_{n-1} + c_2(x)E_{n-2} + c_3(x)E_{n-3} + c_4(x)E_{n-4} \\ &= 2E_{n-3} - 3E_{n-4} + (2E_{n-1} - 4E_{n-2} + E_{n-3} + 5E_{n-4})x \\ &\quad + (3E_{n-2} - 4E_{n-3} - E_{n-4})x^2 + (E_{n-3} - E_{n-4})x^3, \end{aligned} \tag{30}$$

where E_n is the number of edges of Γ_n as given in (24). Expanding (30) in powers of $x - 1$ and simplifying the coefficients, we obtain (28), which holds for $n \geq 2$ as written. Taking the derivative of $I_n(x)$ and evaluating at $x = 1$ gives (29). □

In Table 2 we present the irregularity polynomials of the Fibonacci, Lucas and alternate Lucas cubes for $n \leq 12$.

From the expansion of $I_n(x)$ as given in (30), we can obtain the higher moments of $|\deg_{\mathcal{L}_n}(u) - \deg_{\mathcal{L}_n}(v)|$ over $uv \in E(\mathcal{L}_n)$ as

$$\begin{aligned} &\sum_{uv\in E(\mathcal{L}_n)} |\deg_{\mathcal{L}_n}(u) - \deg_{\mathcal{L}_n}(v)|^m \\ &\quad = (2E_{n-1} - 4E_{n-2} + E_{n-3} + 5E_{n-4}) + (3E_{n-2} - 4E_{n-3} - E_{n-4})2^m \\ &\qquad + (E_{n-3} - E_{n-4})3^m. \end{aligned} \tag{31}$$

Table 2. The irregularity polynomials of the Fibonacci, Lucas and alternate Lucas cubes for $1 \leq n \leq 12$.

n	$I_{\Gamma_n}(x)$ [7]	$I_{\Lambda_n}(x)$ [7]	$I_{\mathcal{L}_n}(x)$
1	1	0	0
2	$2x$	$2x$	$2x$
3	x^2+2x+2	$3x^2$	$3x^2$
4	$2x^2+6x+2$	$4x^2+4$	x^3+2x^2+3x+2
5	$5x^2+10x+5$	$5x^2+10x$	x^3+6x^2+7x+1
6	$10x^2+20x+8$	$12x^2+12x+6$	$3x^3+8x^2+15x+4$
7	$20x^2+36x+15$	$21x^2+28x+7$	$5x^3+15x^2+31x+5$
8	$38x^2+66x+26$	$40x^2+48x+16$	$10x^3+24x^2+60x+10$
9	$71x^2+118x+46$	$72x^2+90x+27$	$18x^3+41x^2+114x+16$
10	$130x^2+210x+80$	$130x^2+160x+50$	$33x^3+68x^2+211x+28$
11	$235x^2+370x+139$	$231x^2+286x+88$	$59x^3+114x^2+385x+47$
12	$420x^2+648x+240$	$408x^2+504x+156$	$105x^3+190x^2+693x+80$

Calculating from (31) and (24), we find in particular that the second moment is given by

$$\sum_{uv\in E(\mathcal{L}_n)} |\deg_{\mathcal{L}_n}(u)-\deg_{\mathcal{L}_n}(v)|^2 = 2E_{n-1}+8E_{n-2}-6E_{n-3}-8E_{n-4}$$

$$= (2n-2)F_{n-1}+8F_{n-2}.$$

The irregularity of Λ_n was computed in [7] (the same was also done for Fibonacci cubes in [2, 7]). For comparison, we have

$$irr(\Lambda_n) = 2nF_{n-2}, \quad irr(\mathcal{L}_n) = 2F_{n-1}+2(n-1)F_{n-2},$$

so that

$$irr(\mathcal{L}_n) = irr(\Lambda_n) + 2F_{n-3}.$$

In conjunction with (26), the asymptotic average imbalance of Λ_n and $\mathcal{L}_n$ are both found to be $\sqrt{5}-1$.

6. Hamiltonicity

A graph G is bipartite when the vertex set can be decomposed into two disjoint nonempty independent subsets X and Y. Here, (X,Y) is called a bipartition of G. Since Q_n (or Γ_n) is bipartite, alternate Lucas cubes $\mathcal{L}_n$ are bipartite. Hence, they can have a Hamiltonian cycle only when the number or vertices L_n is even. Since the Lucas numbers are even only for indices of the form $3k$, the only possibility for having a Hamiltonian cycle is for the graphs $\mathcal{L}_{3k}$. We first show that the vertex parity difference in this case rules out Hamiltonian cycles for alternate Lucas cubes.

Proposition 21. *$\mathcal{L}_n$ never has a Hamiltonian cycle.*

Proof. As we remarked, $\mathcal{L}_n$ cannot have a Hamiltonian cycle unless n is of the form $n=3k$. Let V_n^e and V_n^o denote the number of vertices of even and odd Hamming weight in $\mathcal{L}_n$. Then from the fundamental decomposition

$$V_n^e = V_{n-1}^e + V_{n-2}^o$$

$$V_n^o = V_{n-1}^o + V_{n-2}^e$$

for $n\geq 3$ with initial values $V_1^e = V_2^e = 1$, $V_1^o = 0$, $V_2^o = 2$. We easily find that the vertex parity difference $\Delta_n = V_n^o - V_n^e$ satisfies

$$\Delta_n = \Delta_{n-1} - \Delta_{n-2}$$

with $\Delta_1 = -1, \Delta_2 = 1$. Thus

$$|\Delta_n| = \begin{cases} 1 & \text{if } n\equiv 1,2 \pmod 3, \\ 2 & \text{if } n\equiv 0 \pmod 3. \end{cases}$$

Therefore, $\mathcal{L}_n$ cannot have a Hamiltonian path when $n=3k$ and the proposition follows. □

Proposition 21 leaves open the possibility that the alternate Lucas cubes of the form $\mathcal{L}_{3k+1}$ and $\mathcal{L}_{3k+2}$ can possibly have Hamiltonian paths. For these we need to skip over the graphs with index $3k$ in the fundamental decomposition. For instance we decompose $\mathcal{L}_4$ as

$$\begin{aligned}\mathcal{L}_4 &= 0\mathcal{L}_3 + 10\mathcal{L}_2\\ &= 00\mathcal{L}_2 + 010\mathcal{L}_1 + 10\mathcal{L}_2.\end{aligned}$$

One possible Hamiltonian path in $\mathcal{L}_4$ is

$$(0100), 0000, 0001, (1001, 1000, 1010), 0010$$

where the portion in the first parentheses is from $\mathcal{L}_1$ and the second parenthesis is a copy of a Hamiltonian path in $\mathcal{L}_2$. This Hamiltonian path is inserted inside the other Hamiltonian path copy from $\mathcal{L}_2$, namely after the second vertex in $0000, 0001, 0010$. So this suggests a construction of the following type for $\mathcal{L}_7$ (in general $\mathcal{L}_{3k+1}$)

$$010\mathcal{L}_4 \to part_1(00\mathcal{L}_5) \to 10\mathcal{L}_5 \to part_2(00\mathcal{L}_5)$$

for the Hamiltonian path. We need to find canonical vertices where these chains are glued together, for instance $010\mathcal{L}_4$ can end at the all 0 vertex and picked up from there by $part_1(00\mathcal{L}_5)$, etc.

If we can locate/guess these breakpoints for both types of graphs $\mathcal{L}_{3k+1}$ and $\mathcal{L}_{3k+2}$, we can construct Hamiltonian paths for these indices and prove all by induction.

Proposition 22. *$\mathcal{L}_n$ has a Hamiltonian path if and only if n is not divisible by* 3.

Proof. We know that any Fibonacci cube Γ_n has a Hamiltonian path and Γ_n has a Hamiltonian cycle if $n \equiv 1 \mod 3$ except Γ_1 [22]. Furthermore, in the proof [22, Theorem 3.3] a Hamiltonian cycle using a Gray code is constructed for Γ_n and also in [5, Lemma 1], a Hamiltonian path using a Gray code is presented for Γ_n. In these Hamiltonian paths and Hamiltonian cycles one can observe that the two vertices $(0\ldots0)$ and $(0\ldots010)$ are adjacent vertices. For $n \geq 3$ we can decompose $\mathcal{L}_n$ as

$$\mathcal{L}_n = \Gamma_{n-1}0 + \Gamma_{n-3}001$$

and there is a perfect matching between $\Gamma_{n-3}001$ and $\Gamma_{n-3}000 \subseteq \Gamma_{n-1}0$. First we consider the case when $n = 3k+1$ for some positive integer k. We can write

$$\mathcal{L}_{3k+1} = \Gamma_{3k}0 + \Gamma_{3k-2}001.$$

Since Γ_{3k-2} is Hamiltonian and Γ_{3k} has a Hamiltonian path we can obtain a Hamiltonian path for $\mathcal{L}_{3k+1}$ as follows:

(1) Consider the Hamiltonian path in $\Gamma_{3k} \subseteq \Gamma_{3k}0$ and follow this path all the way to the vertex with label $(0\ldots0000)$ or $(0\ldots010000)$ (assume we reach to $(0\ldots0)$).

(2) Then use the perfect matching between $\Gamma_{3k}0$ and $\Gamma_{3k-2}001$ and pass to the vertex with label $(0\dots 0001) \in \Gamma_{3k-2}001$.
(3) Then use the Hamiltonian cycle in $\Gamma_{3k-2} \subseteq \Gamma_{3k-2}001$ and traverse through all vertices in $\Gamma_{3k-2}001$ until the last vertex with label $(0\dots 010001) \in \Gamma_{3k-2}001$ which corresponds to the vertex $(0\dots 010) \in \Gamma_{3k-2}$ and it is neighbor of $(0\dots 0) \in \Gamma_{3k-2}$.
(4) Following this, use the perfect matching between $\Gamma_{3k}0$ and $\Gamma_{3k-2}001$ and pass to the vertex with label $(0\dots 010000) \in \Gamma_{3k}0$.
(5) Use the Hamiltonian path in $\Gamma_{3k} \subseteq \Gamma_{3k}0$ and pass through all the remaining vertices in $\Gamma_{3k}0$.

Similarly, if n is of the form $n = 3k+2$, we use the decomposition

$$\mathcal{L}_{3k+2} = \Gamma_{3k+1}0 + \Gamma_{3k-1}001.$$

In this decomposition Γ_{3k+1} is Hamiltonian and Γ_{3k-1} has a Hamiltonian path. We then use the above argument for $n = 3k+1$ to construct a Hamiltonian path for $\mathcal{L}_{3k+2}$. □

Example 23. We give the details for the construction of the Hamiltonian path for $\mathcal{L}_7$ described in the proof of Proposition 22. Let C^R denote the reverse sequence of the sequence C. If C_{n-1} and C_{n-2} denote the Gray codes for Γ_{n-1} and Γ_{n-2} respectively, then the Gray code sequence $C_n = \{0C^R_{n-1}, 10C^R_{n-2}\}$ gives a Hamiltonian path for Γ_n [5] in conjunction with the fundamental decomposition (1). Since the reverse of a Fibonacci string is also a Fibonacci string and consequently $\Gamma_n = \Gamma_{n-1}0 + \Gamma_{n-2}01$, we observe that the sequence $D_n = \{C^R_{n-1}0, C^R_{n-2}01\}$ is also a Hamiltonian path for Γ_n. We know that

$$C_4 = \{0010, 0000, 0100, 0101, 0001, 1001, 1000, 1010\}$$

is a Hamiltonian cycle for Γ_4 and

$$\begin{aligned} C_5 = \{&01010, 01000, 01001, 00001, 00101, 00100, 00000, 00010, \\ &10010, 10000, 10100, 10101, 10001\} \end{aligned}$$

is a Hamiltonian path for Γ_5. Then we get the Hamiltonian paths for Γ_6 as

$$\begin{aligned} C_6 &= \{0C^R_5, 10C^R_4\} \\ &= \{010001, 010101, 010100, 010000, 010010, 000010, 000000, \\ &\quad 000100, 000101, 000001, 001001, 001000, 001010, \\ &\quad 101010, 101000, 101001, 100001, 100101, 100100, 100000, 100010\} \end{aligned}$$

and

$$\begin{aligned} D_6 &= \{C^R_5 0, C^R_4 01\} \\ &= \{100010, 101010, 101000, 100000, 100100, 000100, \underline{000000}, \end{aligned}$$

$$\underline{001000}, 001010, 000010, 010010, 010000, 010100,$$

$$101001, 100001, 100101, 000101, 010101, 010001, 000001, 001001\}.$$

For $n = 7$, using $C_4\mathbf{001}$, $D_6\mathbf{0}$ and the above algorithm we observe that

$$part_1(D_6\mathbf{0}) \to C_4\mathbf{001} \to part_2(D_6\mathbf{0})$$

gives a Hamiltonian path in $\mathcal{L}_7$, which is

$$100010\mathbf{0}, 101010\mathbf{0}, 101000\mathbf{0}, 100000\mathbf{0}, 100100\mathbf{0}, 000100\mathbf{0}, \underline{000000\mathbf{0}},$$

$$0000\mathbf{001}, 0100\mathbf{001}, 0101\mathbf{001}, 0001\mathbf{001}, 1001\mathbf{001}, 1000\mathbf{001}, 1010\mathbf{001}, 0010\mathbf{001},$$

$$\underline{001000\mathbf{0}}, 001010\mathbf{0}, 000010\mathbf{0}, 010010\mathbf{0}, 010000\mathbf{0}, 010100\mathbf{0},$$

$$101001\mathbf{0}, 100001\mathbf{0}, 100101\mathbf{0}, 000101\mathbf{0}, 010101\mathbf{0}, 010001\mathbf{0}, 000001\mathbf{0}, 001001\mathbf{0}.$$

Proposition 22 in particular implies the following result on the independence number of $\mathcal{L}_n$.

Corollary 24. *For any integer* $n \geq 1$ *not divisible by* 3, *the independence number* $\alpha(\mathcal{L}_n)$ *of* $\mathcal{L}_n$ *is equal to* $\lceil \frac{L_n}{2} \rceil$.

Proof. The proof mimics the proof of [13, Corollary 3.2]. As $\mathcal{L}_n$ has a Hamiltonian path, $\alpha(\mathcal{L}_n) \leq \lceil V(\mathcal{L}_n)/2 \rceil = \lceil L_n/2 \rceil$. On the other hand, let $X+Y$ be the bipartition of $\mathcal{L}_n$. Then $\alpha(\mathcal{L}_n) \geq \max\{|X|, |Y|\} \geq \lceil L_n/2 \rceil$ since the vertex parity difference satisfies $|\Delta_n| = 1$. □

7. Further Properties

For a connected graph G, let $d_G(u, v)$ denote the length of the shortest path between the vertices u and v in G. We know that in Q_n, Γ_n and $\mathcal{L}_n$ this distance is the Hamming distance. Let $ecc_G(u)$ denotes the eccentricity of a vertex $u \in G$, defined as the greatest distance between u and any other vertex v in G. The radius $rad(G)$ and the diameter $diam(G)$ of G are the minimum and maximum eccentricity among the vertices of G, respectively. The center $Z(G)$ of G is defined as the set of vertices with eccentricity equal to the radius of G. In [16] the center, diameter and radius of Λ_n is determined. Related properties for the Fibonacci cubes can be found in [13].

For alternate Lucas cubes, we have the following results.

Proposition 25. *For any integer* $n \geq 3$, *alternate Lucas cube* $\mathcal{L}_n$ *satisfies*

(1) $\mathrm{diam}(\mathcal{L}_n) = n - 1$,
(2) $\mathrm{rad}(\mathcal{L}_n) = \lfloor \frac{n}{2} \rfloor$,
(3) $Z(\mathcal{L}_n) = \begin{cases} \{0^n\} & \textit{if } n \textit{ is odd}, \\ \{0^n, 10^{n-1}\} & \textit{if } n \textit{ is even}. \end{cases}$

Proof. For part (1), we know that $diam(\mathcal{L}_n) \leq diam(Q_n) = n$. For any $u \in V(\mathcal{L}_n)$, if $ecc_{\mathcal{L}_n}(u) = ecc(u) = n$ then there exist a $v \in V(\mathcal{L}_n)$ such that $d(u, v) = n$.

But this is impossible for $\mathcal{L}_n$ since if u ends with $000, 001, 010, 100$, then v must end with $111, 110, 101, 011$, respectively. Furthermore, for $n = 2m$ we have $d((10)^m, (01)^{m-1}00) = n-1$ and for $n = 2m+1$ we have $d((10)^m 0, (01)^m 0) = n-1$, which completes the proof. For the results (2) and (3), we follow the lines of the proof of [16, Theorem 1]. $\square$

Remark 26. If $n > 2$ is even, then $diam(\Lambda_n) = n$ and $Z(\Lambda_n) = \{0^n\}$ which are different from the corresponding quantities for $\mathcal{L}_n$.

The boundary enumerator polynomial of hypercubes, as defined and studied in [21] for the class of Fibonacci cubes, can also be determined for the family of alternate Lucas cubes $\mathcal{L}_n$. This is work in progress and will be reported in a subsequent paper [8].

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