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Brick tabloids and the connection matrices between bases of symmetric functions

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Abstract

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Let H_n denote the space of symmetric functions, homogeneous of degree n. In this paper we introduce a new set of combinatorial objects called λ-brick tabloids and its variants, which we use to give combinatorial interpretations of the entries for twelve of the transition matrices between natural bases of H_n. Using these interpretations, it is possible to give purely combinatorial proofs of various identities between these connection matrices. Also as a consequence, the so called forgotten basis of Doubilet and Rota is shown to admit a natural combinatorial description in terms of brick tabloids and the monomial symmetric functions.

Introduction

Let H_n denote the space of symmetric functions, homogeneous of degree n. It is well known that the dimension of H_n is the number of partitions of n. Here we write $\lambda \vdash n$ if λ is a partition of n, i.e., if $\lambda = (0 < \lambda_1 \leq \cdots \leq \lambda_k)$ and $\lambda_1 + \cdots + \lambda_k = n$. We shall also write $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ to mean that λ has α_1 parts of size 1, α_2 parts of size 2, etc. There are five bases of H_n which are normally considered, namely

- $\{m_\lambda\}_{\lambda \vdash n}$ (the monomial symmetric functions),
- $\{e_\lambda\}_{\lambda \vdash n}$ (the elementary symmetric functions),
- $\{h_\lambda\}_{\lambda \vdash n}$ (the homogeneous symmetric functions),

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- $\{p_\lambda\}_{\lambda \vdash n}$ (the power symmetric functions),
- $\{s_\lambda\}_{\lambda \vdash n}$ (the Schur symmetric functions).

One defines the so called Hall inner product on H_n by declaring that

$$\langle m_\lambda, h_\mu \rangle = \chi(\lambda = \mu), \tag{0.1}$$

where for any statement A, we let $\chi(A)=1$ if A is true and $\chi(A)=0$ if A is false. One can then show that under this inner product, the bases $\langle s_\lambda \rangle$ and $\langle p_\lambda/\sqrt{z_\lambda} \rangle$ are self-dual where $z_\lambda = \alpha_1! \cdots \alpha_n! \, 1^{\alpha_1} \cdots n^{\alpha_n}$ if $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$. In other words, we have

$$\langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu), \tag{0.2}$$

and

$$\left\langle \frac{p_\lambda}{\sqrt{z_\lambda}}, \frac{p_\mu}{\sqrt{z_\mu}} \right\rangle = \chi(\lambda = \mu). \tag{0.3}$$

Now a sixth basis for H_n, $\{f_\lambda\}_{\lambda \vdash n}$, may be introduced by declaring that $\{f_\lambda\}_{\lambda \vdash n}$ is the dual basis of $\{e_\lambda\}_{\lambda \vdash n}$, i.e., by the requirement that

$$\langle e_\lambda, f_\mu \rangle = \chi(\lambda = \mu). \tag{0.4}$$

MacDonald [6] calls the basis $\{f_\lambda\}_{\lambda \vdash n}$ the *forgotten* symmetric functions and says "they have no particularly simple direct description". A combinatorial interpretation of $\lambda_1! \lambda_2! \cdots \lambda_n! \, f_\lambda(x_1, \ldots, x_N)$ was given by Doubilet [1]. In Section 3, we will provide a simple combinatorial description of $f_\lambda(x_1, \ldots, x_N)$ as a linear combination of the monomial symmetric functions.

The main purpose of this paper is to introduce certain combinatorial objects, which we call *λ-brick tabloids* (or *λ-domino tabloids*), and show how these tabloids can be used to give combinatorial interpretations to the entries of twelve of the transition matrices between various bases of H_n mentioned above. That is, first impose some total order on the set P_n of all partitions of n, say the lexicographic order. Next, given a basis $\{u_\lambda\}_{\lambda \vdash n}$ of H_n, let $\langle u_\lambda \rangle_{\lambda \vdash n}$ denote the row vector formed by ordering the basis $\{u_\lambda\}_{\lambda \vdash n}$ according to the ordering of P_n. Then given two bases $\{u_\lambda\}_{\lambda \vdash n}$ and $\{v_\lambda\}_{\lambda \vdash n}$ of H_n, we let $M(u,v)$ denote the transition matrix which transforms $\{u_\lambda\}_{\lambda \vdash n}$ into $\{v_\lambda\}_{\lambda \vdash n}$. In other words,

$$\langle v_\lambda \rangle_{\lambda \vdash n} = \langle u_\lambda \rangle_{\lambda \vdash n} M(u,v) \tag{0.5}$$

so that

$$v_\lambda = \sum_{\mu \vdash n} u_\mu M(u,v)_{\mu,\lambda}. \tag{0.6}$$

Equivalently, $M(u,v)_{\mu,\lambda} = \langle v_\lambda, \hat{u}_\mu \rangle$ is the coefficient of u_μ in v_λ, where $\hat{u}$ denotes the dual basis of u with respect to the Hall inner product on H_n.

For various pairs $\langle u_\lambda \rangle_{\lambda \vdash n}$ and $\langle v_\lambda \rangle_{\lambda \vdash n}$ of the six bases of H_n mentioned above, the entries $M(u,v)_{\mu,\lambda}$ have interesting combinatorial interpretations. It is well known that all of these transition matrices can be algebraically expressed in terms of essentially two matrices (see MacDonald [6]). Doubilet in [1] also gave interpretations for transition matrices between various multiples of the natural bases, in terms of the Möbius function of the partition lattice.

It turns out that by introducing certain simple combinatorial objects (brick tabloids and others), direct combinatorial interpretations for the entries $M(u,v)_{\mu,\lambda}$ for all thirty possible transition matrices can be given [3]. In this paper we consider twelve of these matrices which can be interpreted directly by brick tabloids and their variants.

Next we introduce λ-brick tabloids. First we must establish some notation. Given a partition $\lambda=(0<\lambda_1\leq\cdots\leq\lambda_k)$ of n, we let $k(\lambda)=k$ denote the number of parts of λ. The Ferrers' diagram F_λ of λ is the diagram which consists of left justified rows of squares or cells of lengths $\lambda_1,\ldots,\lambda_k$ reading from top to bottom. See for example Fig. 1.

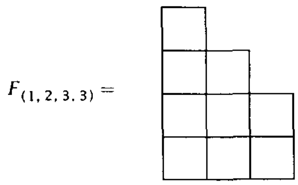

Fig. 1.

Given partitions $\lambda=(0<\lambda_1\leq\cdots\leq\lambda_k)$ and μ, a *λ-brick tabloid T of shape μ* is a filling of F_μ with bricks $b_1,\ldots,b_k$ of lengths $\lambda_1,\ldots,\lambda_k$, respectively, such that

(i) each brick b_i covers exactly λ_i squares of F_μ all of which lie in a single row of F_μ,

(ii) no two bricks overlap.

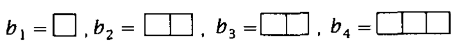

Fig. 2.

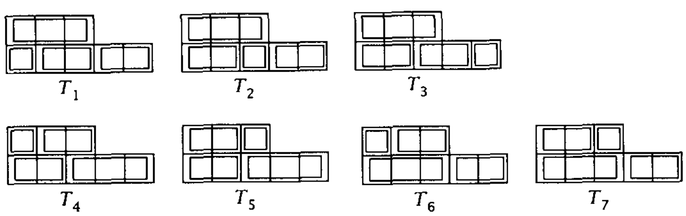

Fig. 3.

For example if $\lambda=(1,2,2,3)$ and $\mu=(3,5)$, then we must cover F_μ with the bricks of Fig. 2. Here, bricks of the same size are indistinguishable. Then there are seven λ-brick tabloids of shape μ (see Fig. 3).

We let $B_{\lambda,\mu}$ denote the set of λ-brick tabloids of shape μ. Define a weight $w(T)$ for each λ-brick tabloid $T\in B_{\lambda,\mu}$ by

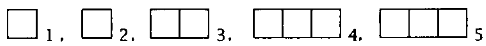

Fig. 4.

$$w(T) = \prod_{b\in T} w_T(b) \tag{0.7}$$

where for each brick b in T, $|b|$ denotes the size of b and

$$w_T(b) = \begin{cases} 1, & \text{if } b \text{ is not at the end of a row in } T, \\ |b|, & \text{if } b \text{ is at the end of a row in } T. \end{cases} \tag{0.8}$$

Thus $w(T)$ is the product of the lengths of the rightmost bricks in T. For example, for our seven $(1,2,2,3)$-brick tabloids of shape $(3,5)$ given in Fig. 3, the weights are computed to be $w(T_1)=6$, $w(T_2)=6$, $w(T_3)=3$, $w(T_4)=6$, $w(T_5)=3$, $w(T_6)=4$, and $w(T_7)=2$.

We let

$$w(B_{\lambda,\mu}) = \sum_{T\in B_{\lambda,\mu}} w(T). \tag{0.9}$$

Finally we introduce one further variation on λ-brick tabloids which we call *ordered λ-brick tabloids*. Ordered λ-brick tabloids differ from λ-brick tabloids in two ways. First, we want to be able to distinguish between various λ-bricks. To this end we subscript all of the λ-bricks with $1,\ldots,\lambda(k)$ in such a way that the subscripts on smaller size bricks precede subscripts on larger size bricks. For example if $\lambda = (1,1,2,3,3)$, then our set of bricks consists of the ones shown in Fig. 4. Second, in ordered λ-brick tabloids we insist that the subscripts on the bricks increase from left to right in each row. More precisely, an *ordered λ-brick tabloid* is a filling of F_μ with subscripted λ-bricks such that

(i) each brick b_i covers exactly λ_i squares of F_μ all of which lie in a single row of F_μ,
(ii) no two bricks overlap,
(iii) in each row, the subscripts on the λ-bricks increase from left to right.

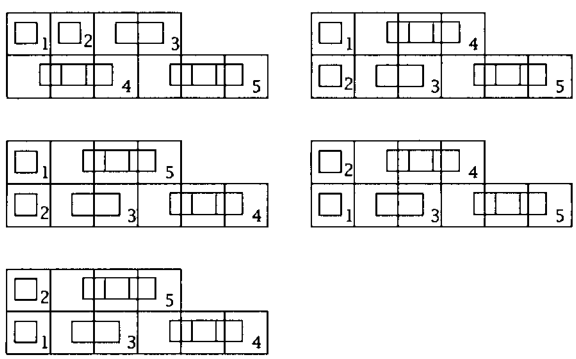

Fig. 5.

For example, if $\lambda=(1,1,2,3,3)$ and $\mu=(4,6)$, then there are five ordered λ-bricks of shape μ as given in Fig. 5.

$OB_{\lambda,\mu}$ denotes the set of ordered λ-brick tabloids of shape μ.

The main purpose of this paper is to show that up to sign, the numbers $|B_{\lambda,\mu}|$, $w(B_{\lambda,\mu})$, and $|OB_{\lambda,\mu}|$ can be used to interpret the entries of the twelve of the transition matrices between various bases of symmetric functions. In particular,

(1) $M(e,h)_{\mu,\lambda}=(-1)^{n-k(\mu)}|B_{\mu,\lambda}|$,
(2) $M(h,e)_{\mu,\lambda}=(-1)^{n-k(\mu)}|B_{\mu,\lambda}|$,
(3) $M(m,f)_{\mu,\lambda}=(-1)^{n-k(\lambda)}|B_{\lambda,\mu}|$,
(4) $M(f,m)_{\mu,\lambda}=(-1)^{n-k(\lambda)}|B_{\lambda,\mu}|$,
(5) $M(e,p)_{\mu,\lambda}=(-1)^{n-k(\mu)}w(B_{\mu,\lambda})$,
(6) $M(h,p)_{\mu,\lambda}=(-1)^{k(\mu)-k(\lambda)}w(B_{\mu,\lambda})$,
(7) $M(m,p)_{\mu,\lambda}=|OB_{\lambda,\mu}|$,
(8) $M(f,p)_{\mu,\lambda}=(-1)^{n-k(\lambda)}|OB_{\lambda,\mu}|$,
(9) $M(p,h)_{\mu,\lambda}=|OB_{\mu,\lambda}|/z_\mu$,
(10) $M(p,e)_{\mu,\lambda}=(-1)^{n-k(\mu)}|OB_{\mu,\lambda}|/z_\mu$,
(11) $M(p,m)_{\mu,\lambda}=(-1)^{k(\lambda)-k(\mu)}w(B_{\lambda,\mu})/z_\mu$,
(12) $M(p,f)_{\mu,\lambda}=(-1)^{n-k(\lambda)}w(B_{\lambda,\mu})/z_\mu$.

The outline of this paper is as follows. In Section 1, we summarize a number of facts about H_n that we shall subsequently need. It turns out that once we can prove (1), (5), and (7) above, all of the other results will follow. Section 2 will be devoted to proving (1), (5), and (7). Then in Section 3 we shall consider a number of consequences of our combinatorial interpretations.

1. The Hall transformation

The definitions of the five bases $\{m_\lambda\}_{\lambda\vdash n}$, $\{e_\lambda\}_{\lambda\vdash n}$, $\{h_\lambda\}_{\lambda\vdash n}$, $\{p_\lambda\}_{\lambda\vdash n}$, and $\{s_\lambda\}_{\lambda\vdash n}$ for H_n mentioned in the introduction can be found in MacDonald [6]. In our treatment we will make use of an important isometry $\omega: H_n\to H_n$ often called the *Hall transformation*. ω is constructed by defining it on the basis $\{h_\lambda\}_{\lambda\vdash n}$ by setting

$$\omega(h_\lambda)=e_\lambda, \tag{1.1}$$

and then extending it to H_n by linearity.

The following result summarizes the basic facts about ω which we shall need. Proofs of all these properties may be found in MacDonald [6].

Theorem 1.1. *$\omega: H_n\to H_n$ has the following properties*:
(i) ω^2 = *identity*; *in particular* $\omega(e_\lambda)=h_\lambda$,
(ii) ω *is an isometry, i.e., for any* $f,g\in H_n$, $\langle f,g\rangle=\langle\omega(f),\omega(g)\rangle$,
(iii) $\omega(p_\lambda)=(-1)^{n-k(\lambda)}p_\lambda$,
(iv) $\omega(m_\lambda)=f_\lambda$ *and* $\omega(f_\lambda)=m_\lambda$.

As we have remarked in the introduction, it is known that the following pairs of bases are dual for the Hall inner product in H_n: h_λ and m_μ, $p_\lambda \backslash z_\lambda$ and p_μ/z_μ, f_μ and e_λ. In conjunction with Theorem 1.1, one can show that the following relations hold between the connection matrices $M(u,v)$:

$$M(u,v)_{\mu,\lambda} = M(\omega(u),\omega(v))_{\mu,\lambda}, \tag{1.2}$$

$$M(u,v)_{\mu,\lambda} = M(\hat{v},\hat{u})_{\lambda,\mu}, \tag{1.3}$$

where the hat denotes the dual basis.

By applying Theorem 1.1, and the identities (1.2) and (1.3), it is not difficult to show that for the combinatorial interpretations of the matrices (1)–(12) of Section 0, we only need to show the following three

(1) $M(e,h)_{\mu,\lambda} = (-1)^{n-k(\mu)}|B_{\mu,\lambda}|$,
(2) $M(e,p)_{\mu,\lambda} = (-1)^{n-k(\mu)}w(B_{\mu,\lambda})$,
(7) $M(m,p)_{\mu,\lambda} = |OB_{\lambda,\mu}|$.

2. The matrices $M(e,h)$, $M(e,p)$, and $M(m,p)$

In the derivation of the combinatorial interpretations of these three matrices, our point of departure is the following pair of simple identities:

$$\sum_{k=0}^{n} (-1)^k h_k e_{n-k} = 0, \tag{2.1}$$

$$\sum_{k=1}^{n} (-1)^{k-1} p_k e_{n-k} = n e_n. \tag{2.2}$$

The proofs of (2.1) and (2.2) can be found in [6]. These and similar identities can also be given purely combinatorial proofs by a tableaux interpretation (see [2,3]).

We start this section by proving

$$M(e,h)_{\mu,\lambda} = (-1)^{n-k(\mu)}|B_{\mu,\lambda}|. \tag{2.3}$$

We shall show that (2.3) can be derived from the identity (2.1). Note that another way of writing (2.1) is

$$h_n = \sum_{k=1}^{n} (-1)^{k-1} e_k h_{n-k}. \tag{2.4}$$

Now (2.4) easily allows us to give recursions for the entries of $M(e,h)$ for H_n as n varies. To state our recursions precisely, we must first establish some notation. Given $\lambda \vdash n$, we let λ/k equal the partition which is formed by removing a part of size k from λ if λ has a part of size k, and we say λ/k is undefined if λ has no part of size k. For any pair of our six bases of H_n we make the convention that $M(u,v)_{\lambda,\mu}$ denotes 0 if either λ or μ is undefined. Given two partitions λ and μ, $\lambda \cup \mu$ denotes the partition of $|\lambda| + |\mu|$ whose parts are the union of the parts of λ and the parts of μ. For example if $\lambda=(1,2,3,3)$ and $\mu=(1,3,4)$, then $\lambda\cup\mu=(1,1,2,3,3,3,3,4)$. Given this, we have the following recursions:

Lemma 2.1. *Let $\lambda=(\lambda_1,\ldots,\lambda_l)$ and $\mu=(\mu_1,\ldots,\mu_k)$ be two partitions of n with at least two parts and let γ be an arbitrary partition of n. Then*

(i) $M(e,h)_{(n),(n)}=(-1)^{n-1}$,

(ii) $M(e,h)_{\lambda,(n)}=\sum_{j=1}^{n-1}(-1)^{j-1}M(e,h)_{\lambda/j,(n-j)}$,

(iii) $M(e,h)_{\gamma,\mu}=\sum_{\alpha\vdash\mu_1,\beta\vdash\mu_2+\cdots+\mu_k}M(e,h)_{\alpha,(\mu_1)}M(e,h)_{\beta,(\mu_2,\ldots,\mu_k)}\chi(\alpha\cup\beta=\gamma)$.

Proof. Note that for $n=1$, $h_1=e_1$ so that $M(e,h)_{(1),(1)}=1$. Next by (2.4),

$$\begin{aligned}h_n &= \left(\sum_{k=1}^{n-1}(-1)^{k-1}e_k h_{n-k}\right)+(-1)^{n-1}e_n\\ &= \left(\sum_{k=1}^{n}(-1)^{k-1}e_k\left(\sum_{\alpha\vdash n-k}M(e,h)_{\alpha,(n-k)}e_\alpha\right)\right)+(-1)^{n-1}e_n\\ &= (-1)^{n-1}e_n+\sum_{\lambda\vdash n}e_\lambda\left(\sum_{j=1}^{n-1}(-1)^{j-1}M(e,h)_{\lambda/j,(n-j)}\right).\end{aligned} \tag{2.5}$$

Thus (2.5) shows that

$$M(e,h)_{(n),(n)}=(-1)^{n-1}\quad\text{for all } n, \tag{2.6}$$

and

$$M(e,h)_{\lambda,(n)}=\sum_{j=1}^{n-1}(-1)^{j-1}M(e,h)_{\lambda/j,(n-j)}\quad\text{if } k(\lambda)\geq 2. \tag{2.7}$$

Finally observe that if $\mu=(\mu_1,\ldots,\mu_k)$ where $k\geq 2$, then

$$\begin{aligned}h_\mu &= \sum_{\gamma\vdash n}M(e,h)_{\gamma,\mu}e_\gamma\\ &= h_{\mu_1}\cdot h_{(\mu_2,\ldots,\mu_k)}\\ &= \left(\sum_{\alpha\vdash\mu_1}M(e,h)_{\alpha,(\mu_1)}e_\alpha\right)\left(\sum_{\beta\vdash n-\mu_1}M(e,h)_{\beta,(\mu_2,\ldots,\mu_k)}h_\beta\right)\\ &= \sum_{\gamma\vdash n}e_\gamma\left(\sum_{\substack{\alpha\vdash\mu_1\\ \beta\vdash n-\mu_1}}M(e,h)_{\alpha,(\mu_1)}M(e,h)_{\beta,(\mu_2,\ldots,\mu_k)}\chi(\alpha\cup\beta=\gamma)\right).\end{aligned} \tag{2.8}$$

Thus (2.8) shows that for such μ

$$M(e,h)_{\gamma,\mu}=\sum_{\substack{\alpha\vdash\mu_1\\ \beta\vdash n-\mu_1}}M(e,h)_{\alpha,(\mu_1)}M(e,h)_{\beta,(\mu_2,\ldots,\mu_k)}\chi(\alpha\cup\beta=\gamma). \tag{2.9}$$

It is easy to see that (2.6), (2.7), and (2.9) completely determine $M(e,h)_{\lambda,\mu}$. To establish our combinatorial interpretation for the entries $M(e,h)_{\lambda,\mu}$, we need only show that the numbers $(-1)^{n-k(\lambda)}|B_{\lambda,\mu}|$ satisfy the same recursions. To this end, let us define the sign $\varepsilon(T)$ of a λ-brick tabloid T by

$$\varepsilon(T)=\prod_{b\in T}\varepsilon(b), \tag{2.10}$$

where for any brick b, $\varepsilon(b)=(-1)^{|b|-1}$. It is then easy to see that if $T\in B_{\lambda,\mu}$, $\varepsilon(T)=(-1)^{n-k(\lambda)}$. Thus

$$(-1)^{n-k(\lambda)}|B_{\lambda,\mu}| = \sum_{T\in B_{\lambda,\mu}} \varepsilon(T). \tag{2.11}$$

Similarly,

$$(-1)^{n-k(\lambda)}w(B_{\lambda,\mu}) = \sum_{T\in B_{\lambda,\mu}} \varepsilon(T)\,w(T). \tag{2.12}$$

For the moment let $E_{\lambda,\mu}=\sum_{T\in B_{\lambda,\mu}}\varepsilon(T)$, and $w(E)_{\lambda,\mu}=\sum_{T\in B_{\lambda,\mu}}\varepsilon(T)\,w(T)$. Then we have the following:

Lemma 2.2. *Let $\lambda=(\lambda_1,\dots,\lambda_l)$ and $\mu=(\mu_1,\dots,\mu_k)$ be two partitions of n with at least two parts and let γ be an arbitrary partition of n. Then*
(a) $E_{(n),(n)}=(-1)^{n-1}$, *and* $w(E)_{(n),(n)}=(-1)^{n-1}n$,
(b) $E_{\lambda,(n)}=\sum_{j=1}^{n-1}(-1)^{j-1}E_{\lambda/j,(n-j)}$, *and* $w(E)_{\lambda,(n)}=\sum_{j=1}^{n-1}(-1)^{j-1}w(E)_{\lambda/j,(n-j)}$,
(c) $E_{\gamma,\mu}=\sum_{\alpha\vdash\mu_1,\beta\vdash n-\mu_1}E_{\alpha,(\mu_1)}E_{\beta,(\mu_2,\dots,\mu_k)}\chi(\alpha\cup\beta=\gamma)$, *and*

$$w(E)_{\gamma,\mu} = \sum_{\substack{\alpha\vdash\mu_1\\ \beta\vdash n-\mu_1}} w(E)_{\alpha,(\mu_1)}\,w(E)_{\beta,(\mu_2,\dots,\mu_k)}\,\chi(\alpha\cup\beta=\gamma).$$

Proof. For (a), note that there is only one brick tabloid T in $B_{(n),(n)}$, namely the one where the brick of size n covers the row of size n. Thus $E_{(n),(n)}=\varepsilon(T)=(-1)^{n-1}$, and $w(E)_{(n),(n)}=\varepsilon(T)\,w(T)=(-1)^{n-1}n$. The right-hand sides in (b) result from organizing the sums $\sum_{T\in B_{\lambda,(n)}}\varepsilon(T)$ and $\sum_{T\in B_{\lambda,(n)}}\varepsilon(T)\,w(T)$ according to the size of the first brick b in T. Note that since $k(\lambda)\geq 2$, b is not at the end of a row in T so that $\varepsilon(b)=\varepsilon(b)\,w(b)=(-1)^{|b|-1}$. Similarly, the right-hand sides in (c) result from organizing the sums $\sum_{T\in B_{\gamma,\mu}}\varepsilon(T)$ and $\sum_{T\in B_{\gamma,\mu}}\varepsilon(T)\,w(T)$ according to the bricks that lie in the top row of T. □

Note that Lemma 2.2. together with (2.11) show that $(-1)^{n-k(\mu)}|B_{\mu,\lambda}|$ satisfy exactly the same recursions as $M(e,h)_{\mu,\lambda}$ and $M(h,e)_{\mu,\lambda}$. Thus we have proved the following:

Theorem 2.3. *For all partitions λ and μ of n,*

$$M(e,h)_{\mu,\lambda} = M(h,e)_{\mu,\lambda} = (-1)^{n-k(\mu)}|B_{\mu,\lambda}|.$$

Next we turn to the proof of

$$M(e,p)_{\mu,\lambda} = (-1)^{n-k(\lambda)}w(B_{\mu,\lambda}). \tag{2.13}$$

Just as was the case for the proof of (2.3), we shall show how (2.13) can be derived from the relation (2.2) between e_k's and p_k's. First we write (2.2) in the form

$$p_n = \left(\sum_{k=1}^{n-1}(-1)^{k-1}e_k p_{n-k}\right) + (-1)^{n-1}ne_n. \tag{2.14}$$

Now (2.14) can be used exactly as (2.4) was used in the proof of Lemma 2.2 to derive the following recursions for $M(e,p)_{\mu,\lambda}$:

Lemma 2.4. *Let $\lambda=(\lambda_1,\dots,\lambda_l)$ and $\mu=(\mu_1,\dots,\mu_k)$ be two partitions of n with at least two parts and let γ be an arbitrary partition of n. Then*

(i) $M(e,p)_{(n),(n)}=(-1)^{n-1}n$,

(ii) $M(e,p)_{\lambda,(n)}=\sum_{j=1}^{n-1}(-1)^{j-1}M(e,p)_{\lambda/j,(n-j)}$,

(iii) $M(e,p)_{\gamma,\mu}=\sum_{\alpha\vdash\mu_1,\beta\vdash\mu_2+\cdots+\mu_k}M(e,p)_{\alpha,(\mu_1)}M(e,p)_{\beta,(\mu_2,\dots,\mu_k)}\chi(\alpha\cup\beta=\gamma)$.

It now follows from (2.12) and Lemma 2.2 that the numbers $(-1)^{n-k(\mu)}w(B_{\mu,\lambda})$ satisfy the same recursions as $M(e,p)_{\mu,\lambda}$. Thus since the recursions in Lemma 2.2 clearly determine $M(e,p)_{\mu,\lambda}$ for all μ and λ, we have proved the following:

Theorem 2.5. *For all partitions μ and λ of n,*

$$M(e,p)_{\mu,\lambda}=(-1)^{n-k(\mu)}w(B_{\mu,\lambda}).$$

Remark 2.6. Using Theorem 1.1(iii) and (1.9), Theorem 2.5 immediately implies

$$M(h,p)_{\mu,\lambda}=(-1)^{k(\mu)-k(\lambda)}w(B_{\mu,\lambda}). \tag{2.15}$$

We can however prove (2.15) directly in much the same way as we proved (2.3) and (2.13). That is, we can make use of the simple relation

$$n h_n=\sum_{k=1}^{n}p_k h_{n-k}, \tag{2.16}$$

which can be obtained by applying ω to the identity (2.2). Then (2.16) can be used to establish recursions satisfied by the numbers $M(h,p)_{\mu,\lambda}$. We can then show that $(-1)^{k(\mu)-k(\lambda)}w(B_{\mu,\lambda})$ satisfies the same recursions as $M(h,p)_{\mu,\lambda}$. The details of such a direct proof can be found in [3,5].

Remark 2.7. We can also use (2.2) and (2.16) to develop simple recursions for $M(p,e)_{\mu,\lambda}$ and $M(p,h)_{\mu,\lambda}$ directly. One defines a new weight $\bar{w}$ on λ-brick tabloids so that up to a sign $M(p,e)_{\mu,\lambda}$ and $M(p,h)_{\mu,\lambda}$ are of the form $\sum_{T\in B_{\mu,\lambda}}\bar{w}(T)$ where for any $T\in B_{\mu,\lambda}$, $\bar{w}(T)=\prod_{b\in T}\bar{w}_T(b)$. In this product, $\bar{w}_T(b)=1/k$, where k is the sum of the lengths of the bricks which are weakly to the right of b in the row of T containing b. However, the weight $\bar{w}_T$ is quite cumbersome to calculate.

Our final result in this section is to establish a combinatorial interpretation for $M(m,p)_{\mu,\lambda}$.

Theorem 2.8. *For $\lambda,\mu\vdash n$, $M(m,p)_{\mu,\lambda}=|OB_{\lambda,\mu}|$.* (2.17)

Proof. For each partition $\lambda=(\lambda_1,\dots,\lambda_l)$ of n, we have

$$p_\lambda = \sum_{\mu \vdash n} m_\mu M(m,p)_{\mu,\lambda}. \tag{2.18}$$

Now if $\mu = (\mu_1, \ldots, \mu_k)$, then $M(m,p)_{\mu,\lambda}$ is just the coefficient of m_μ in the expansion of p_λ, written $p_\lambda|_{m_\mu}$. But clearly $p_\lambda|_{m_\mu} = p_\lambda|_{x_1^{\mu_1}\cdots x_k^{\mu_k}}$. Thus

$$M(m,p)_{\mu,\lambda} = p_\lambda|_{x_1^{\mu_1}\cdots x_k^{\mu_k}}. \tag{2.19}$$

If we now think of using the distributive law to multiply out $p_{\lambda_1}\cdots p_{\lambda_l}$, each term in the expansion is of the form $x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_l}^{\lambda_l}$. Thus

$$p_\lambda|_{x_1^{\mu_1}\cdots x_k^{\mu_k}} = \sum_{i_1,\ldots,i_l} x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_l}^{\lambda_l}|_{x_1^{\mu_1}\cdots x_k^{\mu_k}}. \tag{2.20}$$

We note that each sequence $i_1, i_2, \ldots, i_l$ which occurs on the right-hand side of (2.20)

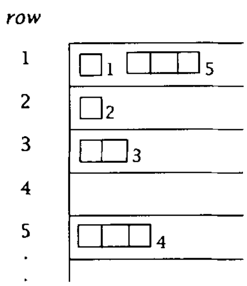

Fig. 6.

naturally corresponds to a placement of λ-bricks in N rows indexed $1, 2, \ldots, N$, where the indices of the bricks increase in each row from right to left. To see this, we simply put brick b_j in row i_j for $j = 1, 2, \ldots, l$. For example the sequence 1, 2, 3, 5, 1 which

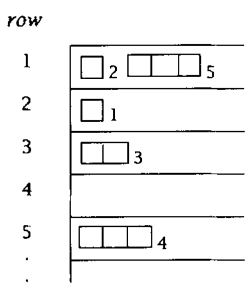

Fig. 7.

gives rise to the monomial $x_1^1 x_2^1 x_3^2 x_5^3 x_1^3$ corresponds to the placement of bricks as shown in Fig. 6. Note that the order does make a difference, for example, the sequence 2, 1, 3, 1, 5 which corresponds to the monomial $x_2^1 x_1^1 x_3^2 x_1^3 x_5^3$ is distinguished from the sequence 1, 2, 3, 5, 1 on the right-hand side of (2.20) even though both sequences give rise to the same monomial. Our placement of bricks reflects this difference. For example, the sequence 2, 1, 3, 1, 5 corresponds to the placement of subscripted λ-bricks as shown in Fig. 7.

Now a sequence $i_1, i_2, \dots, i_l$ gives rise to a monomial of the form $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ if and only if in the corresponding placement of λ-bricks, the lengths of the bricks in row i add up to μ_i for $i = 1, 2, \dots, n$. In other words, the corresponding placement of λ-bricks must be of the form of an ordered λ-brick tabloid of shape μ. Thus

$$p_\lambda \,|_{x_1^{\mu_1} \cdots x_k^{\mu_k}} = |OB_{\lambda, \mu}|. \tag{2.21}$$

Now combining (2.21) with (2.19) proves Theorem 2.8. □

3. Remarks and conclusions

In this section we shall consider a number of consequences of our combinatorial interpretations. We start with the identity (3) of the introduction

$$M(m, f)_{\mu, \lambda} = (-1)^{n - k(\lambda)} |B_{\lambda, \mu}|. \tag{3.1}$$

Thus

$$f_\lambda = (-1)^{n - k(\lambda)} \sum_{\mu \vdash n} |B_{\lambda, \mu}| \, m_\mu. \tag{3.2}$$

If we assign a monomial weight $\bar{m}(T) = m_\mu$ to each λ-brick tabloid T of shape μ, then we arrive at the following expansion for f_λ:

Theorem 3.1.

$$f_\lambda = (-1)^{n - k(\lambda)} \sum_{T \in B_\lambda} \bar{m}(T), \tag{3.3}$$

where B_λ is the collection of all λ-brick tabloids.

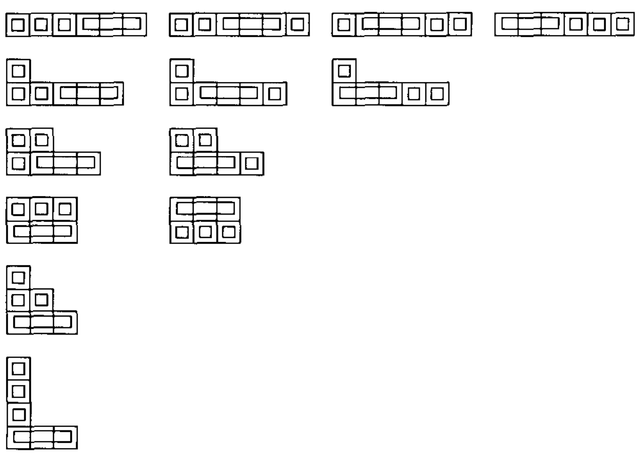

Fig. 8.

Remark 3.2. Theorem 3.1 provides a combinatorial interpretation of f_λ in terms of weights of λ-brick tabloids much like the combinatorial interpretation of s_λ in terms of weights of column strict tableaux. For example, in Fig. 8 we list the set B_λ of all λ-brick tabloids for $\lambda=(1,1,1,3)$. Then for each $T\in B_\lambda$, we simply replace T by the monomial symmetric function $m_{\mathrm{sh}(T)}$ where sh(T) is the shape of T, sum, and then multiply by $(-1)^{n-k(\lambda)}$ to get f_λ. Thus in our example,

$$f_{(1^3,3)}=(-1)^{6-4}(4m_{(6)}+3m_{(1,5)}+2m_{(2,4)}+2m_{(3,3)}+m_{(1,2,3)}+m_{(1^3,3)}). \quad (3.4)$$

From the expressions (9) and (11) of the matrices $M(p,h)$, and $M(p,m)$ of the introduction, we have

$$M(p,h)_{\mu,\lambda}=|OB_{\mu,\lambda}|/z_\mu, \qquad M(p,m)_{\mu,\lambda}=(-1)^{k(\lambda)-k(\mu)}w(B_{\lambda,\mu})/z_\mu. \quad (3.5)$$

We note that in general, $M(p,h)_{\mu,\lambda}$ is not an integer. For example, if $\lambda=(n)$, then there is exactly one ordered μ-brick tabloid of shape (n), so that $|OB_{\mu,(n)}|=1$. Thus $M(p,h)_{\mu,(n)}=1/z_\mu$, which is never an integer for $n>1$. Similarly, $M(p,m)_{\mu,\lambda}$ is not in general an integer. For example, if $\lambda=(1^n)$, then it is easy to see that there is precisely one (1^n)-brick tabloid of shape μ for any μ, and the weight of such a (1^n)-brick tabloid is 1. Thus $w(B_{(1^n),\mu})=1$ and $M(p,m)_{\mu,(1^n)}=(-1)^{n-k(\mu)}/z_\mu$, which is never an integer for $n>1$.

We conclude this paper with a few remarks about the twelve connection matrices we have considered. Given partitions $\lambda=(\lambda_1,\dots,\lambda_l)$ and $\mu=(\mu_1,\dots,\mu_k)$, we say that λ is a *refinement* of μ, written $\lambda\leq_{\mathrm{r}}\mu$, if and only if there is a set partition $(A_1,\dots,A_k)$ of $\{1,\dots,l\}$ such that for each $i\leq k$, $\mu_i=\sum_{j\in A_i}\lambda_j$. It is easy to prove that there is a λ-brick tabloid or an ordered λ-brick tabloid of shape μ if and only if $\lambda\leq_{\mathrm{r}}\mu$. Thus we have the following:

Theorem 3.3. *For partitions λ and μ of n,*
(i) $|B_{\lambda,\mu}|\neq 0$ *if and only if* $\lambda\leq_{\mathrm{r}}\mu$,
(ii) $w(B_{\lambda,\mu})\neq 0$ *if and only if* $\lambda\leq_{\mathrm{r}}\mu$,
(iii) $|OB_{\lambda,\mu}|\neq 0$ *if and only if* $\lambda\leq_{\mathrm{r}}\mu$.

Note that since the lexicographic ordering on partitions is a linear extension of $\leq_{\mathrm{r}}$, Theorem 3.3 implies that each of the twelve matrices we have considered is either upper or lower triangular.

We should also mention that the quantities $w(B_{\lambda,\mu})/z_\mu$ and $|OB_{\mu,\lambda}|/z_\mu$ which appear in matrices (9)–(12) of the introduction have other combinatorial interpretations. Given a partition λ of n, $\lambda=(\lambda_1,\dots,\lambda_l)=(1^{\alpha_1}\cdots n^{\alpha_n})$, let $\lambda!=\lambda_1!\cdots\lambda_l!$ as before, and $\alpha(\lambda)!=\alpha_1!\cdots\alpha_n!$. Let $S_\lambda=S_{\lambda_1}\times\cdots\times S_{\lambda_l}$ be the Young subgroup [7] of the symmetric group S_n where for each i, S_{λ_i} is the subgroup of all permutations on

$$\left\{1+\sum_{j=1}^{i-1}\lambda_j, 2+\sum_{j=1}^{i-1}\lambda_j,\dots,\lambda_i+\sum_{j=1}^{i-1}\lambda_j\right\}. \quad (3.6)$$

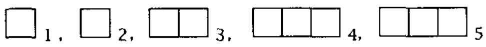

Fig. 9.

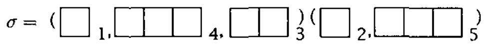

Fig. 10.

We let C_μ denote the conjugacy class corresponding to the partition μ. Thus C_μ is set of all $\sigma \in S_n$ such that the partition of n induced by the cycles of σ is μ. Then it is proved in [3] that

$$\frac{|OB_{\mu,\lambda}|}{z_\mu} = \frac{|S_\lambda \cap C_\mu|}{\lambda!}. \tag{3.7}$$

For a second interpretation for $w(B_{\lambda,\mu})/z_\mu$, we need to define the concept of λ-brick permutations. Given $\lambda = (0 < \lambda_1 \leq \cdots \leq \lambda_k)$, we start with the set of subscripted λ-bricks as in the definition of ordered λ-brick tabloids. For example, if $\lambda = (1,1,2,3,3)$, then our set of bricks consists of the ones shown in Fig. 9. We then let $S(\lambda)$ denote the set of all permutations of these bricks. We shall write a λ-brick permutation in its cycle structure. For example, if σ is as shown in Fig. 10, then σ corresponds to the λ-brick permutation whose cycle diagram is given in Fig. 11. In one line notation, σ would be written as in Fig. 12.

Given a cycle $c = (b_{i_1}, \ldots, b_{i_l})$, we define the *shape* of c by $\mathrm{sh}(c) = \sum_{j=1}^{l} |b_{i_j}|$ where $|b_{i_j}|$ denotes the size of the brick b_{i_j}. The shape, $\mathrm{sh}(\sigma)$, of a λ-brick permutation σ which consists of the cycles $c_1, \ldots, c_l$ in its cycle decomposition, is defined to be the partition which is the increasing rearrangement of the sequence $\mathrm{sh}(c_1), \ldots, \mathrm{sh}(c_l)$. For example for σ pictured in Fig. 10, $\mathrm{sh}(\sigma) = (4,6)$. We let $S(\lambda)^\mu$ denote the set of all λ-brick permutations of shape μ. Then it is proved in [5] that

$$\frac{w(B_{\lambda,\mu})}{z_\mu} = \frac{|S(\lambda)^\mu|}{\alpha(\lambda)!}. \tag{3.8}$$

Identities such as (3.7) and (3.8) which provide alternate interpretations for the entries of a number of transition matrices for bases of symmetric functions turn out to be useful in combinatorial proofs of matrix identities that arise in this setting [3].

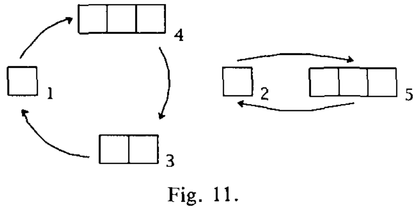

Fig. 11.

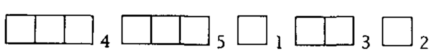

Fig. 12.

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