A Catalan-Hankel Determinant Evaluation

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Abstract

Let $C_k = \binom{2k}{k}/(k+1)$ denote the k-th Catalan number and put $a_k(x) = C_k + C_{k-1}x + \cdots + C_0 x^k$. Define the $(n+1)\times(n+1)$ Hankel determinant by setting $H_n(x) = \det[a_{i+j}(x)]_{0\le i,j\le n}$. Even though $H_n(x)$ does not admit a product form evaluation for arbitrary x, the recently introduced technique of γ-operators is applicable. We illustrate this technique by evaluating this Hankel determinant as

$$H_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n+i}{n-i} x^i \ .$$

The Hankel determinant of the polynomial sequence where the coefficients are central binomial coefficients instead of Catalan numbers can also be evaluated in a similar form.

1 Introduction

Given a sequence $a_0, a_1, \ldots$, the sequence of Hankel determinants $H_0, H_1, \ldots$, defined in terms of the a_k's by

$$H_n = \det[a_{i+j}]_{0\le i,j\le n}$$

is sometimes referred to as the *Hankel transform* of the original sequence [12]. Many interesting properties and instances of the Hankel transform are known [3, 16, 2]. In particular, for the Catalan sequence

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

taking $a_k = C_k = \binom{2k}{k}/(k+1)$ for $k \ge 0$, it is well-known that the Hankel transform is the constant sequence $H_n = 1$ [18]. Hankel matrices of Catalan and Catalan-like numbers of combinatorial interest have been studied by many authors; we can only give a few references: [18, 1, 20, 17, 8, 11, 5, 7]. Define

$$a_k(x) = \sum_{m=0}^{k} C_{k-m} x^m \tag{1}$$

so that $a_k(0) = C_k$. We have

$$\begin{aligned}
a_0(x) &= 1 \\
a_1(x) &= x+1 \\
a_2(x) &= x^2+x+2 \\
a_3(x) &= x^3+x^2+2x+5 \\
a_4(x) &= x^4+x^3+2x^2+5x+14 \\
a_5(x) &= x^5+x^4+2x^3+5x^2+14x+42
\end{aligned}$$

Let

$$H_n(x) = \det[a_{i+j}(x)]_{0\le i,j\le n} \ . \tag{2}$$

First few of these Hankel determinants are as follows:

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= -x+1 \\
H_2(x) &= x^2-3x+1 \\
H_3(x) &= -x^3+5x^2-6x+1 \\
H_4(x) &= x^4-7x^3+15x^2-10x+1 \\
H_5(x) &= -x^5+9x^4-28x^3+35x^2-15x+1
\end{aligned}$$

Since $a_k(x) - x a_{k-1}(x) = C_k$, elementary row operations give

$$H_n(x) = \det \begin{bmatrix} a_0(x) & a_1(x) & \cdots & a_n(x) \\ C_1 & C_2 & \cdots & C_{n+1} \\ C_2 & C_3 & \cdots & C_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ C_n & C_{n+1} & \cdots & C_{2n} \end{bmatrix} \tag{3}$$

and therefore $H_n(x)$ is a polynomial of degree n. In this paper we make use of the technique of γ-operators introduced in [6] to explicitly evaluate $H_n(x)$ in the following form:

Theorem 1 *Suppose $a_k(x)$ and the $H_n(x)$ are as defined in (1) and (2). Then*

$$H_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n+i}{n-i} x^i \ . \tag{4}$$

The γ-operator technique to find the explicit form of $H_n(x)$ relies on differential-convolution equations, and in this case establishes a second order differential equation for $H_n(x)$. We show that $y = H_n(x)$ satisfies

$$x(x-4)y'' + 2(x-1)y' - n(n+1)y = 0 \tag{5}$$

from which the solution (4) follows by the Frobenius method.

For product form evaluations of various combinatorially interesting determinants, LU decomposition, continued fractions and Dodgson condensation are the classical methods that have been used successfully. There exists an extensive literature on this topic, going back to the treatise of Muir [13, 14]. We direct the reader to Krattenthaler [9, 10] for a modern treatment of the theory of determinant evaluation including Hankel determinants.

The γ-operator method introduced in [6] is applicable to a number of Hankel determinant evaluations including those that do not admit a product form evaluation, but which can be evaluated as an *almost product*; a sum of a small number of products [5]. Examples of such evaluations in product and almost product forms involving various binomial coefficients can be found in [8, 5, 6, 7].

2 Preliminaries

A *partition* λ of an integer $m > 0$ is a weakly decreasing sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0)$ with $m = \lambda_1 + \lambda_2 + \cdots + \lambda_p$. Each λ_i is a *part* of λ. We use the notation $\lambda = m^{\alpha_m} \cdots 2^{\alpha_2} 1^{\alpha_1}$, indicating that λ has α_i parts of size i. Given $n > 0$, each partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0)$ with $p \leq n+1$ defines a determinant of a matrix obtained from the $(n+1) \times (n+1)$ Hankel matrix $A_n = [a_{i+j}]_{0 \leq i,j \leq n}$ by shifting the column indices of the entries up according to λ as follows: Let $\mu_i = \lambda_i$ for $i = 1, \ldots, p$ and $\mu_i = 0$ for $i > p$. Then

$$H_n(x, \lambda) = \det[a_{i+j+\mu_{n+1-j}}]_{0 \leq i,j \leq n} \ . \tag{6}$$

We use the special notation 0 to denote the sequence $\mu_i = 0$ for $i = 1, \ldots, n+1$. For example when $n = 3$,

$$H_n(x, 0) = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}, \ H_n(x, 31^2) = \det \begin{bmatrix} a_0 & a_2 & a_3 & a_6 \\ a_1 & a_3 & a_4 & a_7 \\ a_2 & a_4 & a_5 & a_8 \\ a_3 & a_5 & a_6 & a_{10} \end{bmatrix}.$$

Note that with this notation, $H_n(x)$ of (4) is denoted by $H_n(x, 0)$. We will keep the notation $H_n(x)$ for the determinant itself instead of $H_n(x, 0)$.

Let $a_k(x)$ be as in (1). We define the convolution polynomials $c_n = c_n(x)$ by

$$c_n = \sum_{k=0}^{n} a_k a_{n-k}$$

with $c_{-1} = 0$.

The γ-operator is a multilinear operator defined on m-tuples of matrices:

Definition 1 *Given $(n+1)\times(n+1)$ matrices A and $X_1, X_2, \ldots, X_m$ with $m \geq 1$, define $\gamma_A(\,) = \det(A)$ and*

$$\gamma_A(X_1, \ldots, X_m) = \\ \partial_{t_1}\partial_{t_2}\cdots\partial_{t_m} \det(A + t_1X_1 + t_2X_2 + \cdots + t_mX_m)|_{t_1=\cdots=t_m=0}$$

where $t_1, t_2, \ldots, t_m$ are variables that do not appear in A or $X_1, X_2, \ldots, X_m$.

The γ-operators behave well with respect to differentiation; the derivative of a γ is a sum of γ's.

Proposition 1 *For $m \leq n$,*

$$\begin{aligned} \frac{d}{dx}\gamma_A(X_1, \ldots, X_m) &= \gamma_A(\frac{d}{dx}A, X_1, \ldots, X_m) \\ &\quad + \sum_{j=1}^{m} \gamma_A(X_1, \ldots, X_{j-1}, \frac{d}{dx}X_j, X_{j+1}, \ldots, X_m)\ . \end{aligned}$$

The reader is referred to [6] for the proofs of various properties of γ-operators. It is worth mentioning that the values of the γ-operators need not be calculated from scratch for different Hankel determinant evaluations. Let $A_n = [a_{i+j}]_{0\leq i,j\leq n}$ be a Hankel matrix in the generic symbols a_k. Extensive tables of of values of γ-operators on various kinds of matrices as well as a computationally feasible combinatorial interpretation of $\gamma_A(X_1, \ldots, X_m)$ for small m can be found in [6]. Suppressing the dependence on n for the moment for notational convenience, denote the shifted Hankel matrices defined via the right hand side of (6) by H_λ. The γ-operator evaluations needed for the derivations in this paper are given in Tables 1 and 2 with this notation.

3 Identities and expansions

The bulk of the work for the derivation of the differential equation (5) relies on three essential identities, which are characteristic of the method. The first is a differential-convolution equation. The second involves convolutions and the a_k but no derivatives. The third identity is a relation between shifted columns of the matrix A_n.

$$\begin{aligned}
\gamma_A([a_{i+j}]) &= (n+1)H_0 \\
\gamma_A([a_{i+j+1}]) &= H_1 \\
\gamma_A([a_{i+j+2}]) &= H_2 - H_{1^2} \\
\gamma_A([(i+j)a_{i+j}]) &= n(n+1)H_0 \\
\gamma_A([(i+j)a_{i+j+1}]) &= 2nH_1 \\
\gamma_A([(i+j)a_{i+j+2}]) &= 2nH_2 - 2(n-1)H_{1^2} \\
\gamma_A([c_{i+j-1}]) &= 0 \\
\gamma_A([c_{i+j}]) &= (2n+1)a_0H_0 \\
\gamma_A([c_{i+j+1}]) &= 2a_0H_1 + 2na_1H_0
\end{aligned}$$

Table 1: $\gamma_A(*)$ computations.

$$\begin{aligned}
\gamma_A([a_{i+j+1}],[a_{i+j}]) &= nH_1 \\
\gamma_A([a_{i+j+1}],[a_{i+j+1}]) &= 2H_{1^2} \\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j}]) &= n(n-1)H_1 \\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+1}]) &= 2(2n-1)H_{1^2} \\
\gamma_A([a_{i+j+1}],[c_{i+j-1}]) &= -2na_0H_0 \\
\gamma_A([a_{i+j+1}],[c_{i+j}]) &= (2n-1)a_0H_1 - (2n-1)a_1H_0
\end{aligned}$$

Table 2: $\gamma_A(([a_{i+j+1}],*)$ computations.

Lemma 1 *(First Identity (FI))*

$$x(x-4)\frac{d}{dx}a_n = (n+1)a_{n+1} - (2+4n+x)a_n + c_n - xc_{n-1} \ .$$

Lemma 2 *(Second Identity (SI))*

$$2(2n+3)xa_n - (6+4n+3x+nx)a_{n+1} + (n+3)a_{n+2} = 0 \ . \quad (7)$$

Lemma 3 *(Third Identity (TI))*

$$\sum_{j=0}^{n+2} w_{n,j}(x)a_{i+j}(x) = 0 \quad (8)$$

for $i = 0, 1, \ldots, n$ *where*

$$w_{n,j} = (-1)^{n-j}\left[\binom{n+j+1}{2j-1} + \binom{n+j+2}{2j+1}x\right] \ . \quad (9)$$

From (9), we have that in particular

$$\begin{aligned}
w_{n,n+2} &= 1 \\
w_{n,n+1} &= -(2+2n+x) \\
w_{n,n} &= n+2n^2+2x+2nx\ .
\end{aligned} \tag{10}$$

Proofs of the lemmas
We sketch the proofs of these three lemmas. FI and SI can be proved readily by generating function methods. We compute

$$\begin{aligned}
f(x,y) &= \sum_{n\geq 0} a_k(x)y^k \\
&= \sum_{k\geq 0} x^k \sum_{n\geq k} \frac{1}{n-k+1}\binom{2(n-k)}{n-k} y^n \\
&= \sum_{k\geq 0} x^k y^k \left(\frac{1-\sqrt{1-4y}}{2y}\right) \\
&= \frac{1-\sqrt{1-4y}}{2y(1-xy)}
\end{aligned}$$

Denoting differentiation with respect to y by primes below, we make the substitutions

$$\begin{aligned}
\frac{d}{dx}a_n &\rightarrow \frac{d}{dx}f \\
a_n &\rightarrow f \\
na_n &\rightarrow yf' \\
a_{n+1} &\rightarrow (f-1)/y \\
na_{n+1} &\rightarrow y((f-1)/y)' \\
a_{n+2} &\rightarrow (f-1-(1+x)y)/y^2 \\
na_{n+2} &\rightarrow y((f-1-(1+x)y)/y^2)' \\
c_{n-1} &\rightarrow yf^2 \\
c_n &\rightarrow f^2
\end{aligned}$$

in the first two lemmas. The proofs of the FI and the SI become algebraic verification of two functional identities involving $f(x,y)$. These verifications can be carried out easily on any symbolic algebra package such as Mathematica or Maple.

The weights in Lemma 3 are typical of the λ-operator method for Hankel determinant evaluation. We do not give the proof of the third identity in Lemma 3 but remark that once the weights are guessed as we have in (9), the proofs of the identities can be left to automatic binomial identity provers such as `MultiZeilberger` supplied by Doron Zeilberger (in Maple [21]), and `MultiSum` by Wegschaider (in Mathematica [19]). •

3.1 Equation from the SI

Apply $\gamma_A(*)$ to the $(n+1)\times(n+1)$ matrix whose (i,j)-th entry is obtained from the second identity (7) evaluated at $i+j$ and expand using linearity. In other words, if we denote the matrix so obtained from the second identity by $[SI(i+j)]$, then the computation is the expansion of $\gamma_A([SI(i+j)])=0$. For notational convenience, we can write $[SI(n)]$ for the $(n+1)\times(n+1)$ matrix $[SI(i+j)]$, $[nSI(n)]$ for the matrix $[(i+j)SI(i+j)]$, etc. Making use of the entries in the $\gamma_A(*)$ computations from Table 1, we get

$$\begin{aligned} 0 &= 6x\gamma_A([a_n]) + 4x\gamma_A([na_n]) - (3x+6)\gamma_A([a_{n+1}]) \\ &\quad -(4+x)\gamma_A([na_{n+1}]) + 3\gamma_A([a_{n+2}]) + \gamma_A([na_{n+2}]). \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= 6x(n+1)H_n(x) + 4xn(n+1)H_n(x) - (3x+6)H_n(x,1) \\ &\quad -(4+x)2nH_n(x,1) + 3(H_n(x,2) - H_n(x,1^2)) \\ &\quad +2nH_n(x,2) - 2(n-1)H_n(x,1^2) \end{aligned}$$

and this can be simplified as

$$\begin{aligned} 0 &= 2(n+1)(2n+3)xH_n(x) - (6+8n+3x+2nx)H_n(x,1) \\ &\qquad -(2n+1)H_n(x,1^2) + (2n+3)H_n(x,2). \end{aligned} \tag{11}$$

The identity (11) is a linear relation between the four determinants $H_n(x)$, $H_n(x,1)$, $H_n(x,2)$ and $H_n(x,1^2)$.

3.2 Equation from the TI

We will use the third identity in a determinantal form. Let

$$v_j = [a_j, a_{j+1}, \ldots, a_{j+n}]^T \ .$$

The third identity (8) says that the vectors $v_0, v_1, \ldots, v_{n+2}$ are linearly dependent with coefficients $w_{n,j}$, i.e.

$$\sum_{j=0}^{n+2} w_{n,j} v_j = 0 \ . \tag{12}$$

Now consider the determinant of the $(n+1)\times(n+1)$ matrix whose first n columns are the columns of A, and whose last column is the zero vector. Writing the zero vector in the form (12) and expanding the determinant by linearity, we find

$$w_{n,n+2}H_n(x,2) + w_{n,n+1}H_n(x,1) + w_{n,n}H_n(x) = 0 \ . \tag{13}$$

Substituting the particular weights (9) of the third identity we have

$$H_n(x,2) - (2+2n+x)H_n(x,1) + (n+2n^2+2x+2nx)H_n(x) = 0. \quad (14)$$

This is a linear relation between $H_n(x)$, $H_n(x,1)$ and $H_n(x,2)$. Solving the system (11), and (14) for $H_n(x,1^2), H_n(x,2)$ in terms of $H_n(x)$ and $H_n(x,1)$,

$$\begin{aligned} H_n(x,1^2) &= -n(3+2n)H_n(x) + 2nH_n(x,1) \\ H_n(x,2) &= -(n+2n^2+2x+2nx)H_n(x) + (2+2n+x)H_n(x,1) \end{aligned} \quad (15)$$

4 The derivatives of $H_n(x)$ and $H_n(x,1)$

We now calculate the derivatives of $H_n(x)$ and $H_n(x,1)$. We will find an expression for $\frac{d}{dx}H_n(x)$ in terms of $H_n(x)$ and $H_n(x,1)$; and then find an expression for $\frac{d}{dx}H_n(x,1)$ in terms of $H_n(x)$, $H_n(x,1)$, $H_n(x,2)$ and $H_n(x,1^2)$. In view of (15), we can eliminate $H_n(x,2)$ and $H_n(x,1^2)$. This results in a linear system involving $\frac{d}{dx}H_n(x)$, $\frac{d}{dx}H_n(x,1)$, $H_n(x)$ and $H_n(x,1)$, from which the second order differential equation for $H_n(x)$ can be constructed.

4.1 The derivative of $H_n(x)$

Since $H_n(x) = \gamma_A(\cdot)$, using Proposition 1 we obtain

$$\frac{d}{dx}H_n(x) = \gamma_A([\frac{d}{dx}a_{i+j}]).$$

From the first identity at $i+j$ (i.e. $FI(n)$) we obtain

$$\begin{aligned} x(x-4)\frac{d}{dx}H_n(x) &= -(2+x)\gamma_A([a_n]) - 4\gamma_A([na_n]) \\ &\quad +\gamma_A([a_{n+1}]) + \gamma_A([na_{n+1}]) - x\gamma_A([c_{n-1}]) + \gamma_A([c_n]). \end{aligned}$$

Making the replacements from the γ tables, we have

$$\begin{aligned} x(x-4)\frac{d}{dx}H_n(x) &= -(2+x)(n+1)H_n(x) - 4n(n+1)H_n(x) \\ &\quad +H_n(x,1) + 2nH_n(x,1) + (2n+1)H_n(x) \end{aligned}$$

and therefore

$$\begin{aligned} x(x-4)\frac{d}{dx}H_n(x) &= -(1+4n+4n^2+x+nx)H_n(x) \\ &\quad +(2n+1)H_n(x,1). \end{aligned} \quad (16)$$

4.2 The derivative of $H_n(x,1)$

To differentiate $H_n(x,1)$ we use the expression $H_n(x,1) = \gamma_A([a_{i+j+1}])$ from Table 1 and Proposition 1:

$$\frac{d}{dx}H_n(x,1) = \gamma_A([a_{i+j+1}],[\frac{d}{dx}a_{i+j}]) + \gamma_A([\frac{d}{dx}a_{i+j+1}]) \ .$$

Therefore, to compute $\frac{d}{dx}H_n(x,1)$

$$\gamma_A([a_{i+j+1}],[FI(i+j)]) \ \text{ and } \ \gamma_A([FI(i+j+1)])$$

are needed. For the first one of these we obtain

$$\begin{aligned}
&x(x-4)\gamma_A([a_{n+1}],\frac{d}{dx}A) = -(x+2)\gamma_A([a_{n+1}],[a_n]) \\
&-4\gamma_A([a_{n+1}],[na_n]) + \gamma_A([a_{n+1}],[a_{n+1}]) \\
&+\gamma_A([a_{n+1}],[na_{n+1}]) - x\gamma_A([a_{n+1}],[c_{n-1}]) + \gamma_A([a_{n+1}],[c_n]).
\end{aligned}$$

Using the entries in the $\gamma_A([a_{i+j+1}],*)$ computations from Table 2, we get

$$\begin{aligned}
&x(x-4)\gamma_A([a_{n+1}],\frac{d}{dx}A) = -(x+2)nH_n(x,1) - 4n(n-1)H_n(x,1) \\
&+2H_n(x,1^2) + 2(2n-1)H_n(x,1^2) - x(-2nH_n(x)) \\
&+(2n-1)H_n(x,1) - (2n-1)(x+1)H_n(x).
\end{aligned} \tag{17}$$

We evaluate $\gamma_A(\frac{d}{dx}[a_{i+j+1}])$ as

$$\begin{aligned}
&x(x-4)\gamma_A(\frac{d}{dx}[a_{n+1}]) = -(x+6)\gamma_A([a_{n+1}]) - 4\gamma_A([na_{n+1}]) \\
&+2\gamma_A([a_{n+2}]) + \gamma_A([na_{n+2}]) - x\gamma_A([c_n]) + \gamma_A([c_{n+1}]).
\end{aligned}$$

Therefore

$$\begin{aligned}
&x(x-4)\gamma_A(\frac{d}{dx}[a_{n+1}]) = -(x+6)H_n(x,1) - 8nH_n(x,1) \\
&+2(H_n(x,2) - H_n(x,1^2)) + 2nH_n(x,2) - 2(n-1)H_n(x,1^2) \\
&-x(2n+1)H_n(x) + 2H_n(x,1) + 2n(x+1)H_n(x).
\end{aligned} \tag{18}$$

Adding the expressions in (18) and (19),

$$\begin{aligned}
x(x-4)\frac{d}{dx}H_n(x,1) \quad &= \quad H_n(x) - (5+4n+4n^2+x+nx)H_n(x,1) \\
&\quad +2(1+n)H_n(x,2) + 2nH_n(x,1^2).
\end{aligned} \tag{19}$$

5 The differential equation for $H_n(x)$

Differentiating (16) and substituting $\frac{d}{dx}H_n(x,1)$ as given above in (19), and eliminating $H_n(x,2)$ and $H_n(x,1^2)$ via (15),

$$x^2(x-4)^2\frac{d^2}{dx^2}H_n(x) = (x-4)x(3-4n-4n^2-3x-nx)\frac{d}{dx}H_n(x)$$
$$+(1-16n^2-32n^3-16n^4-12nx-20n^2x-8n^3x-x^2-nx^2)H_n(x)$$
$$-(1+2n)(1-4n-4n^2-x-nx)H_n(x,1).$$

Using this identity and the expression for $\frac{d}{dx}H_n(x)$ in (16), we eliminate $H_n(x,1)$ to obtain

$$x(x-4)\frac{d^2}{dx^2}H_n(x)+2(x-1)\frac{d}{dx}H_n(x)-n(n+1)H_n(x)=0,$$

which is (5). After some manipulation, the Frobenius solution is found to be

$$H_n(x) = k_0\sum_{i=0}^{n}(-1)^i\binom{n+i}{n-i}x^i\ . \tag{20}$$

The constant of integration $k_0 = H_n(0)$ is the Hankel transform of the Catalan sequence. Therefore $k_0 = 1$ for all n [18] and we have (4).

6 Evaluation at special values and additional results

The general result on Hankel determinants ([5], Section 3, Proposition 1) in our case becomes:

Proposition 2

$$H_{n-1}(x)H_{n+1}(x) = H_n(x)H_n(x,2)+H_n(x)H_n(x,1^2)-H_n(x,1)^2\ . \tag{21}$$

This identity allows us to evaluate $H_n(x)$ easily for special values of x. For example at $x=0$, from (16), we obtain

$$H_n(0,1) = 2n+1\ .$$

Using (19) and (15), we also find the evaluations

$$\begin{aligned} H_n(0,1^2) &= n(2n-1) \\ H_n(0,2) &= (n+2)(2n+1). \end{aligned}$$

Note that at $x = 0$ identity (21) reads

$$1 = H_n(0,2) + H_n(0,1^2) - H_n(0,1)^2$$

which of course agrees with the values computed. The same equations we used for the above specialization at $x = 4$ give

$$\begin{aligned} H_n(4,1) &= \frac{4n^2+8n+5}{2n+1} H_n(4) \\ H_n(4,1^2) &= \frac{n(4n^2+8n+7)}{2n+1} H_n(4) \\ H_n(4,2) &= \frac{4n^3+20n^2+33n+22}{2n+1} H_n(4). \end{aligned} \tag{22}$$

Let $\gamma_n = H_n(4,0)$. Using (22) and identity (21) we get

$$\gamma_{n-1}\gamma_{n+1} = \frac{(2n-1)(2n+3)}{(2n+1)^2} \gamma_n^2.$$

Solving the resulting recurrence relation for $\frac{\gamma_n}{\gamma_{n-1}}$ with $\gamma_0 = 1$ and $\gamma_1 = -3$, we find

$$\frac{\gamma_n}{\gamma_{n-1}} = -\frac{2n+1}{2n-1}$$

from which it follows that

$$\gamma_n = (-1)^n(2n+1)\ .$$

In this case

$$a_k(4) = 2^{2k+1} - \binom{2k+1}{k+1}$$

and this particular evaluation of the Hankel determinant is known [15].

We collect these specializations of $H_n(x)$ in the following theorem:

Theorem 2 *Suppose $a_k(x)$ and the $H_n(x)$ are as defined in (1) and (2). Then*

$$\begin{aligned} \det[a_{i+j}(0)]_{0\le i,j\le n} &= 1, \\ \det[a_{i+j}(1)]_{0\le i,j\le n} &= \begin{cases} -1 & \textit{if } n \equiv 2,3 \pmod 6 \\ 0 & \textit{if } n \equiv 1,4 \pmod 6 \\ 1 & \textit{if } n \equiv 0,5 \pmod 6), \end{cases} \\ \det[a_{i+j}(2)]_{0\le i,j\le n} &= (-1)^{\frac{n(n+1)}{2}}, \\ \det[a_{i+j}(4)]_{0\le i,j\le n} &= (-1)^n(2n+1). \end{aligned}$$

6.1 The generating function of $H_n(x)$

It is useful to have the generating function of the sequence $H_n(x)$ itself. Specializations of the determinant at various x can be found directly from the generating function. We have

$$\sum_{n\geq 0}\binom{n+k}{n-k}y^n=\frac{y^k}{(1-y)^{2k+1}}$$

and using this we compute that

$$\sum_{n\geq 0}H_n(x)y^n=\frac{1-y}{1+(x-2)y+y^2}\ .$$

6.2 The central binomial case

It can be shown that taking

$$a_k(x)=\sum_{m=0}^{k}\binom{2k-2m}{k-m}x^m, \tag{23}$$

the corresponding Hankel determinant $H_n(x)$ also satisfies the differential equation (5). Therefore $H_n(x)$ is of the form (20). The constant of integration k_0 is now the Hankel transform of the sequence $a_k=\binom{2k}{k}$, which is known to be $H_n(0)=2^n$ [4]. We record this evaluation in the following theorem.

Theorem 3 *Suppose $a_k(x)$ and the $H_n(x)$ are as defined in (23) and (2). Then*

$$H_n(x)=2^n\sum_{i=0}^{n}(-1)^i\binom{n+i}{n-i}x^i\ .$$

6.3 Interlacing of the zeros

Since we have an explicit expression for $H_n(x)$ in terms of binomial coefficients, and we know the differential equation that it satisfies, we can derive additional properties of the determinants. We used Mathematica for alternate expressions for $H_n(x)$. $H_n(x)$ can be written in terms of the trigonometric functions as

$$H_n(x)=\frac{2\cos\left((2n+1)\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\right)}{\sqrt{4-x}}\ . \tag{24}$$

In view of the differential equation (5), this can also be written in terms of the associated Legendre function of the second kind. Additional properties such as the simplicity and the location of its roots and interleaving of the zeros of $H_n(x)$ and $H_{n+1}(x)$ can also be fairly directly found. For example the $H_n(x)$ has n real simple zeros on the interval $(0,4)$ and forms a Sturm sequence. For the interlacing property, we can show that $H'_{n+1}H_n - H_{n+1}H'_n < 0$ on $(0,4)$ directly. Using the expression (24) for $H_n(x)$, we compute the expression on the left hand side as

$$\frac{-2}{(4-x)\sqrt{(4-x)x}}\left[(n+1)\sqrt{(4-x)x} + \sin\left(4(n+1)\csc^{-1}(2/\sqrt{x})\right)\right]$$

Therefore the interlacing property is a consequence of the inequality

$$(n+1)\sqrt{(4-x)x} + \sin\left(4(n+1)\csc^{-1}\left(\frac{2}{\sqrt{x}}\right)\right) > 0$$

valid for $n \geq 0$ and $0 < x < 4$.

6.4 Alternate expansion of $H_n(x)$

Let D_j denote the determinant of the $n \times n$ matrix obtained by omitting the j-th column of the $n \times (n+1)$ matrix

$$\begin{bmatrix} C_1 & C_2 & \cdots & C_{n+1} \\ C_2 & C_3 & \cdots & C_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ C_n & C_{n+1} & \cdots & C_{2n} \end{bmatrix}. \tag{25}$$

Expanding the determinant (3) for $H_n(x)$ by the first row and using (4),

$$\sum_{j=0}^{n}(-1)^j a_j(x) D_{j+1} = \sum_{j=0}^{n}(-1)^j \binom{n+j}{n-j} x^j .$$

Equating coefficients of x^i,

$$\sum_{j=0}^{n}(-1)^j C_{j-i} D_{j+1} = (-1)^i \binom{n+i}{n-i} .$$

This can be rewritten as the matrix equation

$$\begin{bmatrix} C_0 & C_1 & C_2 & \cdots & C_n \\ 0 & C_0 & C_1 & \cdots & C_{n-1} \\ 0 & 0 & C_0 & \cdots & C_{n-2} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C_0 \end{bmatrix} \begin{bmatrix} D_1 \\ -D_2 \\ D_3 \\ \vdots \\ (-1)^n D_{n+1} \end{bmatrix} = \begin{bmatrix} \binom{n}{n} \\ -\binom{n+1}{n-1} \\ \binom{n+2}{n-2} \\ \vdots \\ (-1)^n \binom{2n}{0} \end{bmatrix}.$$

By Cramer's rule, $(-1)^n D_{n+1} = (-1)^n$, which is a restatement of the known [18] result $\det[C_{i+j-1}]_{1\le i,j\le n} = 1$.

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