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A note on iterated galileo sequences

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Galileo sequences are generalizations of a simple sequence of integers that Galileo used in early 17th century for describing his law of falling bodies. The curious property he noted happens to be exactly what is needed to quantify his observation that the acceleration of falling bodies is uniform. Among the generalizations and extensions later given are *iterated* Galileo sequences. We show that these are closely related to polynomials that arise in enumerating integers by their Hamming weight.

Keywords: Galileo sequence; iterated Galileo sequence; Hamming weight.

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1. Galileo Sequences

A sequence a_n, $n = 1, 2, \ldots$ of positive integers with partial sums $S_n = a_1 + a_2 + \cdots + a_n$ is a (*standard*) *Galileo sequence* if the S_n satisfy

$$S_{2n} = qS_n, \quad n = 1, 2, \ldots$$

for a fixed constant q. In this case, the original sequence satisfies

$$a_{2n-1} + a_{2n} = qa_n, \quad n = 1, 2, \ldots. \tag{1.1}$$

The first such sequence was provided by Galileo in early 17th century. His sequence is $a_n = 2n - 1$, and so $S_n = n^2$ and $q = 4$. This came about in the context of his inclined plane experiments which led to the remarkable discovery of the law of falling bodies. By "slowing motion" and using a rolling ball on an inclined plane which triggered bells as it rolled down, he found that in equal time intervals, the ball covered distances of $1, 3, 5, 7, \ldots$ units, regardless of its weight and the angle of inclination. So the distance covered by the ball depended on the square of time, and the velocity of the ball increased linearly with time, as he expounded in his *Dialogues of Two New Sciences* of 1638 [1].

Galileo noted that this last property of velocity, i.e., that the acceleration of falling bodies is uniform, can be expressed as

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)} = \cdots$$

and in fact, almost magically, $a_n = 2n-1$ with $q = 4$ is the only arithmetic progression with $a_1 = 1$ which satisfies this condition.

If we do not insist on a_n being an arithmetic progression or that the sequence be integers or be nonnegative, whole classes of Galileo sequences exist, as was shown by Zeitlin. In [3], he extended and generalized standard Galileo sequences to many other types, writing a few years after Kenneth May pointed out the usefulness of this sequence as a teaching tool because of its curious numerical properties [2].

2. Iterated Galileo Sequences

Zeitlin also considered the interesting problem of the enumeration of *iterated Galileo sequences.* These are Galileo sequences a_n, whose partial sums S_n, $n = 1, 2, \ldots$ themselves form Galileo sequences. So here, $S_n = a_1 + \cdots + a_n$ and $S_n^* = S_1 + \cdots + S_n$ with a_n and S_n both Galileo sequences. Letting $S_{2n} = qS_n$ and $S_{2n}^* = q^* S_n^*$, this time the sequence a_n must satisfy

$$a_{2n-2} + 2a_{2n-1} + a_{2n} = q^* a_n \quad \text{and} \quad a_{2n-1} + a_{2n} = q\, a_n. \tag{2.1}$$

Zeitlin shows that we must have $q^* = q + 1$ and $S_{2n-1} = S_n$ and $S_{2n} = qS_{2n-1}$. With $a_1 = 1$ (and taking $a_0 = 0$), we can view a_n as a polynomial in q. Using (2.1),

$$a_n = \begin{cases} q\, a_{\frac{n}{2}} - a_{n-1}, & n \text{ even}, \\ a_{\frac{n+1}{2}} - a_{n-1}, & n \text{ odd}. \end{cases} \tag{2.2}$$

For $q = 2, 3, \ldots$, the first eight elements of the sequences a_n, S_n, S_n^* for iterated Galileo sequences were given by Zeitlin in [3]. Longer lists to $n = 16$ of these sequences in terms of the parameter q are as follows:

$$\begin{aligned}
a_n &= (1, q-1, 0, q^2-q, q-q^2, q^2-q, 0, q^3-q^2, q-q^3, q^2-q, 0, q^3-q^2, q^2-q^3, \\
&\qquad q^3-q^2, 0, q^4-q^3, \ldots) \\
S_n &= (1, q, q, q^2, q, q^2, q^2, q^3, q, q^2, q^2, q^3, q^2, q^3, q^3, q^4, \ldots) \\
S_n^* &= (1, q+1, 2q+1, (q+1)^2, q^2+3q+1, (q+1)(2q+1), 3q^2+3q+1, (q+1)^3, \\
&\qquad q^3+3q^2+4q+1, (q+1)(q^2+3q+1), q^3+5q^2+4q+1, (q+1)^2(2q+1), \\
&\qquad 2q^3+6q^2+4q+1, (q+1)(3q^2+3q+1), (2q+1)(2q^2+2q+1), (q+1)^4, \ldots).
\end{aligned}$$

Elements of these sequences are curious polynomials in q. It is natural to ask where they come from and what the formula for the nth one in each case is. We show that they are closely related to the *Hamming weight* of binary strings.

3. Hamming Weight and Iterated Galileo Sequences

Definition 3.1. Given a binary string $b = b_n \cdots b_1 b_0$, the Hamming weight $H(b)$ of b is the number of 1's in b. In other words,

$$H(b) = \sum_{i=0}^{n} b_i.$$

For a nonnegative integer n, $H(n)$ is defined to be the Hamming weight of its binary expansion.

Remark 3.2. Note that $H(2^k n) = H(n)$ for any $k \geq 0$, since the the binary expansion of $2^k n$ is the binary expansion of n shifted to the left by k digits. Therefore, if the integer $c \geq 0$ has binary expansion of no more than k digits, then $H(2^k n + c) = H(n) + H(c)$.

We have the following result:

Proposition 3.3. *If sequences a_n with $a_1 = 1$ and its partial sums S_n with $S_{2n} = qS_n$ are both Galileo sequences, then for $n > 1$*

$$a_n = q^{H(n-1)} - q^{H(n-2)} \tag{3.1}$$

with

$$S_n = q^{H(n-1)} \quad \text{and} \quad S_n^* = \sum_{k=0}^{n-1} q^{H(k)}.$$

Proof. For any $m \geq 0$, $H(2m+1) = H(m)+1$, therefore taking $m = n-1$ proves that $S_{2n} = qS_n$. To prove that $S_{2n}^* = (q+1)S_n^*$, we need to show

$$\sum_{k=0}^{2n-1} q^{H(k)} = (q+1) \sum_{k=0}^{n-1} q^{H(k)}.$$

This is equivalent to showing

$$\sum_{k=0}^{n-1} q^{H(k+n)} = \sum_{k=0}^{n-1} q^{H(k)+1},$$

which can be written as

$$q^{H(2n-1)} + q^{H(2n-2)} - q^{H(n-1)} + \sum_{k=0}^{n-2} q^{H(k+n-1)} = q^{H(n-1)+1} + \sum_{k=0}^{n-2} q^{H(k)+1}.$$

Therefore, $S_{2n}^* = (q+1)S_n^*$ follows by induction on n if it is true that

$$q^{H(2n-1)} + q^{H(2n-2)} - q^{H(n-1)} - q^{H(n-1)+1} = 0. \tag{3.2}$$

This latter identity (3.2) is an immediate consequence of the properties of the Hamming weight in Remark 3.2. □

The expression for a_n in (3.1) explains the reason why the entries for $n = 3, 7, 11, 15, \ldots$, i.e., for the indices of the form $n = 4k + 3$, $k = 0, 1, \ldots$ are all 0. This is simply because for any $k \geq 0$, $H(4k + 1) = H(4k + 2)$ $(= H(k) + 1)$, and therefore (3.1) vanishes.

For a_n given by (3.1), the value of S_n is as claimed since the sum telescopes:

$$S_n = 1 + (q^{H(1)} - q^{H(0)}) + (q^{H(2)} - q^{H(1)}) + \cdots + (q^{H(n-1)} - q^{H(n-2)}) = q^{H(n-1)}.$$

Furthermore, adding up the values of S_n, the polynomial S_n^* in q is the enumerator of non-negative integers less than n by their Hamming weight. Some of these are easy to write down in closed from. As an example, $S_n^* = (q + 1)^k$ for $n = 2^k$, since for such n, S_n^* is the enumerator of binary strings of length k by their number of 1's, and this is $(q + 1)^k$ by the binomial theorem.

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