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A Combinatorial Proof of the Giambelli Identity for Schur Functions

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The Giambelli identity provides a formula for expressing an arbitrary Schur function of shape λ as a determinant of Schur functions of hook shapes $(l, 1^k)$. We give the first complete combinatorial proof of the Giambelli identity. We show how to derive various hook formulas from the Giambelli identity and show how to extend our methods to derive extensions of the Giambelli identity and the hook formula for the number of standard tableaux to certain skew shapes.

INTRODUCTION

In this paper, we shall give the first completely combinatorial proof of the Giambelli identity [9] which provides a formula for expressing an arbitrary Schur function $S_\lambda(x)$ as a determinant of Schur functions of hook shapes, $S_{(l,1^k)}(x)$. In recent years, there has been considerable success in providing combinatorial proofs of symmetric function identities which express some given symmetric function as a determinant of other symmetric functions. Most notably, Gessel [7] and Gessel and Viennot [8] used the approach that various determinants of symmetric functions could be interpreted as a weighted sum of k-tuples of paths. They then defined certain natural involutions on this weighted sum which has the effect of pairing off various terms with opposite sign. Then after cancelling those terms which were paired, one is left with only a subclass of the original set of paths which is shown to correspond to the given symmetric function. To date, a combinatorial proof of the Giambelli identity has proved resistant to this type of weighted path approach. The new feature of our approach to the Giambelli identity is that we interpret the terms that arise from the determinant of Schur functions of hook shapes directly as the sum of

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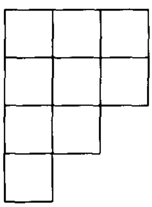
weights of various fillings of the Ferrers diagram of shape λ. Then as before we define a simple involution which pairs off terms of opposite signs leaving only the column strict tableaux of shape λ which correspond to $S_\lambda(x)$. We note that this approach of interpreting the terms arising from a determinant of symmetric functions directly as the weight of certain fillings of a Ferrers diagram can be used to prove a number of other symmetric function identities of the type mentioned above. We shall not pursue these variants here but we refer the reader to the first author's thesis [1] written under the direction of A. Garsia for such proofs.

Our methods yield a surprisingly simple and elegant proof of the Giambelli identity. Indeed in our opinion, the proof given here is much simpler than any of the algebraic proofs of the identity with which we are familiar. Moreover, as is often the case with good combinatorial proofs, our methods easily extend to prove various extensions of the Giambelli identity. For example, in Section 4, we shall extend the Giambelli identity to certain classes of skew Schur functions which we call winged Schur functions and this extension will enable us to derive a hook-type formula for the number of standard tableaux for some special skew shapes. Moreover, we also refer the reader to [1], where an extension of the Giambelli identity is developed for arbitrary skew shapes.

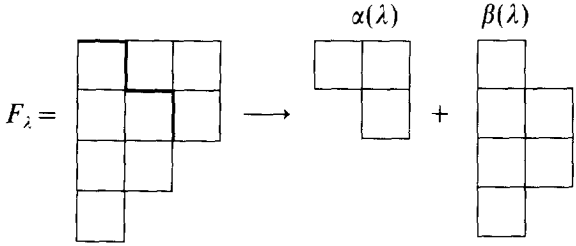
The outline of this paper is as follows: In Section 1 we deal with preliminaries and establish our notation. In Section 2, we give our combinatorial proof of the Giambelli identity. Then in Section 3, we show how we can rather easily derive from the Giambelli identity the hook formulas for the number of standard tableaux of shape λ and the number of column strict tableaux of shape λ with bounded entries as well as give a new extension of the generating function for the number of reverse plane partitions of shape λ. Finally, in Section 4, we describe our extension of the Giambelli identity to skew Schur functions of winged shape.

1. Preliminaries

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0)$ be a partition of n, i.e., $n = |\lambda| = \sum_{i=1}^{k} \lambda_i$. The Ferrers diagram of shape λ, denoted by F_λ, is the set of left justified rows of squares or cells with λ_i cells in the ith row for $i = 1, \ldots, k$. For example,

$F_{(3,3,2,1)} =$ 

The cells of F_λ which are to the right of the North–East boundary of the main diagonal will be denoted by $\alpha(\lambda)$ and the remaining cells will be denoted by $\beta(\lambda)$. For example, we have the decomposition



If we denote the rows of $\alpha(\lambda)$ by $\alpha_1 > \alpha_2 > \cdots > \alpha_d \geqslant 0$ and the columns of $\beta(\lambda)$ by $\beta_1 > \beta_2 > \cdots > \beta_d > 0$, then we arrive at the following alternative notation for λ:

$$\lambda = (\alpha \mid \beta),$$

where $\alpha = (\alpha_1, ..., \alpha_d)$ and $\beta = (\beta_1, ..., \beta_d)$. In such a notation d represents the length of the side of the so-called Durfee square D_λ of shape λ which is the set of cells corresponding to the largest square which fits inside F_λ; for example, if $\lambda = (2, 1 \mid 4, 2)$ then D_λ is 2×2. We shall refer to this alternative notation as the Fröbenius notation for λ but we should remark that our conventions differ from the usual Fröbenius notation in that the cells on the main diagonal are accounted for in the β-vector. We let λ' denote the *conjugate* of λ, i.e., the partition that results by reading the columns of F_λ, and $\hat{\lambda}$ denote the *augmented partition* corresponding to λ, i.e., $\hat{\lambda} = (\hat{\lambda}_1, ..., \hat{\lambda}_k)$, where $\hat{\lambda}_i = \lambda_i + k - i$. For example, if $\lambda = (3, 3, 2, 1)$, $\lambda' = (4, 2, 2)$ and $\hat{\lambda} = (6, 5, 3, 1)$.

Let (i, j) denote the cell in the ith row and jth column of F_λ. The *hook* $H_{i,j}$ corresponding to (i, j) consists of the cell (i, j) plus all the cells directly below or directly to the right of (i, j). The *hook number* of cell (i, j), denoted by $h_{i,j}$, equals the number of cells in $H_{i,j}$ and the *content number* of (i, j), denoted by $c_{i,j}$, equals $j - i$. For example, the hook and content numbers for $\lambda = (3, 3, 2, 1)$ are given below.

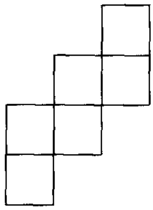
Hook numbers

6	4	2
5	3	1
3	1	
1		

Content numbers

0	1	2
−1	0	1
−2	−1	
−3		

Given partitions $\lambda=(\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0)$ and $\mu=(\mu_1 \geqslant \cdots \geqslant \mu_l > 0)$, we write $\mu \leqslant \lambda$ if $l \leqslant k$ and $\mu_i \leqslant \lambda_i$ for $i = 1, ..., l$. The *skew diagram* $F_{\lambda/\mu}$ of shape λ/μ will consist of the cells of F_λ that remain after the cells of F_μ are removed. For example,

$F_{(3,3,2,1)/(2,1)} =$ 

Of course, we can think of any partition λ as the skew shape λ/ϕ.

A *tabloid* T of shape λ/μ is a filling of $F_{\lambda/\mu}$ with nonnegative integers. We let $T_{i,j}$ denote the entry in the (i, j)th cell of T. We define the *ordinary weight* of T, $W_0(T)$, by

$$W_0(T) = \sum_{(i,j) \in \lambda/\mu} T_{i,j} \tag{1.1}$$

and we define the *monomial weight* or simply the *weight* of T by

$$w(T) = \prod_{(i,j) \in \lambda/\mu} x_{T_{i,j}}, \tag{1.2}$$

where we write $\sum_{(i,j) \in \lambda/\mu}$ and $\prod_{(i,j) \in \lambda/\mu}$ for the sum and product, respectively, over all cells in $F_{\lambda/\mu}$. We say a tabloid T of shape λ/μ is a

(i) *reverse plane partition* if the entries of T are weakly increasing from left to right in each row and weakly increasing from top to bottom in each column,

(ii) *column-strict tableau* or simply a *tableau* if $T_{i,j} \geqslant 1$ for all $(i, j) \in F_{\lambda/\mu}$ and the entries of T are weakly increasing in each row from left to right and strictly increasing in each column from top to bottom, and

(iii) *standard tableau* if T is a column-strict tableau whose entries are $1, ..., n$, where $n = |\lambda/\mu| = |\lambda| - |\mu|$.

We let $\mathbb{P}(\lambda/\mu)$, $\mathbb{T}(\lambda/\mu)$, and $\mathbb{S}(\lambda/\mu)$ denote the set of all reverse plane partitions of shape λ/μ, the set of a column strict tableau of shape λ/μ, and the set of all standard tableaux of shape λ/μ, respectively. We also let $\mathbb{T}^a(\lambda/\mu)$ denote the set of $T \in \mathbb{T}(\lambda/\mu)$ such that $T_{i,j} \leqslant a$ for all $(i, j) \in F_{\lambda/\mu}$.

The Schur function $S_\lambda(x)$ of shape λ is defined by

$$S_\lambda(x) = \sum_{T \in \mathbb{T}(\lambda)} w(T) \tag{1.3}$$

and the skew Schur function $S_{\lambda/\mu}(x)$ of shape λ/μ is defined by

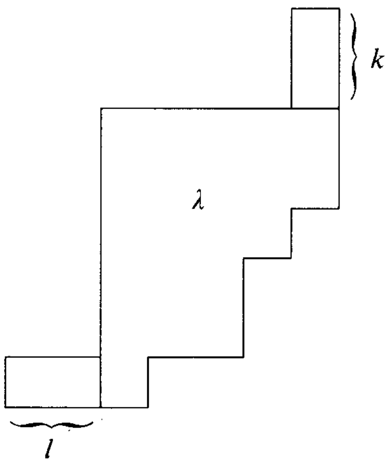
$$S_{\lambda/\mu}(x) = \sum_{T \in \mathbb{T}(\lambda/\mu)} w(T) \tag{1.4}$$

whenever $\mu \leqslant \lambda$.

A partition λ of the form $(l, 1^k)$ or $(l-1 \mid k+1)$ in Fröbenius notation will be called a *hook* and $S_\lambda(x)$ will be called a *hook Schur function.* A skew diagram which results from a Ferrers diagram of shape λ by adding a column of height $k \geqslant 0$ on top of the rightmost cell of the first row and a row of length $l \geqslant 0$ to the left of the bottom cell of the first column will be called a *winged skew shape* and will be denoted by $[\lambda, k, l]$ (see below) and $S_{[\lambda,k,l]}(x)$ will be called a *winged skew Schur function.*

Winged skew shapes

$[\lambda, k, l] =$



Finally, $\mathscr{S}_d$ will denote the symmetric group on $\{1, ..., d\}$.

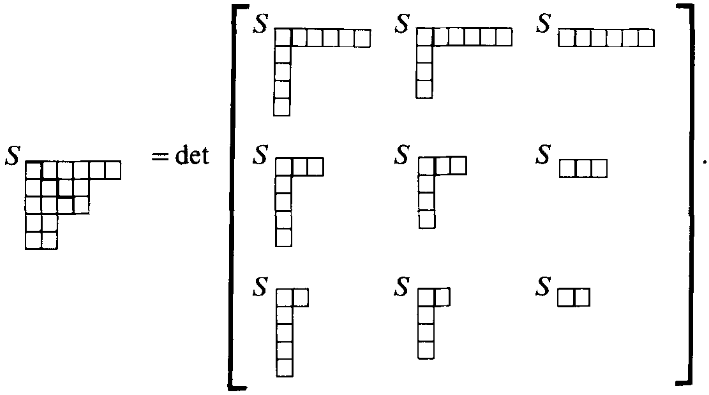
2. The Giambelli Determinantal Formula

In this section we first give a combinatorial proof of Giambelli's determinantal formula [9] for the expansion of S_λ in terms of hook Schur functions. A number of consequences are derived in the following sections.

THEOREM 2.1.

$$S_{(\alpha \mid \beta)} = \det \| S_{(\alpha_i \mid \beta_j)} \|. \tag{2.1}$$

EXAMPLE.



(2.2)

Proof. Before proceeding with the proof we need some preliminary notions.

Put $(\alpha \mid \beta) = \lambda$ and suppose that the Durfee square of λ is $d \times d$. Thus the matrix $\| S_{(\alpha_i \mid \beta_j)} \|$ is $d \times d$.

Given a permutation $\sigma \in \mathcal{S}_d$ and a tabloid T of shape λ, we shall say that T is a *σ-tabloid* if it satisfies the following three conditions:

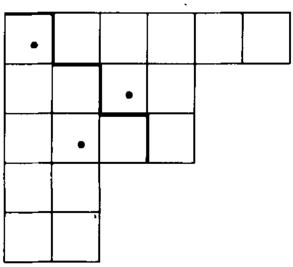
(i) The entries of T are weakly increasing along the rows of $\alpha(\lambda)$.

(ii) The entries of T are strictly increasing down the columns of $\beta(\lambda)$.

(iii) $T_{\sigma_i, \sigma_i} \leqslant T_{i, i+1}$ for $i = 1, 2, ..., d$ whenever $i + 1 \leqslant \lambda_i$.

Condition (iii) is better understood pictorially as follows:

We can represent a permutation $\sigma \in \mathcal{S}_d$ by identifying its matrix with the Durfee square of λ and marking the positions of the 1's. For instance $\sigma = (1)(23)$ would be represented as



(2.3)

We then pair off the ith row of $\alpha(\lambda)$ with the σ_ith column of $\beta(\lambda)$. Condition (iii) simply says that the topmost element of T in a column of $\beta(\lambda)$ is no larger than the leftmost entry of T that appears in the corresponding of $\alpha(\lambda)$. For example, we have below a (1)(23)-tabloid T_1 and a (132)-tabloid T_2, both of shape $\lambda = (64^2 2^2)$:

$$T_1 = \begin{array}{|c|c|c|c|c|c|}\hline 1\bullet & 2 & 2 & 4 & 4 & 5 \\ \hline 2 & 1 & 3\bullet & 4 \\ \cline{1-4} 3 & 2\bullet & 2 & 1 \\ \cline{1-4} 5 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2\bullet & 3 & 3 & 4 \\ \hline 3\bullet & 4 & 3 & 4 \\ \cline{1-4} 6 & 6\bullet & 1 & 7 \\ \cline{1-4} 8 & 7 \\ \cline{1-2} 9 & 9 \\ \cline{1-2}\end{array} \tag{2.4}$$

Remark. A 1-tabloid T is a column-strict tableau of shape λ, where 1 denotes the identity permutation in $\mathscr{S}_d$.

This given we let $\mathbb{X}$ be the space of all pairs (σ, T), where $\sigma \in \mathscr{S}_d$ and T is a σ-tabloid of shape λ. $\mathbb{X}$ is made into a weighted, signed space by setting

$$w(\sigma, T) = \prod_{(i,j) \in \lambda} x_{T_{ij}}$$
$$\operatorname{sign}(\sigma, T) = (-1)^{i(\sigma)}.$$

For example, we have

$$w((1)(23), T_1) = x_1^3 x_2^5 x_3^2 x_4^4 x_5^2 x_7 x_8$$
$$\operatorname{sign}((1)(23), T_1) = -1,$$

where T_1 is as shown in (2.4).

Note that

$$\begin{aligned} \det \| S_{(\alpha_i \mid \beta_j)} \| &= \sum_{\sigma \in \mathscr{S}_d} (-1)^{i(\sigma)} \prod_{i=1}^{d} S_{(\alpha_i \mid \beta_{\sigma_i})} \\ &= \sum_{\sigma \in \mathscr{S}_d} (-1)^{i(\sigma)} \prod_{i=1}^{d} \sum_{T_i \in \mathbb{T}(\alpha_i \mid \beta_{\sigma_i})} w(T_i), \end{aligned} \tag{2.5}$$

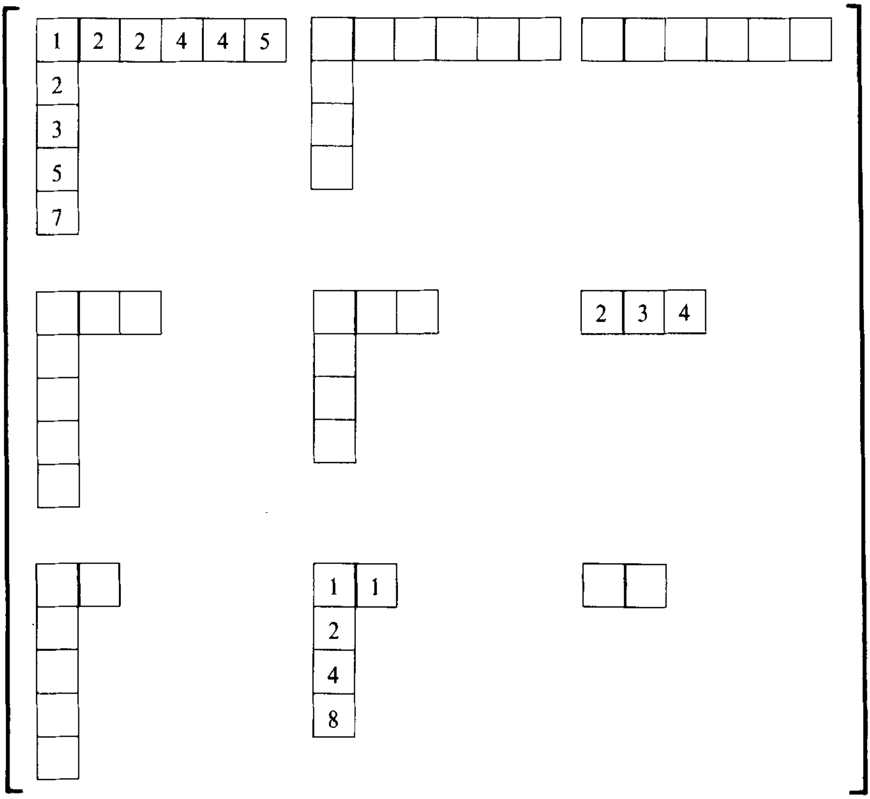
where $w(T_i)$ is the weight of the tableau T_i as defined in (1.2). It follows from (2.5) that a typical monomial that appears in the expansion of $\det \| S_{(\alpha_i \mid \beta_j)} \|$ is of the form

$$(-1)^{i(\sigma)}\, w(T_1)\, w(T_2) \cdots w(T_d),$$

where T_i is a column-strict tableau of shape $(\alpha_i \mid \beta_{\sigma_i})$. But

$$w(T_1)\, w(T_2) \cdots w(T_d) = w(\sigma, T) \qquad \text{and} \qquad (-1)^{i(\sigma)} = \operatorname{sign}(\sigma, T),$$

where T is the σ-tabloid of shape λ obtained by filling the ith row of $\alpha(\lambda)$ and the σ_ith column of $\beta(\lambda)$ with the entries of T_i, $i = 1, 2, ..., d$. For example, the (1)(23)-tabloid T_1 depicted in (2.4) corresponds to picking the tableaux



in the expansion of the determinant that appears in (2.2),

It follows from (2.5) that

$$\mathbb{W}_{\mathbb{X}} = \sum_{(\sigma, T) \in \mathbb{X}} \operatorname{sign}(\sigma, T)\, w(\sigma, T) = \det \| S_{(\alpha_i | \beta_j)} \|. \tag{2.6}$$

We proceed to construct a weight-preserving, sign-reversing involution Θ on $\mathbb{X}$, where the set of fixed points of Θ, $\mathbb{X}^{\Theta}$ is given by

$$\mathbb{X}^{\Theta} = \{(1, T) : T \in \mathbb{T}(\lambda)\}. \tag{2.7}$$

This will prove the theorem in view of (2.6), since we clearly have

$$\mathbb{W}_{\mathbb{X}^{\Theta}} = \sum_{(1, T) \in \mathbb{X}^{\Theta}} w(T) = S_{(\alpha | \beta)}.$$

Given $(\sigma, T) \in \mathbb{X}$, T can fail to be a column-strict tableau in one of the following three ways:

(1) There is a violation of column-strictness in $\alpha(\lambda)$.

(2) There is no violation of column-strictness in $\alpha(\lambda)$ but there is a violation of monotonicity along the rows of $\beta(\lambda)$.

(3) The entries of T in $\alpha(\lambda)$ and $\beta(\lambda)$ both form column-strict tableaux but there is a violation along the North–East boundary of the main diagonal of λ.

The involution Θ is defined separately according to these three cases as follows:

Case 1. Locate the rightmost and then the topmost violation in $\alpha(\lambda)$. If this violation occurs in the rows i and $i+1$ of $\alpha(\lambda)$ then we have the following situation:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c}
\hline
x_0 & x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_n & \cdots \\
\hline
\multicolumn{1}{c|}{} & & & & & \not\curlywedge & \curlywedge & \curlywedge & \curlywedge & \\
\multicolumn{1}{c|}{} & y_1 & y_2 & \cdots & y_{k-1} & y_k & y_{k+1} & \cdots & y_n & \\
\cline{2-9}
\end{array}$$

where $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n$. Also

$$T_{\sigma_i, \sigma_i} \leqslant x_0,$$
$$T_{\sigma_{i+1}, \sigma_{i+1}} \leqslant y_1$$

since T is a σ-tabloid.

Now set $\Theta(\sigma, T) = (\sigma', T')$, where $\sigma' = (i\; i+1)\,\sigma$ and T' is obtained from T by rearranging the entries in the rows i and $i+1$ as

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c}
\hline
y_1 & y_2 & y_3 & \cdots & y_k & x_k & x_{k+1} & \cdots & x_n & \cdots \\
\hline
\multicolumn{1}{c|}{} & & & & & \not\curlywedge & \curlywedge & \curlywedge & \curlywedge & \\
\multicolumn{1}{c|}{} & x_0 & x_1 & \cdots & x_{k-2} & x_{k-1} & y_{k+1} & \cdots & y_n & \\
\cline{2-9}
\end{array}$$

Then

$$y_1 \leqslant y_2 \leqslant \cdots \leqslant y_k \leqslant x_k \leqslant \cdots \leqslant x_n$$
$$x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{k-1} \leqslant y_{k+1} \leqslant \cdots \leqslant y_n.$$

We also have

$$T'_{\sigma'_i, \sigma'_i} \leqslant y_1,$$
$$T'_{\sigma'_{i+1}, \sigma'_{i+1}} \leqslant x_0$$

so that T' is a σ'-tabloid. Since the location of the rightmost and then the topmost violation of column-strictness is preserved, Θ is an involution. Clearly Θ preserves the weight of (σ, T) and reverses its sign.

EXAMPLE. The (1)(23)-tabloid T_1 depicted in (2.4) is mapped under Θ to the (132)-tabloid

$T'_1 =$

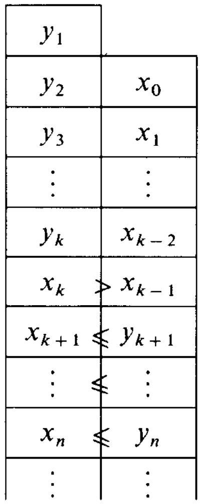
1	3	4•	4	4	5
2•	1	2	2		
3	2•	2	1		
5	4				
7	8				

Case 2. Locate the bottommost and then the leftmost violation in $\beta(\lambda)$. If this violation occurs in the columns j and $j+1$ of $\beta(\lambda)$, then we have

x_0		
x_1		y_1
x_2		y_2
$\vdots$		$\vdots$
x_{k-1}		y_{k-1}
x_k	$>$	y_k
x_{k+1}	$\leqslant$	y_{k+1}
$\vdots$	$\leqslant$	$\vdots$
x_n	$\leqslant$	y_n
$\vdots$		

with $x_0 < x_1 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$.

We set $\Theta(\sigma, T) = (\sigma', T')$, where T' is obtained from T by rearranging the entries of T in the jth and $(j+1)$st columns as



and σ' is obtained from σ by interchanging the σ_jth and the σ_{j+1}st columns of the matrix of σ. In other words, $\sigma' = (\sigma_j \sigma_{j+1})\,\sigma$. It is not difficult to see that T' is a σ'-tabloid and that Θ has the desired properties.

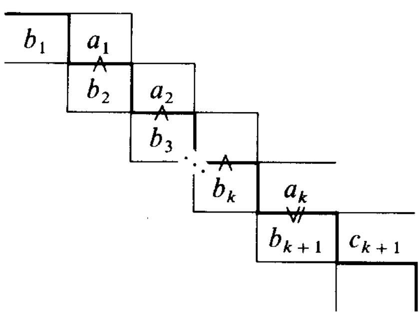
EXAMPLE. The (132)-tabloid T_2 in (2.4) is mapped under Θ to the (13)(2)-tabloid T'_2 below:

$T'_2 =$

4	2	2•	3	3	4
6	1•	3	4		
7•	3	1	7		
8	6				
9	9				

Case 3. In this case we make use of the fact that the entries of T in $\alpha(\lambda)$ and $\beta(\lambda)$ form column-strict tableaux.

Pick the smallest index k such that there is a violation of the following form across the North–East boundary of the main diagonal of λ:



First of all observe that we must necessarily have $\sigma_p = p$ for $p = 1, 2, ..., k-1$. For if $\sigma_1 = r$ say, then $a_1 \geqslant T_{r,r} = b_r$ since T is a σ-tabloid. Also the entries of T in $\beta(\lambda)$ form a tableau and therefore we have $b_1 < b_2 < \cdots$. Now the condition $a_1 < b_2$ forces $r = 1$ and $b_1 \leqslant a_1$. A similar argument goes through for $p = 2, 3, ..., k-1$.

Suppose now $\sigma_i = k$ and $\sigma_j = k+1$. By the above remark we necessarily have $i \geqslant k$. Thus $b_{k+1} \leqslant a_k \leqslant a_i$ and $b_k \leqslant b_{k+1} \leqslant a_j$.

Set $\Theta(\sigma, T) = (\sigma', T)$, where $\sigma' = (ij)\,\sigma$. Since $b_{k+1} = b_{\sigma'_i} \leqslant a_i$ and $b_k = b_{\sigma'_j} \leqslant a_j$, T is also a σ'-tabloid. The desired properties of Θ are trivially verified.

Note that the argument given above implies that the fixed points of Θ are precisely of the form given in (2.7). Thus Theorem 2.1 follows. ∎

Remark. The involution Θ leaves invariant the sum of the entries of a σ-tabloid T along each diagonal of λ. Therefore we have

$$\prod_{(i,j)\in\lambda} x_{j-i}^{T_{ij}} = \prod_{(i,j)\in\lambda} x_{j-i}^{T'_{ij}}, \tag{2.8}$$

where $\Theta(\sigma, T) = (\sigma', T')$.

Given a partition μ and a tabloid T of shape μ we set

$$W(T) = \prod_{(i,j)\in\mu} x_{j-i}^{T_{ij}}. \tag{2.9}$$

COROLLARY 2.2.

$$\sum_{T\in\mathbb{T}(\alpha\mid\beta)} W(T) = \det\left\| \sum_{T\in\mathbb{T}(\alpha_i\mid\beta_j)} W(T)\right\|. \tag{2.10}$$

Proof. This is a consequence of Theorem 2.1 and the invariance property of Θ. ∎

3. DERIVATIONS OF HOOK FORMULAS FROM THE GIAMBELLI IDENTITY

In this section we shall show the power of the Giambelli identity by showing that the hook formulas for the number of standard tableaux, the ordinary weight generating function for the set of column-strict tableaux with bounded entries, and a new multivariate extension of the generating function for reverse plane partition all easily follow from it. The idea of all the derivations which follow is to show that the formula for hook shapes can be derived directly. Then we employ one of two basic determinantal identities to derive the general case. The two basic determinantal identities we shall need are given in our next lemma.

LEMMA 3.1. *Let u_i and v_j be indeterminates for $1 \leqslant i, j \leqslant d$. Then*

$$\det \left\| \frac{1}{1-u_i v_j} \right\| = \Delta(u)\, \Delta(v) \prod_{i,j} \frac{1}{1-u_i u_j} \tag{3.1}$$

$$\det \left\| \frac{1}{1-u_i-v_j} \right\| = \Delta(u)\, \Delta(v) \prod_{i,j} \frac{1}{1-u_i-v_j}, \tag{3.2}$$

where $\Delta(u) = \det \|u_i^{j-1}\| = \prod_{i<j} (u_i - u_j)$ is the Vandermonde determinant.

Proof. We note that (3.1) is just the classical Cauchy determinant; see [15, p. 38]. Equation (3.2) is an additive version of the Cauchy determinant and can be proved as follows. Let

$$w_i = (1-u_i-v_1)(1-u_i-v_2) \cdots (1-u_i-v_d)$$

for $i = 1, 2, ..., d$ and put

$$D = w_1 w_2 \cdots w_d \det \left\| \frac{1}{1-u_i-v_j} \right\| = \det \left\| \frac{w_i}{1-u_i-v_j} \right\|. \tag{3.3}$$

Since D is alternating in the u's and alternating in the v's it is divisible by $\Delta(u)\, \Delta(v)$. Moreover we have

$$\deg \frac{w_i}{1-u_i-v_j} = \deg \prod_{k \neq j} (1-u_i-v_k) = n-1.$$

Thus $\deg D \leqslant n(n-1) = \deg \Delta(u)\, \Delta(v)$. We conclude that

$$\frac{D}{\Delta(u)\, \Delta(v)} = c \qquad \text{(constant)}. \tag{3.4}$$

Specialize the left-hand side of (3.4) by putting $v_k = 1 - u_k$ for $i = 1, 2, ..., d$. With this specialization the denominator becomes $(-1)^{n(n-1)/2} \Delta^2(u)$ and the numerator D reduces to

$$\det \left\| \prod_{k \neq j} (u_k - u_i) \right\| = \det \left\| \delta_{ij} \prod_{k \neq j} (u_k - u_i) \right\| = (-1)^{n(n-1)/2} \Delta^2(u).$$

Therefore $c = 1$. ∎

Let us first consider the hook formula for the number of standard tableaux. So let λ be a partition of N and n_λ denote the number of standard tableaux of shape λ. Fröbenius [4, 5] and Young [21] independently showed that

$$n_\lambda = N! \frac{\prod_{i<j} (\hat{\lambda}_i - \hat{\lambda}_j)}{\hat{\lambda}_1!\, \hat{\lambda}_2! \cdots \hat{\lambda}_d!} \tag{3.5}$$

in connection with the representation theory of the symmetric group.

In 1954, Frame, Robinson, and Thrall [2] introduced the notion of hooks and derived the identity

$$\frac{\hat{\lambda}_1!\,\hat{\lambda}_2!\cdots\hat{\lambda}_d!}{\prod_{i<j}(\hat{\lambda}_i-\hat{\lambda}_j)}=\prod_{(i,j)\in\lambda} h_{ij}. \tag{3.6}$$

Combining (3.5) and (3.6) gives a simpler expression for n_λ: the hook formula for the number of standard tableaux

$$n_\lambda=\frac{N!}{\prod_{(i,j)\in\lambda} h_{ij}}. \tag{3.7}$$

Since 1943, Hillman and Grassl [11] and later Greene, Nijenhuis, and Wilf [10] gave different proofs of (3.7). The first bijective proof of the hook formula was constructed by Remmel and Whitney [18]. This was followed by bijective proofs by Franzblau and Zeilberger [3] and Zeilberger [22].

The fundamental step is (3.6), from which it follows that

$$\frac{\alpha_1!\,\alpha_2!\cdots\alpha_d!}{\prod_{i<j}(\alpha_i-\alpha_j)}=\prod_{(i,j)\in\alpha(\lambda)-D_\lambda} h_{ij} \tag{3.8}$$

$$\frac{(\beta_1-1)!\,(\beta_2-1)!\cdots(\beta_d-1)!}{\prod_{i<j}(\beta_i-\beta_j)}=\prod_{(i,j)\in\beta(\lambda)-D_\lambda} h_{ij}, \tag{3.9}$$

where D_λ denotes the Durfee square of $\lambda=(\alpha\mid\beta)$.

Note that by Theorem 2.1 we have

$$n_\lambda=\sum_{\sigma\in\mathscr{S}_d}(-1)^{i(\sigma)}\prod_{i=1}^{d}\binom{N}{s_{\sigma_1},s_{\sigma_2},\ldots,s_{\sigma_d}}n_{(\alpha_i\mid\beta_{\sigma_i})}, \tag{3.10}$$

where $s_{\sigma_i}=\alpha_i+\beta_{\sigma_i}$ for $i=1,2,\ldots,d$. Clearly

$$n_{(\alpha_i\mid\beta_{\sigma_i})}=\binom{s_{\sigma_i}-1}{\alpha_i}$$

so that the product in (3.10) simplifies to

$$\frac{N!}{\alpha_1!\,\alpha_2!\cdots\alpha_d!\,(\beta_1-1)!\,(\beta_2-1)!\cdots(\beta_d-1)!\,s_{\sigma_1}s_{\sigma_2}\cdots s_{\sigma_d}}.$$

Therefore from (3.10), we obtain

$$n_\lambda=\frac{N!}{\alpha_1!\,\alpha_2!\cdots\alpha_d!\,(\beta_1-1)!\,(\beta_2-1)!\cdots(\beta_d-1)!}\det\left\|\frac{1}{\alpha_i+\beta_j}\right\|. \tag{3.11}$$

We can now apply (3.2) with $u_i = -\alpha_i$ and $v_j = -\beta_j + 1$ in (3.11) to obtain

$$n_\lambda = N! \frac{\prod_{i<j} (\alpha_i - \alpha_j)}{\alpha_1! \cdots \alpha_d!} \frac{\prod_{i<j} (\beta_i - \beta_j)}{(\beta_1 - 1)! \cdots (\beta_d - 1)!} \prod_{i,j} \frac{1}{\alpha_i + \beta_j}.$$

Now using (3.8) and (3.9) and the fact that the numbers $\alpha_i + \beta_j$ for $1 \leqslant i, j \leqslant d$ represent the hook numbers of F_λ in the Durfee square D_λ, we obtain

$$\begin{aligned} n_\lambda &= N! \prod_{(i,j) \in \alpha(\lambda) - D_\lambda} h_{i,j}^{-1} \prod_{(i,j) \in \beta(\lambda) - D_\lambda} h_{i,j}^{-1} \prod_{(i,j) \in D_\lambda} h_{i,j}^{-1} \\ &= \frac{N!}{\prod_{(i,j) \in \lambda} h_{i,j}}. \end{aligned}$$

Next we consider the ordinary weight generating function for column-strict tableaux with bounded entries. An explicit formula for the generating function $\sum_{T \in \mathbb{T}^a(\lambda)} q^{W_0(T)}$ was first derived by Littlewood [14] and later Stanley [19] transformed Littlewood's formula to give the following explicit expression in terms of content and hook numbers,

$$\sum_{T \in \mathbb{T}^m(\lambda)} q^{W_0(T)} = q^{\sum i\lambda_i} \prod_{(i,j) \in \lambda} \frac{[m + c_{i,j}]}{[h_{i,j}]}, \tag{3.12}$$

where $[n] = 1 + q + \cdots + q^{n-1}$ is the q-analogue of n. A bijective proof of (3.12) has been given by Remmel and Whitney [17].

We can establish (3.12) in much the same way as we established (3.7). The main difference is that the special case of (3.12) for hook shapes is not quite so obvious. However, as our next proposition will show, this special case follows from rather straightforward manipulations of q-binomial coefficients.

PROPOSITION 3.2.

$$\sum_{T \in \mathbb{T}^m((a|b))} q^{W_0(T)} = q^{a+b+\binom{b}{2}} \frac{[m-b+1]_{a+b}}{[a]!\,[b-1]!\,[a+b]}, \tag{3.13}$$

where $[n]! = [n][n-1] \cdots [2][1]$ *and* $[n]_k = [n][n+1] \cdots [n+k-1]$.

Proof. Let $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = [n]!/[n-k]!\,[k]!$ be the q-analogue of the binomial coefficients. Then it is well known that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant n-k} q^{j_1 + \cdots + j_k}. \tag{3.14}$$

Next by grouping the terms appearing in the LHS of (3.13) by the value i of $T_{1,1}$ and applying (3.14), it is easy to see that

$$\sum_{T\in \mathbb{T}^m((a\mid b))} q^{W_0(T)} = \sum_{i=1}^{m-b+1} q^{(a+b)i}\begin{bmatrix} m-i+a \\ a\end{bmatrix} q^{\binom{b}{2}}\begin{bmatrix} m-i \\ b-1\end{bmatrix}$$
$$= q^{a+b+\binom{b}{2}} \sum_{i=1}^{m-b+1} q^{(a+b)(i-1)} \frac{[m-i+a]!}{[a]!\,[m-i]!}\,\frac{[m-i]!}{[b-1]!\,[m-i-b+1]!}. \tag{3.15}$$

Comparing (3.15) with the RHS of (3.13) and multiplying both expressions by $q^{-(a+b+\binom{b}{2})}\,[a]!\,[b-1]!$, we see we need only prove

$$\frac{[m-b+1]_{a+b}}{[a+b]} = \sum_{i=1}^{m-b+1} q^{(a+b)(i-1)}[m-i-b+2]_{a+b-1}. \tag{3.16}$$

But (3.16) easily follows from the basic recursion for q-binomial coefficients. That is, if we iterate

$$\begin{bmatrix} n+1 \\ k+1\end{bmatrix} = q^{k+1}\begin{bmatrix} n \\ k+1\end{bmatrix} + \begin{bmatrix} n \\ k\end{bmatrix}, \tag{3.17}$$

we obtain

$$\begin{bmatrix} n+1 \\ k+1\end{bmatrix} = \sum_{i=0}^{n-k} q^{i(k+1)}\begin{bmatrix} n-i \\ k\end{bmatrix}. \tag{3.18}$$

Then multiplying both sides of (3.18) by $[k]!$, we obtain

$$\frac{[n-k+1]_{k+1}}{[k+1]} = \sum_{i=0}^{n-k} q^{(k+1)i}[n-i-k]_k \tag{3.19}$$

of which (3.16) is a special case. ∎

Now returning to the general case of (3.12), we see by Theorem 2.1 and Lemma 3.2 that we can write

$$\sum_{T\in \mathbb{T}^m((\alpha\mid \beta))} q^{W_0(T)} = \sum_{\sigma\in \mathscr{S}_d} (-1)^{i(\sigma)} \prod_{i=1}^{d} q^{\alpha_i+\beta_{\sigma_j}+\binom{\beta_{\sigma_j}}{2}} \frac{[m-\beta_{\sigma_j}+1]_{\alpha_i+\beta_{\sigma_j}}}{[\alpha_i]!\,[\beta_{\sigma_j}-1]!\,[\alpha_i+\beta_{\sigma_j}]}. \tag{3.20}$$

Due to the fact that the content number of a cell (i, j) depends only on its distance to the main diagonal, it follows that the product in (3.20) simplifies to

$$\frac{q^{\sum \alpha_i + \sum \beta_i + \binom{\beta_i}{2}} \prod_{(i,j)\in\lambda} [m+c_{i,j}]}{[\alpha_1]!\,[\alpha_2]!\cdots[\alpha_d]!\,[\beta_1-1]!\,[\beta_2-1]!\cdots[\beta_d-1]!\,[s_{\sigma_1}][s_{\sigma_2}]\cdots[s_{\sigma_d}]},$$

where $s_{\sigma_i} = \alpha_i + \beta_{\sigma_j}$. Thus from (3.20) we obtain

$$\sum_{T\in\mathbb{T}^m((\alpha\,|\,\beta))} q^{W_0(T)} = \frac{q^{\sum \alpha_j + \sum \beta_i + \binom{\beta_i}{2}} \prod_{(i,j)\in\lambda} [m+c_{i,j}]}{[\alpha_1]!\cdots[\alpha_d]!\,[\beta_1-1]!\cdots[\beta_d-1]!} \det \left\| \frac{1}{[\alpha_i+\beta_j]} \right\|. \tag{3.21}$$

Now we claim that it follows from (3.2) that

$$\det \left\| \frac{1}{[\alpha_i+\beta_j]} \right\| = q^{\sum_{i=1}^{d} (i-1)(\alpha_i+\beta_i)} \frac{\prod_{i<j} [\alpha_i-\alpha_j] \prod_{i<j} [\beta_i-\beta_j]}{\prod_{i,j} [\alpha_i+\beta_j]}. \tag{3.22}$$

Thus combining (3.21) and (3.22) and observing that if $\lambda = (\alpha\,|\,\beta)$, then $\sum i\lambda_i = \sum_{i=1}^{d} i\alpha_i + \binom{\beta_i}{2}$, we see that

$$\begin{aligned}
&\sum_{T\in\mathbb{T}^m((\alpha\,|\,\beta))} q^{W_0(T)} \\
&\quad = q^{\sum i\lambda_i} \prod_{(i,j)\in\lambda} [m+c_{i,j}] \frac{\prod_{i<j} [\alpha_i-\alpha_j]}{[\alpha_1]!\cdots[\alpha_d]!} \frac{\prod_{i<j} [\beta_i-\beta_j]}{[\beta_1-1]!\cdots[\beta_d-1]!} \prod_{i,j} \frac{1}{[\alpha_i+\beta_j]} \\
&\quad = q^{\sum i\lambda_i} \prod_{(i,j)\in\lambda} \frac{[m+c_{i,j}]}{[h_{i,j}]}.
\end{aligned}$$

Note: To see that (3.22) follows from (3.2), observe that the LHS of (3.22) can be expressed as

$$q^{-(\sum \alpha_i)} \det \left\| \frac{1}{[\alpha_i]/q^{\alpha_i} + [\beta_j]} \right\| \tag{3.23}$$

so that applying (3.2) with $u_i = -[\alpha_i]/q^{\alpha_i}$ and $v_j = -[\beta_j]+1$, we obtain

$$\begin{aligned}
&\det \left\| \frac{1}{[\alpha_i+\beta_j]} \right\| \\
&\quad = q^{-\sum \alpha_i} \prod_{i<j} \left(\frac{[\alpha_i]}{q^{\alpha_i}} - \frac{[\alpha_j]}{q^{\alpha_j}} \right) \prod_{i<j} ([\beta_i]-[\beta_j]) \prod_{i,j} \frac{1}{[\alpha_i]/q^{\alpha_i} + [\beta_j]} \\
&\quad = q^{-\sum \alpha_i} \prod_{i<j} q^{-\alpha_i} [\alpha_i-\alpha_j] \prod_{i<j} q^{\beta_j} [\beta_i-\beta_j] \prod_{i,j} \frac{q^{\alpha_i}}{[\alpha_i+\beta_j]} \\
&\quad = q^{\sum (i-1)\alpha_i} q^{\sum (i-1)\beta_i} \prod_{i<j} [\alpha_i-\alpha_j] \prod_{i<j} [\beta_i-\beta_j] \prod_{i,j} \frac{1}{[\alpha_i+\beta_j]}.
\end{aligned}$$

The so-called hook formula for reverse plane partitions is

$$\sum_{P \in \mathbb{P}(\lambda)} q^{W_0(P)} = \prod_{(i,j) \in \lambda} \frac{1}{1-q^{h_{ij}}}, \tag{3.24}$$

where W_0 is the ordinary weight of P defined in (1.1).

This result has been established in several ways. It appears first in a paper of Stanley [19] and two combinatorial proofs have also been given: a very elegant one by Hillman and Grassl [11] and one which uses the Garsia–Milne involution principle [6] by Remmel and Whitney [17].

Using the invariance property of the involution Θ in conjunction with Theorem 2.1, we shall obtain a multivariate version of this hook formula which reduces to (3.24) by specializing the variables to q.

Before we proceed we need some preliminary observations.

First of all given $P \in \mathbb{P}(\lambda)$, we define its weight $W(P)$ by (2.8). Clearly $W(P)$ reduces to $q^{W_0(P)}$ when we replace each of the variables x_{j-i} by q. Now observe that

$$(x_0 x_1 \cdots x_{\lambda_1 - 1})(x_{-1} x_0 \cdots x_{\lambda_2 - 2})^2 \cdots (x_{-k+1} x_{-k+2} \cdots x_{\lambda_k - k})^k = \prod_{(i,j) \in \lambda} x_{j-i}^i \tag{3.25}$$

so that

$$\sum_{T \in \mathbb{T}(\lambda)} W(T) = \prod_{(i,j) \in \lambda} x_{j-i}^i \sum_{P \in \mathbb{P}(\lambda)} W(P). \tag{3.26}$$

For $n > 0$, it will be useful to define

$$\begin{aligned} x_n! &= x_n x_{n-1} \cdots x_1, \\ x_{-n}! &= x_{-n+1} x_{-n+2} \cdots x_0. \end{aligned} \tag{3.27}$$

We put $x_0! = 1$.

This given, we have the following version of Corollary 2.2:

COROLLARY 3.3.

$$C_{(\alpha \mid \beta)} \sum_{P \in \mathbb{P}(\alpha \mid \beta)} W(P) = \det \left\| \sum_{P \in \mathbb{P}(\alpha_i \mid \beta_j)} W(P) \right\|, \tag{3.28}$$

where

$$C_{(\alpha \mid \beta)} = (x_{\alpha_2}!)(x_{\alpha_3}!)^2 \cdots (x_{\alpha_d}!)^{d-1} (x_{-\beta_2}!)(x_{-\beta_3}!)^2 \cdots (x_{-\beta_d}!)^{d-1}.$$

Proof. We only need to note that for a hook $(\alpha_i \mid \beta_j)$ we have

$$\sum_{T \in \mathbb{T}(\alpha_i \mid \beta_j)} W(T) = (x_{\alpha_i}!)\, x_0 x^2_{-1} x^3_{-2} \cdots x^{\beta_j}_{-\beta_j+1}. \tag{3.29}$$

Now factoring out the proper expressions from the determinant in (2.10) and using (2.2) and (2.3) yields (3.28) as desired. ∎

It will be good to introduce some further notation here. With a partition $(\alpha \mid \beta)$ in mind we set

$$\begin{aligned} \{x\}_i &= (1 - x_1 x_2 \cdots x_{\alpha_i})(1 - x_2 x_3 \cdots x_{\alpha_i}) \cdots (1 - x_{\alpha_i}) \\ \{x\}_{-j} &= (1 - x_{-1} x_{-2} \cdots x_{-\beta_j+1})(1 - x_{-2} x_{-3} \cdots x_{-\beta_j+1}) \cdots (1 - x_{-\beta_j+1}). \end{aligned} \tag{3.30}$$

Given $(i, j) \in \lambda$ the weight of its hook H_{ij} is defined as

$$W(H_{ij}) = \prod_{(r,s) \in H_{ij}} x_{s-r}.$$

In other words, we first label each cell $(r, s) \in \lambda$ by x_{s-r}. Then the weight of a hook is simply the product of the labels of its cells.

EXAMPLES.

x_0	x_1	x_2	x_3	x_4	x_5
x_{-1}	x_0	x_1	x_2		
x_{-2}	x_{-1}	x_0	x_1		
x_{-3}	x_{-2}				

$$\begin{aligned} \{x\}_2 &= (1 - x_1 x_2)(1 - x_2) \\ \{x\}_{-2} &= (1 - x_{-1} x_{-2} x_{-3})(1 - x_{-2} x_{-3})(1 - x_{-3}) \\ W(H_{32}) &= x_{-2} x_{-1} x_0 x_1. \end{aligned}$$

Suppose now $\lambda = (\alpha_i \mid \beta_j)$ is a hook. The following proposition has a straightforward bijective proof:

PROPOSITION 3.4.

$$\sum_{P \in \mathbb{P}(\lambda)} W(P) = \prod_{(r,s) \in \lambda} \frac{1}{1 - W(H_{rs})}. \tag{3.31}$$

Proof. Writing the right-hand side of (3.31) as a product of geometric series we have

$$\sum_{P \in \mathbb{P}(\lambda)} W(P) = \prod_{(r,s) \in \lambda} \sum_{k \geqslant 0} W(H_{rs})^k. \tag{3.32}$$

We shall prove (3.31) by constructing a weight-preserving bijection between $\mathbb{P}(\lambda)$ and the monomials that arise from the product (3.32). This is best illustrated by an example. Take $\alpha_i = \beta_j = 4$:

x_0	x_1	x_2	x_3	x_4
x_{-1}				
x_{-2}				
x_{-3}				

Suppose

$P =$

2	3	3	4	6
2				
5				
5				

We successively reduce the entries of P until they are all zeros and record each reduction as a monomial that appears in a $(1 - W(H_{rs}))^{-1}$ in the following way.

Subtract $2 = P_{11}$ from each of the entries of P. This yields the pair

$P_1 =$

0	1	1	2	4
0				
3				
3				

$\rightarrow (x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3x_4)^2,$

where the monomial is a term in the expansion of $(1 - W(H_{11}))^{-1}$. Now subtract $3 = P_{31}$ from the nonnegative entries in the first column of P_1.

This gives

$$P_2 = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 1 & 2 & 4 \\ \hline 0 \\ \cline{1-1} 0 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \quad \to (x_{-3}x_{-2})^3,$$

where the monomial is a term in the expansion of $(1 - W(H_{31}))^{-1}$. Continuing in this fashion we obtain the following sequence of reverse plane partitions and monomials:

$$P_3 = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 1 & 3 \\ \hline 0 \\ \cline{1-1} 0 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \quad \to (x_1x_2x_3x_4) \qquad (\text{from } (1 - W(H_{12}))^{-1})$$

$$P_4 = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 2 \\ \hline 0 \\ \cline{1-1} 0 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \quad \to (x_3x_4) \qquad (\text{from } (1 - W(H_{14}))^{-1})$$

$$P_5 = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 \\ \cline{1-1} 0 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array} \quad \to (x_4)^2 \qquad (\text{from } (1 - W(H_{15}))^{-1}).$$

Thus P corresponds to the term

$$(x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3x_4)^2\,(x_{-3}x_{-2})^3\,(x_1x_2x_3x_4)(x_3x_4)(x_4)^2 = W(P)$$

on the right-hand side of (3.32).

Note that using the notation introduced in (3.28) and (3.30), (3.31) can be restated as

$$\sum_{P \in \mathbb{P}(\lambda)} W(\lambda) = \frac{1}{(1 - x_{\alpha_i}!\,x_{-\beta_j}!)\{x\}_i\{x\}_{-j}}. \tag{3.33}$$

Our aim is to show that (3.31) holds for an arbitrary partition λ. In order to be able to state our results in terms of hooks, we also need the following result:

LEMMA 3.5. *Suppose we are given a partition* $\lambda=(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k)$ *and indeterminates* $1=f^i_0, f^i_1, f^i_2, \ldots$ *for* $i=1, 2, \ldots, k$. *Then*

$$\prod_{(i,j)\in\lambda} f^i_{h_{ij}} = \frac{\prod_{i=1}^k (f^i_{\lambda_i})!}{\prod_{i<j} f^i_{\lambda_i-\lambda_j}},$$

where the factorial is defined as in (3.27).

Proof. By a fundamental lemma of Frame, Robinson, and Thrall [2, Lemma 1], given $(i, j)\in\lambda$, the integers

$$h_{it} \qquad \text{and} \qquad h_{ij}-h_{sj}$$

for $j\leqslant t\leqslant \lambda_i$ and $i\leqslant s\leqslant \lambda'_j$ form a permutation of $1, 2, \ldots, h_{ij}$. It follows that

$$f^i_1 f^i_2 \cdots f^i_{h_{\lambda_i 1}} = \prod_{i=1}^{\lambda_i} f^i_{h_{it}} \prod_{i<s} f^i_{h_{i1}-h_{s1}}$$

or equivalently

$$(f^i_{\lambda_i})! = \prod_{t=1}^{\lambda_i} f^i_{h_{it}} \prod_{i<s} f^i_{\lambda_i-\lambda_s}.$$

The proposition now follows by multiplying these equations for $i=1, 2, \ldots, k$. ∎

As a consequence of Lemma 3.5 we have that in particular

$$\frac{f^1_{\alpha_1}!\, f^2_{\alpha_2}! \cdots f^d_{\alpha_d}!}{\prod_{i<j} f^i_{\alpha_i-\alpha_j}} = \prod_{(i,j)\in\beta(\lambda)-D_\lambda} f^i_{h_{ij}} \tag{3.34}$$

$$\frac{f^1_{\beta_1-1}!\, f^2_{\beta_2-1}! \cdots f^d_{\beta_d-1}!}{\prod_{i<j} f^i_{\beta_i-\beta_j}} = \prod_{(i,j)\in\beta(\lambda)-D_\lambda} f^i_{h_{ij}}, \tag{3.35}$$

where D_λ denotes the Durfee square of λ. Now we can state and prove our multivariate extension of (3.24).

THEOREM 3.5.

$$\sum_{P\in\mathbb{P}(\lambda)} W(P) = \prod_{(i,j)\in\lambda} \frac{1}{1-W(H_{ij})}. \tag{3.36}$$

Proof. By Corollary 3.3 and Eq. (3.32) we have

$$C_{(\alpha|\beta)} \sum_{P \in \mathbb{P}(\alpha|\beta)} W(P) = \det \left\| \frac{1}{(1 - x_{\alpha_i}!\, x_{-\beta_j}!)\{x\}_i \{x\}_{-j}} \right\|$$
$$= \frac{1}{\prod_{i=1}^{d} \{x\}_i \prod_{j=1}^{d} \{x\}_{-j}} \det \left\| \frac{1}{1 - u_i v_j} \right\|, \tag{3.37}$$

where $u_i = x_{\alpha_i}!$ and $v_j = x_{-\beta_j}!$.

Now by (3.1), we have

$$\det \left\| \frac{1}{1 - u_i v_j} \right\| = \Delta(u)\, \Delta(v) \prod_{i,j} \frac{1}{1 - u_i v_j}. \tag{3.38}$$

Therefore

$$C_{(\alpha|\beta)} \sum_{P \in \mathbb{P}(\alpha|\beta)} W(P) = \frac{\Delta(u)}{\prod \{x\}_i} \frac{\Delta(v)}{\prod \{x\}_{-j}} \prod \frac{1}{1 - u_i v_j}. \tag{3.39}$$

By taking $f_s^i = 1 - x_s x_{s+1} \cdots x_{\alpha_i}$ for $i = 1, 2, ..., k$ in (3.34), we have

$$\frac{\Delta(u)}{\prod \{x\}_i} = (x_{\alpha_2}!)(x_{\alpha_s}!)^2 \cdots (x_{\alpha_d}!)^{d-1} \prod_{(i,j) \in \alpha(\lambda) - D_\lambda} \frac{1}{1 - W(H_{ij})}.$$

Similarly, (3.35) yields

$$\frac{\Delta(v)}{\prod \{x\}_{-j}} = (x_{-\beta_2}!)(x_{-\beta_3}!)^2 \cdots (x_{-\beta_d}!)^{d-1} \prod_{(i,j) \in \beta(\lambda) - D_\lambda} \frac{1}{1 - W(H_{ij})}.$$

Finally, noting that

$$\prod_{i,j} \frac{1}{1 - u_i v_j} = \prod_{(i,j) \in D_\lambda} \frac{1}{1 - W(H_{ij})}$$

and cancelling $C_{(\alpha|\beta)}$ from both sides of (3.39), we obtain (3.36). ∎

EXAMPLE. Let $\lambda = (32)$. Then the cells of λ are labelled as

x_0	x_1	x_2
x_{-1}	x_0	

We have

$$\sum_{P \in \mathbb{P}(32)} W(P) = \frac{1}{(1 - x_2)(1 - x_0 x_1 x_2)(1 - x_{-1} x_0 x_1 x_2)(1 - x_0)(1 - x_{-1} x_0)}$$

and

$$\sum_{T \in \mathbb{T}(32)} W(T) = \frac{x_{-1}^2 x_0^3 x_1 x_2}{(1-x_2)(1-x_0x_1x_2)(1-x_{-1}x_0x_1x_2)(1-x_0)(1-x_{-1}x_0)}.$$

We note that Theorem 3.5 seems to have not been previously observed in the literature; however, it is not difficult to see that the Hillman–Grassl bijection [11] which proves (3.24) also proves Theorem 3.5. Also, a number of results will follow from Theorem 3.5 by specializing the variables of (3.36), a typical example of which is the following.

COROLLARY 3.6. *Let* tr(P) *denote the sum of the main diagonal entries of a reverse plane partition* P. *Then*

$$\sum_{P \in \mathbb{P}(n^n)} q^{\operatorname{tr}(P)} = \frac{1}{(1-q)^{n^2}}.$$

Proof. Let $x_0 = q$ and $x_i = 1$ for $i \neq 0$ in (3.36). ∎

4. Extensions to Winged Skew Schur Functions

The main purpose of this section is to extend the Giambelli determinantal formula to winged skew Schur functions. Now recall that we obtain a winged skew shape from a Ferrers diagram of shape λ by adding a column of length $k \geqslant 0$ on top of the last cell in the first row of F_λ and a row of length $l \geqslant 0$ to the left of the bottom cell in the first column. The main fact to observe is that in the proof of Theorem 2.1 the involution Θ never modifies either the rightmost cell in the first row or bottom cell in the first column of any σ-tabloid T. Thus the proof goes through verbatim if we add a column on top of the last cell for every row corresponding to α_1 and a row to the left of the bottom cell of every column corresponding to β_1. The result will yield an expansion of an arbitrary winged skew Schur function $S_{[\lambda,k,l]}(x)$ as a determinant of winged skew Schur functions derived from hook Schur functions. That is, we have the following.

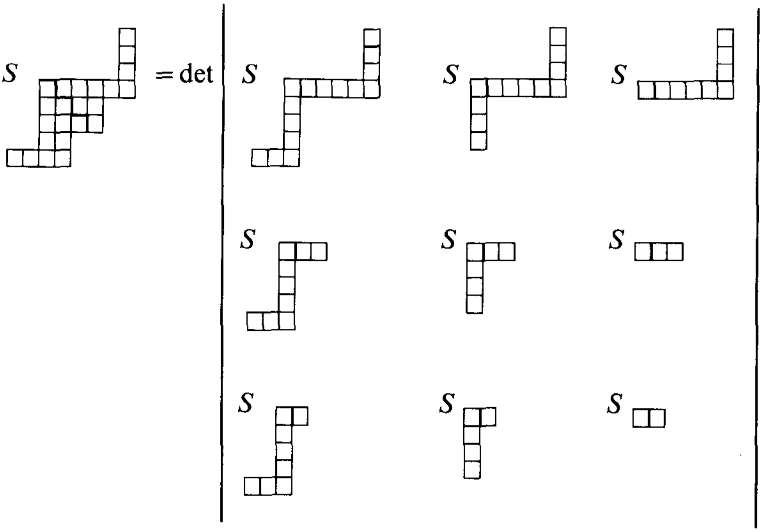
THEOREM 4.1.

$$S_{[(\alpha\mid\beta),k,l]} = \det \| S_{[(\alpha_i\mid\beta_j),k\chi(i=1),l\chi(j=1)]} \|, \tag{4.1}$$

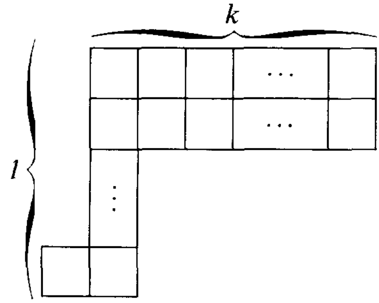
where for a statement A, $\chi(A) = 1$ *if* A *is true and* $\chi(A) = 0$ *if* A *is false.*

EXAMPLE.

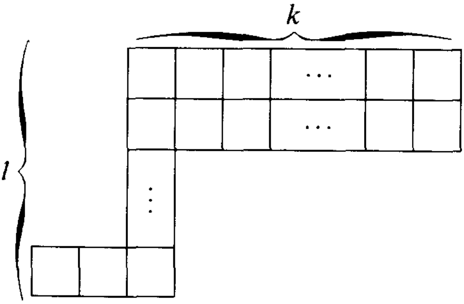
$S_{[(5,2,1\,|\,5,4,1),3,2]}$



One advantage of formula (4.1) over the general expansion of a skew Schur function as determinant of homogeneous symmetric functions or as a determinant of elementary symmetric functions via the Jacobi–Trudi identities [12, 20] is that in certain cases the size of the determinant we must use is greatly reduced. As examples of this phenomenon, we shall end with the calculation of the number of standard skew tableaux $n_{\lambda/\mu}$ via (4.1) for the shapes $\lambda/\mu = [(k, k-1\,|\,l, 1), 0, 1]$ and $\lambda/\mu = [(k, k-1\,|\,l, 1), 0, 2]$, which are pictured below.

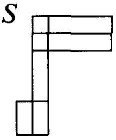


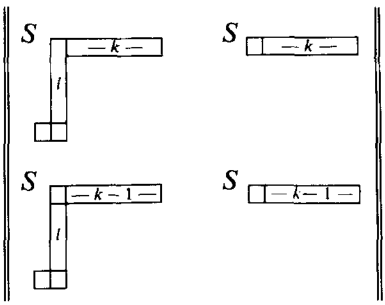
$[(k, k-1\,|\,l, 1), 0, 1]$



$[(k, k-1\,|\,l, 1), 0, 2]$

Now according to (4.1),

$S_{[(k,k-1\,|\,l,1),0,1]} = S$ 

$= \det$ 

It thus follows that

$$n_{[(k,k-1\,|\,l,1),0,1]} = \binom{2k+l+1}{k} n_{[(k\,|\,l),0,1]} - \binom{2k+l+1}{k+1} n_{[(k-1\,|\,l),0,1]}. \qquad (4.2)$$

We note that a similar calculation of $n_{\lambda/\mu}$ for the shapes pictured above using the Jacobi–Trudi identities would require the evaluation of either an $l \times l$ or $(k+2) \times (k+2)$ determinant of binomial coefficients. Now we can calculate the number of standard tableaux T of shape $[(k\,|\,l), 0, 1]$ by observing that either 1 is in the bottommost leftmost cell and hence 2 is in the upper corner square so that there are $\binom{k+l-1}{k}$ such standard tableaux or 1 is in the upper corner square so that we must choose k elements from $2, ..., k+l+1$ to put in the first row and then have $l-1$ ways to arrange the remaining elements resulting in $\binom{k+l}{k}(l-1)$ standard tableaux of the second type. That is,

$$n_{[(k\,|\,l),0,1]} = \binom{k+l-1}{k} + \binom{k+l}{k}(l-1). \qquad (4.3)$$

Then using (4.3) in (4.2) and simplifying results in the following product formula,

$$n_{[(k,\,k-1\,|\,l,1),0,1]} = \binom{2k+l+1}{k+1}\binom{k+l}{k}\frac{l}{(k+l+1)_3}\,[(l-1)(l+k)-k], \qquad (4.4)$$

where $(n)_k = n(n+1)\cdots(n+k-1)$. An entirely similar analysis may be applied to show

$$n_{[(k,k-1|l,1),0,2]} = \binom{2k+l+2}{k} n_{[(k|l),0,2]} - \binom{2k+l+2}{k+1} n_{[(k-1|l),0,2]} \tag{4.5}$$

$$n_{[(k|l),0,2]} = n_{[(k|l),0,1]} + \binom{k+l+1}{k}\binom{l}{2}$$
$$= \binom{k+l-1}{k} + \binom{k+l}{k}(l-1) + \binom{k+l+1}{k}\binom{l}{2}, \tag{4.6}$$

and

$$n_{[(k,k-1|l,1),0,2]} =$$
$$\binom{2k+l+2}{k+1}\binom{k+l+1}{k}\frac{(l+1)}{(k+l-1)_4}\left[\binom{l}{2}(l(l-1)+k(2l+2k-1))-k\right]. \tag{4.7}$$

We note that these formulas show the impossibility of any simple product formula for $n_{\lambda/\mu}$ for arbitrary skew shapes. For example, the final factors on the RHS of (4.4) and (4.7) cannot be simplified or eliminated since if $k=7$ and $l=11$, the factor $[(l-1)(l+k)-k]$ in (4.4) equals 173, which is prime, and similarly if $k=9$ and $l=5$, the factor $[(\binom{l}{2})(l(l-1)+k(2l+2k-1))-k]=2621$, which is also prime.

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