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A Combinatorial Interpretation of the Inverse Kostka Matrix

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The Kostka matrix K relates the homogeneous and the Schur bases in the ring of symmetric functions where $K_{\lambda,\mu}$ enumerates the number of column strict tableaux of shape λ and type μ. We make use of the Jacobi–Trudi identity to give a combinatorial interpretation for the inverse of the Kostka matrix in terms of certain types of signed rim hook tabloids. Using this interpretation, the matrix identity $KK^{-1}=I$ is given a purely combinatorial proof. The generalized Jacobi–Trudi identity itself is also shown to admit a combinatorial proof via these rim hook tabloids. A further application of our combinatorial interpretation is a simple rule for the evaluation of a specialization of skew Schur functions that arises in the computation of plethysms.

INTRODUCTION

In this paper, we first give a combinatorial interpretation to K^{-1}: the inverse of the Kostka matrix. This interpretation is then utilized to show combinatorially that $KK^{-1}=I$. Our point of departure is the Jacobi–Trudi identity, which gives a determinantal expansion of a Schur symmetric function in terms of the homogeneous symmetric functions. In turn, we provide a combinatorial proof of the generalized Jacobi–Trudi identity itself by a natural sign reversing involution on a class of signed tabloids.

In recent years, there has been considerable success in providing combinatorial proofs of symmetric function identities which express some given symmetric function as a determinant of other symmetric functions. Most notably, Gessel and Viennot [6], [7] used the approach that various determinants that arise in a number of combinatorial settings could be interpreted as weighted sum of k-tuples of paths. Certain natural involutions on these spaces have the effect of canceling out various terms with opposite sign, leaving a subclass of the original set of paths which is then shown to correspond to the objects under consideration. In certain other expansion formulae involving symmetric functions such as the Giambelli identity [8], sign reversing involutions are defined directly on classes of signed tabloids [4], [5]. In

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both settings, natural involutions usually provide extra combinatorial information and interesting generalizations that are not evident in purely algebraic treatments.

Our interpretation of K^{-1} is in terms of a special class of rim hook tabloids and the involution that we construct to give a combinatorial proof of $KK^{-1} = I$ relies on a sign reversing involution on pairs of adjacent special rim hooks. Indeed, we show that the basic idea of our sign reversing involution extends to filled rim hook tabloids to give a proof of the Jacobi–Trudi identity itself. That is, we show that the terms arising from the determinant of the generalized Jacobi–Trudi identity can be interpreted as special rim hook tabloids where each rim hook is filled with a weakly increasing sequence. We then give a sign reversing involution on such filled rim hook tabloids whose fixed points are precisely the column strict tableaux. We note that our involution to prove the generalized Jacobi–Trudi identity can be viewed as a translation of the Gessel–Viennot proof, where instead of using k-tuples of weighted paths to interpret the terms of the determinant, we interpret them directly as fillings of the Ferrers' diagram.

The outline of this paper is as follows: We deal with preliminaries and establish our notation in Section 1. In Section 2, the combinatorial interpretation of K^{-1} is given, followed by a combinatorial proof that $KK^{-1} = I$ in Section 3. In Section 4 we construct our sign reversing involution and prove the generalized Jacobi–Trudi identity. In Section 5, we give an application of our results by describing a simple algorithm to compute $S_{\lambda/\mu}(1, \omega, \ldots, \omega^{p-1})$ where ω is any primitive pth root of unity.

1. PRELIMINARIES

Let $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ be a partition of n, i.e., $n = |\lambda| = \sum_{i=1}^{k} \lambda_i$. Each one of the integers λ_i is called a part of λ. We take $\lambda_i = 0$ for $i \leqslant 0$. If λ is a partition of n, this is denoted by $\lambda \vdash n$. An alternate notation for λ is $\lambda = 1^{q_1}2^{q_2} \cdots k^{q_k}$ where q_i is the number of parts of λ of size i.

The Ferrers' diagram of shape λ, denoted by F_λ, is the set of left justified rows of squares of cells with λ_i cells in the ith row from the top for $i = 1, \ldots, k$. For example,

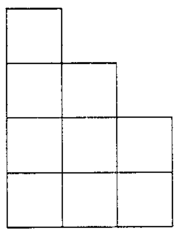
$F_{(1,2,3,3)} =$ 

FIGURE 1.1.

In this context, the pair (i, j) denotes the cell in the ith row and the jth column of F_λ, where we label the rows from bottom to top and the columns from left to right.

Given partitions $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ and $\mu = (0 < \mu_1 \leqslant \cdots \leqslant \mu_l)$, we write $\mu \subseteq \lambda$

if $l \leqslant k$ and $\mu_{l-(i-1)} \leqslant \lambda_{k-(i-1)}$ for $i = 1, \ldots, l$. The skew diagram $F_{\lambda/\mu}$ of shape λ/μ will consist of cells of F_λ that remain after the cells of F_μ are removed. For example

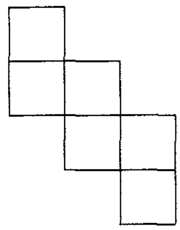
$F_{(1,2,3,3)/(1,2)} =$ 

FIGURE 1.2.

We can think of any partion λ as the skew shape $\lambda/\varnothing$. λ *dominates* μ if $\lambda_k + \lambda_{k-1} + \cdots + \lambda_{k-i} \geqslant \mu_l + \mu_{l-1} + \cdots + \mu_{l-i}$ for all $i \geqslant 0$.

A *rim hook* H of a partition λ is a consecutive sequence of cells on its North-Eastern rim such that any two adjacent cells of H share a common edge, and the removal of H from F_λ leaves a legal diagram. H is a *special rim hook* if it starts in the cell $(\lambda_1, 1)$. The number of rows of a rim hook H is referred to as its *height*, denoted by $\mathrm{ht}(H)$. The special rim hooks of the partition $\lambda = 123^2$ are illustrated in Figure 1.3. These notions generalize naturally to skew shapes.

A *tabloid* T of shape λ/μ is a filling of $F_{\lambda/\mu}$ with positive integers. T is of *type* $\rho = 1^{q_1} \cdots i^{q_i} \cdots$, if i has frequency q_i in T. T_{ij} denotes the entry in the (i, j)th cell of T. A tabloid T of shape λ/μ is a *column strict tableau* if the entries of T are weakly increasing in each row from left to right and strictly increasing in each column from bottom to top. For any tabloid T of shape λ/μ, we define the weight of T by $w(T) = \prod_{(i,j) \in F_\lambda} x_{T_{ij}}$.

The Schur function S_λ of shape λ is defined by

$$S_\lambda(x) = \sum_T w(T), \tag{1.1}$$

where the summation is over all column strict tableaux T of shape λ. Similarly, whenever $\mu \subseteq \lambda$, the skew Schur function $S_{\lambda/\mu}(x)$ of shape λ/μ is defined by

$$S_{\lambda/\mu}(x) = \sum_T w(T), \tag{1.2}$$

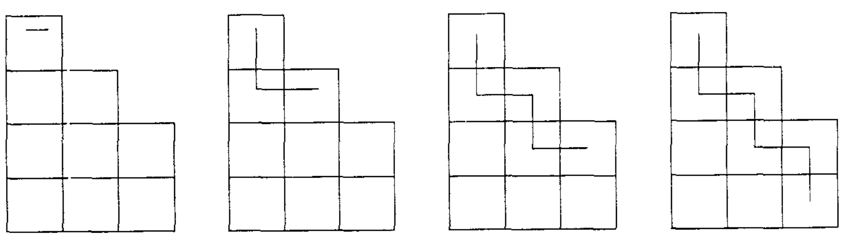


FIGURE 1.3.

where the summation is over all column strict tableaux T of shape λ/μ. The *homogeneous* symmetric function $h_\lambda(x)$ corresponding to a partition $\lambda \vdash n$ is given by

$$h_\lambda(x) = \prod_{i=1}^{k} h_{\lambda_i}(x), \tag{1.3}$$

where

$$h_r(x) = \sum_{0 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r} x_{i_1} x_{i_2} \cdots x_{i_r}. \tag{1.4}$$

Both $\langle S_\lambda \rangle_{\lambda \vdash n}$ and $\langle h_\lambda \rangle_{\lambda \vdash n}$ are integral bases for the space of symmetric functions, homogeneous of degree n. The integral transition matrix K relating the Schur and the homogeneous bases via $\langle h_\lambda \rangle_{\lambda \vdash n} = \langle S_\lambda \rangle_{\lambda \vdash n} K$ is known as the *Kostka* matrix [10]. If $\mu = (0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_n)$ and λ are partitions of n, then $K_{\lambda/\mu}$ is the number of column strict tableaux of shape λ and type $1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n}$. A fact which is well known and which we shall use later is that for any permutation σ of the index set $\{1, 2, \ldots, n\}$, we have

$$\begin{aligned} K_{\lambda/\mu} &= \textit{number of column strict tableaux of shape } \lambda \textit{ and type } 1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n} \\ &= \textit{number of column strict tableaux of shape } \lambda \textit{ and type } 1^{\mu_{\sigma(1)}} 2^{\mu_{\sigma(2)}} \cdots n^{\mu_{\sigma(n)}}. \end{aligned} \tag{1.5}$$

A combinatorial proof that the number of column strict tableaux of shape λ is invariant under permutations of the type may be found in [1]. For the combinatorics of Schur functions see Stanley [12]. The theory of symmetric functions is covered in detail in Macdonald [11].

2. A COMBINATORIAL INTERPRETATION OF K^{-1}

We define a *special rim hook tabloid* T of shape μ and type $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$ as a filling of the Ferrers' diagram F_μ of μ repeatedly with rim hooks of sizes $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ such that each rim hook is special, i.e. has at least one cell in the first column. To be more precise, a special rim hook tabloid T is constructed recursively as follows. First we pick a special rim hook H_1 in F_μ and remove the cells of H_1 to produce a Ferrers' diagram of shape $\mu^{(1)}$. Then the process is repeated for $F^{\mu^{(1)}}$, i.e., we pick a special rim hook H_2 of $F^{\mu^{(1)}}$, remove the cells of H_2 to produce a Ferrers' diagram of shape $F^{\mu^{(2)}}$, etc. We continue this process until we produce a filling T of all the cells of F_μ with special rim hooks $H_1, H_2, \ldots, H_l$. The type of T is λ if $(|H_1|, |H_2|, \ldots, |H_l|)$ arranged in weakly increasing order produces the partition λ.

Note that the above definition can be extended to skew shapes λ by requiring that each rim hook have a cell bordering the Western boundary of λ.

Example There are 2 special rim hook tabloids T of shape $\mu = 3^3 4$ and type $\lambda = 2^2 4 5$:

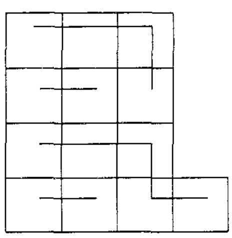
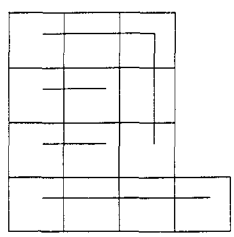
$T_1 =$  $T_2 =$ 

FIGURE 2.1.

The sign of a rim hook H is taken to be $(-1)^{\mathrm{ht}(H)-1}$ as usual. The sign of T is the product of the signs of the rim hooks of T. Thus

$$\begin{aligned} \operatorname{sign}(T_1) &= (-1)^{2-1}(-1)^{1-1}(-1)^{2-1}(-1)^{1-1} = 1, \\ \operatorname{sign}(T_2) &= (-1)^{3-1}(-1)^{1-1}(-1)^{1-1}(-1)^{1-1} = 1. \end{aligned} \tag{2.1}$$

Note that for a special rim hook tabloid, we are only interested in the sizes of the rim hooks and not their order. This is in contrast to the usual notion of a rim hook tableau of type λ. For example, when $\lambda = 2^2 45$ and $\mu = 3^3 4$, the rim hook tableaux in the usual sense are built from

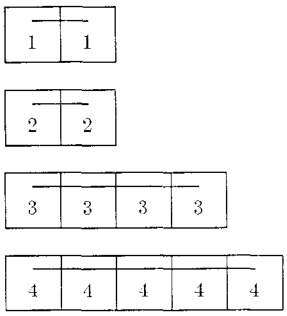


FIGURE 2.2.

where these are placed in their natural order in F_μ so that in each step we have a legitimate shape. Thus the usual rim hook tableaux of type λ and shape μ would be the following:

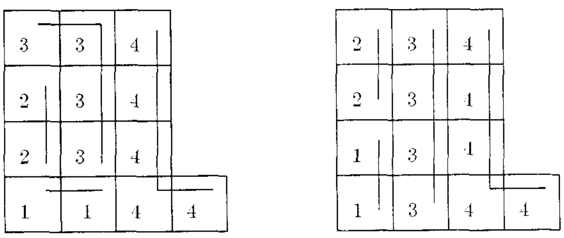


FIGURE 2.3.

A tabloid like T_1, i.e.

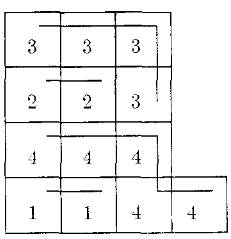


FIGURE 2.4.

is not legal in the usual definition.

Note that if T is a special rim hook tabloid of shape μ and type λ, then λ necessarily dominates μ. The matrices K and K^{-1} for $n = 5$ are given in Figure 2.5.

	5	14	23	1^23	12^2	1^32	1^5
5	1	1	1	1	1	1	1
14	0	1	1	2	2	3	4
23	0	0	1	1	2	3	5
1^23	0	0	0	1	1	3	6
12^2	0	0	0	0	1	2	5
1^32	0	0	0	0	0	1	4
1^5	0	0	0	0	0	0	1

K

	5	14	23	1^23	12^2	1^32	1^5
5	1	-1	0	1	0	-1	1
14	0	1	-1	-1	1	1	-2
23	0	0	1	-1	-1	2	-2
1^23	0	0	0	1	-1	-1	3
12^2	0	0	0	0	1	-2	3
1^32	0	0	0	0	0	1	-4
1^5	0	0	0	0	0	0	1

K^{-1}

FIGURE 2.5.

THEOREM 1

$$K_{\lambda,\mu}^{-1} = \sum_T \operatorname{sign}(T) \tag{2.2}$$

where the summation is over all special rim hook tabloids of type λ and shape μ.

Proof We have that $\langle h_\lambda \rangle = \langle S_\lambda \rangle K$ so that $\langle S_\lambda \rangle = \langle h_\lambda \rangle K^{-1}$. Hence $K_{\lambda,\mu}^{-1}$ is the coefficient of h_λ in the expansion of the Schur function S_μ in terms of the homogeneous symmetric functions.

Recall that the classical Jacobi–Trudi identity [9], [13] furnishes a determinantal formula for S_μ in terms of the homogeneous symmetric functions. More precisely, if $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, then the Jacobi–Trudi identity reads

$$S_\mu = \det \| h_{\mu_j + j - i} \| . \tag{2.3}$$

Expanding the determinant with respect to the first row we find that

$$S_{(\mu_1, \mu_2, \ldots, \mu_k)} = \sum_{j=1}^{k} (-1)^{j-1} h_{\mu_j + j - 1} S_{(\mu_1 - 1, \ldots, \mu_{j-1} - 1, \mu_{j+1}, \ldots, \mu_k)} \tag{2.4}$$

Note that for each j with $1 \leqslant j \leqslant k$, the quantity $\mu_j + j - 1$ is the length of the rim hook that starts in cell $(\mu_1, 1)$ of μ and ends in the jth row from the top. The shape $(\mu_1 - 1, \ldots, \mu_{j-1} - 1, \mu_{j+1}, \ldots, \mu_k)$ is the resulting diagram obtained by removing this rim hook from the Ferrers' diagram of μ.

Example

$$S_{(2,2,3,4)} = \det \begin{bmatrix} h_2 & h_3 & h_5 & h_7 \\ h_1 & h_2 & h_4 & h_6 \\ h_0 & h_1 & h_3 & h_5 \\ 0 & h_0 & h_2 & h_4 \end{bmatrix}$$

$$= h_2 \det\begin{bmatrix} h_2 & h_4 & h_6 \\ h_1 & h_3 & h_5 \\ h_0 & h_2 & h_4 \end{bmatrix} - h_3 \det\begin{bmatrix} h_1 & h_4 & h_6 \\ h_0 & h_3 & h_5 \\ 0 & h_2 & h_4 \end{bmatrix} + h_5 \det\begin{bmatrix} h_1 & h_2 & h_6 \\ h_0 & h_1 & h_5 \\ 0 & h_0 & h_4 \end{bmatrix} - h_7 \det\begin{bmatrix} h_1 & h_2 & h_4 \\ h_0 & h_1 & h_3 \\ 0 & h_0 & h_2 \end{bmatrix}$$

$$= h_2 S_{(2,3,4)} - h_3 S_{(1,3,4)} + h_5 S_{(1,1,4)} - h_7 S_{(1,1,2)},$$

with the correspondences

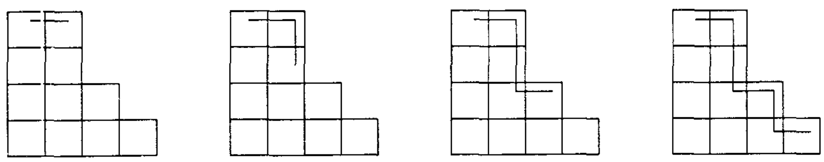


FIGURE 2.6.

The jth term $(-1)^{j-1} h_{\mu_j+j-1} S_{(\mu_1-1,\ldots,\mu_{j-1}-1,\mu_{j+1},\ldots,\mu_k)}$ in the summation (2.4) has sign $(-1)^{j-1}$ and j is precisely the height of the special rim hook which starts in the top cell of the first column of F_μ, and ends at the end of the jth row of F_μ starting from the top. Thus the sign of the jth term in the sum (2.4) is just the sign of the corresponding special rim hook.

Now taking the coefficient of h_λ on both sides of (2.4), we see that $K^{-1}_{\lambda,\mu}$ satisfies the recursion

$$K^{-1}_{\lambda,\mu} = \sum_{i=1}^{k} (-1)^{i-1} K^{-1}_{\lambda/[\mu_i+i-1],(\mu_1-1,\ldots,\mu_{i-1}-1,\mu_{i+1},\ldots,\mu_k)} \tag{2.5}$$

where $\lambda/[j]$ denotes the partition which results from λ by removing a part of size j if λ has such a part. If λ has no part of size j, then we make the convention that $K^{-1}_{\lambda/[j],\rho} = 0$ for any partition ρ.

In fact, it is easy to see that the recursion (2.5) for $K^{-1}_{\lambda,\mu}$ together with the fact that

$$K^{-1}_{\lambda,(k)} = \chi(\lambda = (k)) \tag{2.6}$$

(since $S_{(k)} = h_k$) completely determine $K^{-1}_{\lambda,\mu}$ for all λ and μ. Here $\chi(S)$ is the indicator of the statement S, i.e., $\chi(S) = 1$ if S is a true statement and $\chi(S) = 0$ otherwise.

Now define

$$D_{\lambda,\mu} = \sum_T \operatorname{sign}(T) \tag{2.7}$$

where the summation is over all special rim hook tabloids T of type λ and shape μ. Then it is not difficult to see that

$$D_{\lambda,\mu} = \sum_{i=1}^{k} (-1)^{i-1} D_{\lambda/[\mu_i+i-1],(\mu_1-1,\ldots,\mu_{i-1}-1,\mu_{i+1},\ldots,\mu_k)} \tag{2.8}$$

and

$$D_{\lambda,(k)} = \chi(\lambda = (k)). \tag{2.9}$$

Thus $K^{-1}_{\lambda,\mu} = D_{\lambda,\mu}$ as claimed. ∎

One of the advantages of having such a combinatorial interpretation for $K^{-1}_{\lambda,\mu}$ is that it allows one to easily calculate $K^{-1}_{\lambda,\mu}$ for specific λ and μ by hand. Moreover, our interpretation allows us to give exact formulae for $K^{-1}_{\lambda,\mu}$ for many special cases of λ and μ. For example, it is clear that the only special rim hook tabloid of shape λ and type λ is the one where all the rim hooks are horizontal. Thus $K^{-1}_{\lambda,\lambda}=1$. If λ or μ is either a single row or a column, it is easy to see from our interpretation of $K^{-1}_{\lambda,\mu}$ that the following holds:

COROLLARY 1

(i) $K^{-1}_{(1^n),\mu}=\chi(\mu=(1^n))$,

(ii) $K^{-1}_{\lambda,(n)}=\chi(\lambda=(n))$,

(iii) $K^{-1}_{(n),\mu}=\begin{cases}(-1)^k & \textit{if} \quad \mu=(1^k,n-k) \textit{ with } 0\leqslant k<n,\\ 0 & \textit{otherwise},\end{cases}$

(iv) $K^{-1}_{(1^{\alpha_1}\cdots n^{\alpha_n}),(1^n)}=(-1)^{n-(\alpha_1+\cdots+\alpha_n)}\binom{\alpha_1+\cdots+\alpha_n}{\alpha_1,\ldots,\alpha_n}$.

Proof Since each rim hook of a special rim hook tabloid must start in the first column, we see that $K^{-1}_{\lambda,\mu}\neq 0$ implies $k(\lambda)\leqslant k(\mu)$. Now (i) and (ii) follow from this observation. For (iii), note that if there is only one rim hook in T, then T must have the shape of a hook. For (iv), we observe that in a special rim hook tabloid of shape (1^n) all rim hooks are vertical, so the number of special rim hook tabloids of type $(1^{\alpha_1}\cdots n^{\alpha_n})$ is just the number of permutations of the multiset $\{1^{\alpha_1},\ldots,n^{\alpha_n}\}$. ■

We can also give explicit formulas for $K^{-1}_{\lambda,\mu}$ when λ is a hook, a 2-row, or a 2-column shape. A direct application of Theorem 1 where λ is a hook shape gives

COROLLARY 2 *If* $s\geqslant 2$ *then*

$$K^{-1}_{(1^{n-s},s),\mu}=\begin{cases}(-1)^{s-1}(n-s+1) & \textit{if} \quad \mu=(1^n),\\ (-1)^k & \textit{if} \quad \mu=(1^{n-s-k},2^k,s-k) \textit{ with } 0\leqslant k\leqslant s-2,\\ 0 & \textit{otherwise}.\end{cases}$$

For 2-column partitions we have

COROLLARY 3

$$K^{-1}_{(1^k,2^l),\mu}=\begin{cases}(-1)^p\binom{k+p}{k} & \textit{if} \quad \mu=(1^{k+2p},2^{l-p}),\\ 0 & \textit{otherwise}.\end{cases}$$

Proof First observe that if μ has a part of size 3 or greater, then there are no special rim hook tabloids consisting of rim hooks of sizes 1 or 2. Now if $\mu=(1^s2^t)$, it is easy to see that the only way to fill μ is to have horizontal rim hooks of size 2 in the bottom t rows. Thus we must have $t\leqslant l$ and all the remaining $l-t$ rim hooks of size 2 must be placed vertically in the first column. Clearly there are $\binom{l-t+k}{k}$ ways to arrange these $l-t$ rim hooks of size 2 with the k rim hooks of size 1 to fill the remaining cells in the first column. Furthermore the sign associated to all such tabloids is $(-1)^{l-t}$. ∎

The following corollary for 2-row shapes can be proved by a case by case analysis:

COROLLARY 4

$$\text{(i)}\ K^{-1}_{(n,n),\mu}=\begin{cases}(-1)^{l+n-1} & \textit{if}\quad \mu=(1^{n+l},n-1)\\ & \textit{with } 1\leqslant n-1\leqslant n,\\ (-1)^r & \textit{if}\quad \mu=(1^r,2^k,n-r-k,n-k)\\ & \textit{with } 0\leqslant k\leqslant n-1 \textit{ and } 2\leqslant n-r-k,\\ 0 & \textit{otherwise}.\end{cases}$$

If $m<n$,

$$\text{(ii)}\ K^{-1}_{(m,n),\mu}=\begin{cases}(-1)^{l+n-1}2 & \textit{if}\quad \mu=(1^{n+l},m-l)\\ & \textit{with } 1\leqslant m-l\leqslant m,\\ (-1)^{k+m-1} & \textit{if}\quad \mu=(1^{m+k},n-k)\\ & \textit{with } m\leqslant n-k\leqslant n,\\ (-1)^r & \textit{if}\quad \mu=(1^r,2^k,m-r-k,n-k)\\ & \textit{with } 0\leqslant k\leqslant n-1 \textit{ and } 2\leqslant m-r-k,\\ (-1)^{r+1} & \textit{if}\quad \mu=(1^r,2^k,m-k+1,n-k-r+1)\\ & \textit{with } 0\leqslant k\leqslant m-1 \textit{ and } m-k\leqslant n-r-k,\\ (-1)^r & \textit{if}\quad \mu=(1^r,2^k,n-r-k,m-k)\\ & \textit{with } 0\leqslant k\leqslant m-1 \textit{ and } 2\leqslant n-k-r\leqslant m-k,\\ 0 & \textit{otherwise}.\end{cases}$$

Finally, we should note that the fact that $K^{-1}_{(n,n),\mu}$ is either 0, 1, or -1 can be generalized:

COROLLARY 5

$$K^{-1}_{(k^j),\mu}=0,\ 1,\ \textit{or}\ -1.$$

Proof The proof follows by noting that if all rim hooks are of the same size, then there is at most one special rim hook tabloid of shape μ for any μ. ∎

The one drawback to our combinatorial interpretation is that there can be some cancellation in the computation of the sum $K^{-1}_{\lambda,\mu}=\sum_T \operatorname{sign}(T)$ where T runs over all special rim hook tabloids of type λ and shape μ. However for small values of n such cancellations are rare. For example, it is easy to see from the ideas for the proofs of Corollaries 1–4 that this never happens if λ is a hook, a 2-row, or a 2-column shape. Indeed, the smallest cancellation is when $n=6$, in which case there is only one such example, namely $\lambda=(1,2,3)$, and $\mu=(1^2,2^2)$. We then have two special rim hook tabloids of shape λ and type μ which have opposite signs:

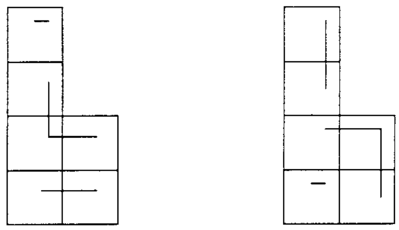


FIGURE 2.7.

Thus $K^{-1}_{(1,2,3),(1^2,2^2)} = 0$. Similarly for $n = 7$, there are exactly two such canceling pairs. Namely, for $\lambda = (1, 2, 4)$ and $\mu = (1, 2^3)$, we have two special rim hook tabloids with opposite signs,

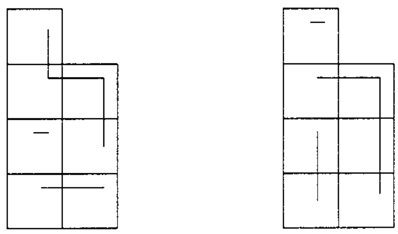


FIGURE 2.8.

and for $\lambda = (1^2, 2, 3)$ and $\mu = (1^3, 2^2)$ there are two special rim hook tabloids with opposite signs:

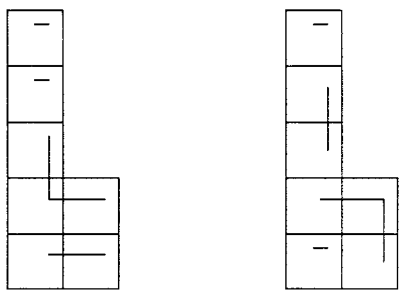


FIGURE 2.9.

3. A COMBINATORIAL PROOF THAT $KK^{-1} = I$

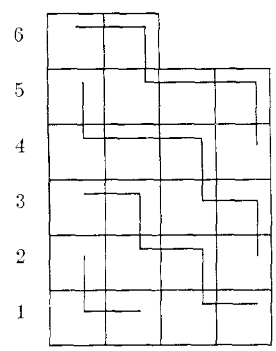
Here we must show that

$$\chi(\lambda = \mu) = \sum_{\nu \vdash n} K_{\lambda,\nu} K^{-1}_{\nu,\mu} = \sum_{(P,T)} \operatorname{sign}(T) \tag{3.1}$$

where the summation runs over all pairs (P, T) in which

(i) T is a special rim hook tabloid of shape $\mu = (0 < \mu_1 \leqslant \cdots \leqslant \mu_k)$ and type ν,
(ii) P is a column strict tableau of shape λ and type $1^{q_1}2^{q_2}\cdots k^{q_k}$, where q_i is the length of the special rim hook that starts in the ith row of μ reading from bottom to top. We put $q_i = 0$ if there is no such special rim hook.

Note that in this section, we are ordering the rows of F_μ from bottom to top. As an example, for the special rim hook tabloid T of Figure 3.1, we require the corresponding column strict tableau P to have type $2^3 3^6 5^7 6^6$.

$T =$ 

$(q_1, q_2, \ldots, q_6) = (0, 3, 6, 0, 7, 6)$

FIGURE 3.1.

Observe that in (3.1), we are explicitly using (1.5), i.e., that for any permutation σ of $\{1, 2, \ldots, n\}$, $K_{\lambda,\mu}$ equals the number of column strict tableau of shape λ and type $1^{\mu_{\sigma(1)}}2^{\mu_{\sigma(2)}}\cdots n^{\mu_{\sigma(n)}}$, because we allow the type of a column strict tableau P in a pair (P, T) to depend on T.

Now consider the following involution on such pairs (P, T).

Consider the first (bottommost) row of P. If the first row of P consists entirely of 1's, then (P, T) will be a fixed point of our involution. Otherwise, let $r+1$ be the largest integer in the first row of P not equal to 1. The involution is defined according whether or not r also appears in P.

Case 1 r is also in P:

In this case, we shall map the pair (P, T) to a new pair (P', T'). T' is produced from T as follows: Consider the two special rim hooks H_r and H_{r+1} which start in the rth and the $r+1$st rows in the first column of μ. Note that H_r and H_{r+1} exist since both r and $r+1$ appear in P.

Suppose H_{r+1} ends in row j and H_r ends in row i in T. Note that we must have $j \leqslant r+1$ and $i \leqslant r$. We claim that there is only one other way to cover the cells occupied by H_r and H_{r+1} by two other special rim hooks H_r' and H_{r+1}' which start in rows r and $r+1$ respectively. A picture will make this clear. There are two subcases to consider:

Subcase (*a*) $i<j$:

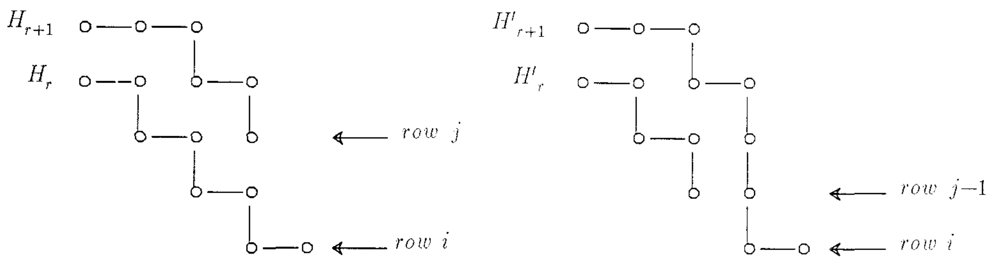


FIGURE 3.2.

Here H'_{r+1} will end in row i and H'_r will end in row $j-1$.

Subcase (*b*) $i \geqslant j$:

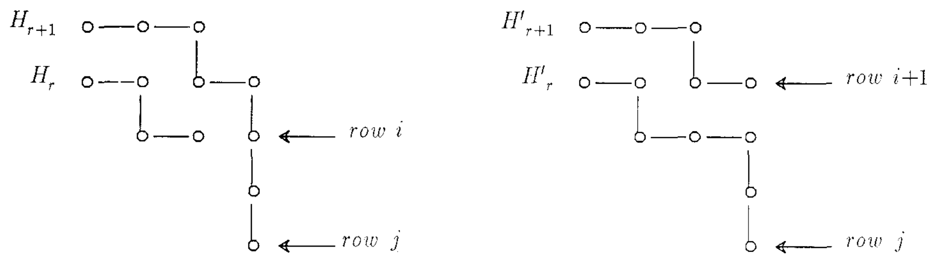


FIGURE 3.3.

In this case, H'_{r+1} ends in row $i+1$ and H'_r ends in row j.

We should also note that a special sub-subcase of Case (a) arises when H_{r+1} consists of a single cell. The switch here takes the form of gluing together H_{r+1} and H_r to obtain H'_{r+1}. We technically think of H'_r as the empty special rim hook. In pictures

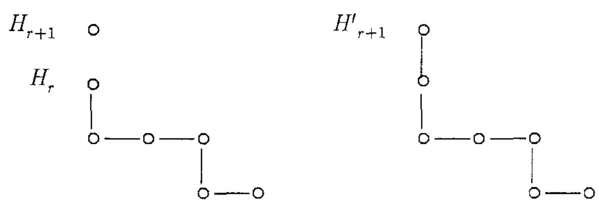


FIGURE 3.4.

Note that in each case the following properties hold:

(1) $\operatorname{sign}(H_r)\operatorname{sign}(H_{r+1}) = -\operatorname{sign}(H'_r)\operatorname{sign}(H'_{r+1})$, (3.2)

(2) $q'_r = q_{r+1} - 1$, $q'_{r+1} = q_r + 1$. (3.3)

This transformation taking H_r and H_{r+1} to the pair H'_r and H'_{r+1} will be referred to as *switching the rth and the* $r+1$*st* special rim hooks in T. Thus switching the rth and the $r+1$st special rim hooks in a special rim hook tabloid T results in a new special rim hook tabloid T' with $\operatorname{sign}(T) = -\operatorname{sign}(T')$.

We must now transform P to a new column strict tableau P' of shape λ so that it can be paired with T'. Note that the relative frequencies $r^{q_r}(r+1)^{q_{r+1}}$ of r and $r+1$ in P should be changed to $r^{q_{r+1}-1}(r+1)^{q_r+1}$ to match the lengths of the special rim hooks in T'.

We produce P' from P in two steps. First we form a new column strict tableau P^+ from P by changing the leftmost $r+1$ in the first row of P to r. Note that the relative frequencies of r and $r+1$ in P^+ now becomes $r^{q_r+1}(r+1)^{q_{r+1}-1}$. Thus to obtain P' from P^+, we need only to switch the relative frequencies of r and $r+1$ in P^+. However, there is a standard procedure due to Bender and Knuth [1] to do this which may be briefly described as follows: First take P^+ and leave fixed all pairs of r and $r+1$ which appear in the same column. Note that $r+1$ necessarily appears immediately above r in such a pair. All other appearances of r and $r+1$ in P^+ are union of consecutive entries in rows of P^+ of the form

$$\boxed{\;r \quad \cdots \quad r \quad r+1 \quad \cdots \quad r+1\;}$$

FIGURE 3.5.

where in each such block, the relative frequencies of r and $r+1$ are $r^s(r+1)^t$ for some $s \geqslant 0$, $t \geqslant 0$ with $s+t>0$. To obtain P' from P^+, we simply alter each such block so that the frequencies of r and $r+1$ become $r^t(r+1)^s$ instead.

Example

$$\begin{array}{|c|}\hline r+1 \\ \hline r \\ \hline\end{array} \longrightarrow \begin{array}{|c|}\hline r+1 \\ \hline r \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline r & r & r \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline r+1 & r+1 & r+1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline r+1 & r+1 & r+1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline r & r & r \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline r & r+1 & r+1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline r & r & r+1 \\ \hline\end{array}$$

FIGURE 3.6.

An example of this procedure in stages is given in Figure 3.7:

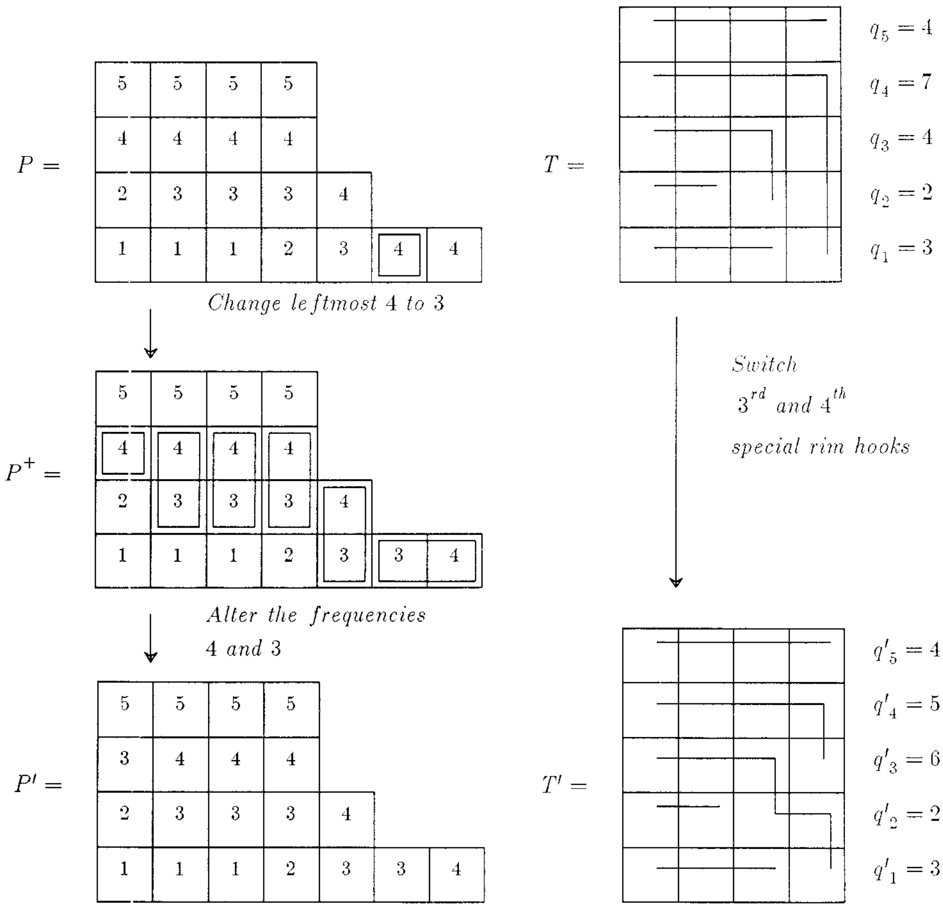


FIGURE 3.7.

Now we consider the case in which r does not appear in P.

Case 2 There is no r in P:

Note that in this case there is no rim hook in T that starts in row r. For example, consider the following pair.

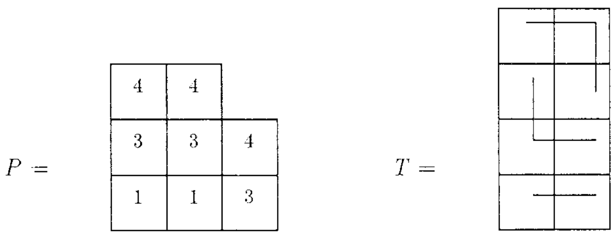


FIGURE 3.8.

Note that the only way this can happen is if the rim hook which starts in row $r+1$ starts by going down. In this case we form two special rim hooks from H_{r+1} by breaking off the top cell of H_{r+1} to form H'_r and H'_{r+1}. This is simply the reverse of the special case singled out in Figure 3.4 where H_r is the empty rim hook. As before, T' is obtained from T by switching the rim hooks H_r and H_{r+1}.

We construct P' from P as before. That is, first the leftmost $r+1$ in the first row of P to r to get P^+ and then interchange the relative frequencies of r and $r+1$ in P^+ to obtain P'. Thus for our example above, the resulting pair (P', T') is pictured below:

$P' =$

4	4	
2	2	4
1	1	3

$T' =$

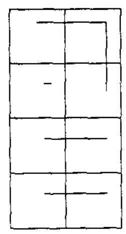


FIGURE 3.9.

This completes our definition of the involution. It is routine but tedious to check that our map is indeed an involution which always maps pairs (P, T) in which there is some element besides 1 in the first row of P to pairs (P', T') in which there is also some entry other than 1 in the first row of P'.

The key fact to note is that when we start with a pair (P, T) in which $r+1$ is the largest element in the first row of P, then when we change the leftmost $r+1$ in the first row of P to r, we end up with a column strict tableau P^+ where there is an r in the first row which is not covered by an $r+1$ in the second row. It follows that when we apply the Bender–Knuth transformation to P^+ which switches the multiplicities of r and $r+1$ in P^+ to produce P', there will be at least one $r+1$ in the first row of P'. Thus we apply our involution to (P', T'), we will once again switch the rth and $r+1$st rim hooks in T'. It then is not difficult to check that our involution will map (P', T') back to (P, T). Finally, it follows from our previous remarks concerning switching adjacent special rim hooks that $\operatorname{sign}(T') = -\operatorname{sign}(T)$. Thus all pairs (P, T) in (3.1) cancel except for those pairs (P, T) in which

$$P \textit{ has all 1's in its first row.} \tag{3.4}$$

Note that (3.4) ensures that if $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_l)$ then $q_1 = \lambda_l$ where q_1 is the length of the special rim hook that starts in the first row of T. But then for each r, there can be at most λ_l r's in P, which means that $q_r \leqslant \lambda_l$ for all special rim hooks in T. Since the rim hook H_1 must be horizontal, there can be no other rim hook which contains cells in the first row. Thus $\mu_k = \lambda_l$ where $\mu = (0 < \mu_1 \leqslant \cdots \leqslant \mu_k)$. But because $\operatorname{sign}(H_1) = 1$, we can simply strip off the last rows of P and T and renumber the labels in P by replacing each occurrence of r by $r-1$ to obtain a pair $(\bar{P}, \bar{T})$ with $\operatorname{sign}(T) = \operatorname{sign}(\bar{T})$, which would appear in the sum corresponding to

$$\sum_{\nu \vdash n - \lambda_l} K_{\lambda/[\lambda_l],\nu} K^{-1}_{\nu,\mu/[\mu_k]} = \sum_{(\bar{P},\bar{T})} \operatorname{sign}(\bar{T}). \tag{3.5}$$

We can then apply our involution inductively to conclude that (3.1) reduces to

$$\chi(\lambda = \mu) = \chi(\lambda/[\lambda_l] = \mu/[\mu_k]) \tag{3.6}$$

which is immediate since $\lambda_l = \mu_k$. ∎

It seems natural to ask for a combinatorial proof of the identity $K^{-1}K = I$ along the same lines as our combinatorial proof of $KK^{-1} = I$. Again, we can reduce the problem of proving $K^{-1}K = I$ to showing that a certain signed sum over pairs (P, T) is either 0 or 1. However, the problem of finding the appropriate sign reversing involution in this case using our combinatorial interpretation of the entries of K^{-1} is still open.

4. A COMBINATORIAL PROOF OF THE GENERALIZED JACOBI–TRUDI IDENTITY

In this section we present a combinatorial proof of the generalized Jacobi–Trudi identity itself based on filled special rim hook tabloids.

Suppose $\lambda = (0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k)$ and $\mu = (0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k)$ are two partitions of n with $\mu \subseteq \lambda$. Putting $h_{-m} = 0$ for $m > 0$, the generalized Jacobi–Trudi identity reads

THEOREM 2

$$S_{\lambda/\mu} = \det \| h_{\lambda_j - \mu_i + j - i} \| . \tag{4.1}$$

In the case that μ is the null partition, (4.1) reduces to the expansion formula given in (2.3) that relates the Schur and the homogeneous bases.

Our technique here is based on the construction of a certain space of filled special rim hook (f.s.r.h.) tabloids of shape λ/μ. First we proceed exactly as in the expansion of (2.3) and show that each nonzero term arising from the determinant on the right-hand side of (4.1) can be coded by a special rim hook tabloid of shape λ/μ. That is, analogous to the expansion of (2.3) with respect to the first row of the matrix, the quantity $\lambda_j - \mu_1 + j - 1$ is the length of the special rim hook that starts in the extreme North-West cell of the diagram of λ/μ, and ends at the jth row from the top. We must be careful however, because for skew shapes it need not be the case that these special rim hooks lie completely in the diagram of λ/μ. For example, consider the case where $\lambda = (2, 2, 3, 4, 5)$ and $\mu = (0, 0, 2, 2, 3)$. Denoting μ by $(2, 2, 3)$ for notational convenience, we have

$$S_{(2,2,3,4,5)/(2,2,3)} = \det \left[\begin{array}{cc|ccc} h_2 & h_3 & h_5 & h_7 & h_9 \\ h_1 & h_2 & h_4 & h_6 & h_8 \\ \hline 0 & 0 & h_1 & h_3 & h_5 \\ 0 & 0 & 1 & h_2 & h_4 \\ 0 & 0 & 0 & 1 & h_2 \end{array}\right]$$

FIGURE 4.1.

Then the terms h_2, h_3, h_5, h_7, h_8 correspond to the following rim hooks:

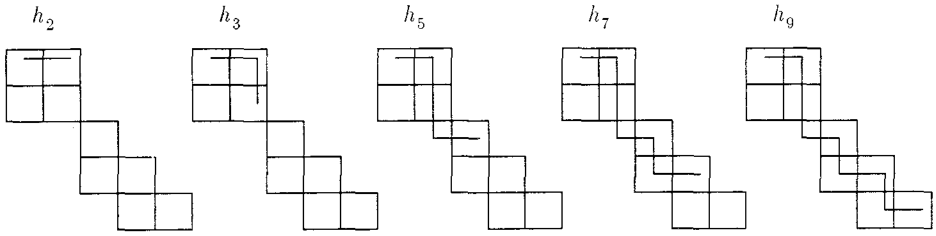


FIGURE 4.2.

Note that only the rim hooks corresponding to h_2 and h_3 lie completely within the diagram of λ/μ. However, it is easy to see in this case that to get a nonzero term in the determinant our choice of elements in the first two columns must come from the top 2×2 minor and our choices for the last three columns must come from the bottom 3×3 minor. In other words, when we expand our determinant about the first row only those terms corresponding to special rim hooks which lie completely in $F_{\lambda/\mu}$ can produce nonzero terms. This is true in general. In other words, if the jth special rim hook does not lie completely in $F_{\lambda/\mu}$, then $\mu_j + j - 1 > \lambda_i + i - 1$ for $i < j$. This means that $h_{\lambda_i - \mu_j + j - i} = 0$ for $i < j$. Thus the entries in the first $j-1$ columns are all zero below row j and hence any nonzero terms in the determinant arise from choosing entries in the first $j-1$ columns from the first $j-1$ rows. It follows that when we expand about the first row, only those terms corresponding to special rim hooks completely contained in $F_{\lambda/\mu}$ contribute anything to the determinant. As was the case with the expansion of (2.1), the sign of the rim hook is the sign associated to the corresponding term in the expansion and the minor is the determinant corresponding to $S_{\alpha/\beta}$ where α/β is the skew shape which results from $F_{\lambda/\mu}$ by stripping off cells corresponding to the special rim hook.

Example

$$S_{(2,2,3,4,5)/(2,2,3)} = \det \begin{bmatrix} h_2 & h_3 & h_5 & h_7 & h_9 \\ h_1 & h_2 & h_4 & h_6 & h_8 \\ 0 & 0 & h_1 & h_3 & h_5 \\ 0 & 0 & 1 & h_2 & h_4 \\ 0 & 0 & 0 & 1 & h_2 \end{bmatrix}$$

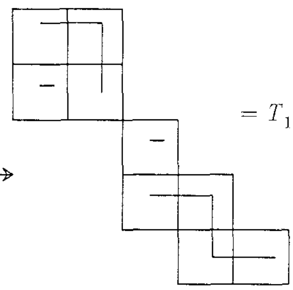
$$= h_2 \det \begin{bmatrix} h_2 & h_4 & h_6 & h_8 \\ 0 & h_1 & h_3 & h_5 \\ 0 & 1 & h_2 & h_4 \\ 0 & 0 & h_1 & h_2 \end{bmatrix} - h_3 \det \begin{bmatrix} h_1 & h_4 & h_6 & h_8 \\ 0 & h_1 & h_3 & h_5 \\ 0 & 1 & h_2 & h_4 \\ 0 & 0 & 1 & h_2 \end{bmatrix}$$

$$= h_2\, S_{(2,3,4,5)/(2,2,3)} - h_3\, S_{(1,3,4,5)/(2,2,3)}$$

FIGURE 4.3.

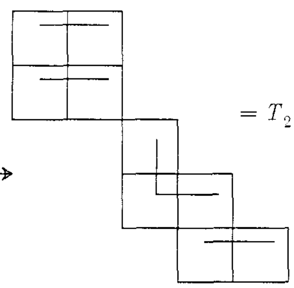
We can continue to expand each of the minors about the first row recursively and show just as in Section 1 that each nonzero term in the determinant, $\mathrm{sign}(\sigma)h_{\lambda_{\sigma_1}-\mu_1+\sigma_1-1}\cdots h_{\lambda_{\sigma_k}-\mu_k+\sigma_k-k}$ corresponds to a special rim hook tabloid T of shape λ/μ and that $\mathrm{sign}(T)=\mathrm{sign}(\sigma)$. Moreover in T if we let H_i be the special rim hook that starts in the ith row from the top of the skew diagram, then $|H_i| = \lambda_{\sigma_i}-\mu_i+\sigma_i-i$. We give two examples of this correspondence below:

$$\begin{bmatrix} h_2 & h_3 & h_5 & h_7 & h_9 \\ h_0 & h_2 & h_4 & h_6 & h_8 \\ 0 & 0 & h_1 & h_3 & h_5 \\ 0 & 0 & 1 & h_2 & h_4 \\ 0 & 0 & 0 & 1 & h_2 \end{bmatrix} \longrightarrow$$



$= T_1$

$$\begin{bmatrix} h_2 & h_3 & h_5 & h_7 & h_9 \\ h & h_2 & h_4 & h_6 & h_8 \\ 0 & 0 & h_1 & h_3 & h_5 \\ 0 & 0 & 1 & h_2 & h_4 \\ 0 & 0 & 0 & 1 & h_2 \end{bmatrix} \longrightarrow$$



$= T_2$

FIGURE 4.4.

Finally, when we take a term like $h_3h_1h_1h_4h_0$ which corresponds to the special rim hook tabloid T_1 above and expand it in terms of monomials, then each monomial in the expansion corresponds to a product of weights of column strict tableaux

$$W(\boxed{i_1 \mid i_2 \mid i_3})\ W(\boxed{j_1})\ W(\boxed{k_1})\ W(\boxed{l_1 \mid l_2 \mid l_3 \mid l_4}) = x_{i_1}x_{i_2}x_{i_3}x_{j_1}x_{k_1}x_{l_1}x_{l_2}x_{l_3}$$

FIGURE 4.5.

whose shapes are single rows corresponding to the lengths of the special rim hooks of T_1. We can then insert the entries of each column strict tableau into the cells occupied by the corresponding rim hook starting at the North-West corner. In this way, we associate to each signed monomial arising from the determinant in (4.1), a filled special rim hook tabloid. For example, in our specific case, we would produce the following f.s.r.h.-tabloid

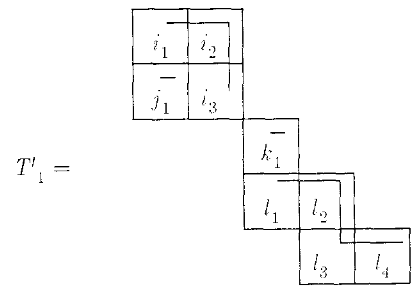


FIGURE 4.6.

The weight of such a f.s.r.h.-tabloid T'_1 is just its weight when viewed as a tabloid, and the sign associated to T'_1 is its sign when viewed as a rim hook tabloid. This given, we let $\mathbf{X}_{\lambda/\mu}$ denote the space of all f.s.r.h.-tabloids T of shape λ/μ. We have shown that

$$\det\|h_{\lambda_j-\mu_i+j-i}\| = \sum_{T\in\mathbf{X}_{\lambda/\mu}} \mathrm{sign}(T)W(T). \tag{4.3}$$

To prove the generalized Jacobi–Trudi identity, it suffices to prove that

$$S_{\lambda/\mu} = \sum_{T\in\mathbf{X}_{\lambda/\mu}} \mathrm{sign}(T)W(T). \tag{4.4}$$

To prove (4.4), we shall construct a weight preserving and sign reversing involution θ on $\mathbf{X}=\mathbf{X}_{\lambda/\mu}$ whose fixed points will consist precisely of all column strict tableaux of shape λ/μ.

Note that since we filled each rim hook with a weakly increasing sequence starting at the extreme North-West cell, it follows that the only way a f.s.r.h.-tabloid T in $\mathbf{X}$ could be a column strict tableau is if all the rim hooks are horizontal and the entries in each column are strictly increasing from bottom to top.

For the construction of θ, suppose we are given a $T\in\mathbf{X}$ which is not column strict. Consider the leftmost and then the topmost violation of column strictness in T. We claim that this violation is necessarily of the form

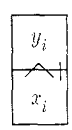


FIGURE 4.7.

For if the leftmost and then the topmost violation involves two adjacent entries x_i and y_i in a row, then $x_i > y_i$ and by our coding these two entries necessarily belong to two distinct special rim hooks H_{r+1} and H_r, respectively. Note that in this case H_{r+1} must be *horizontal*, i.e. all of its cells lie in a single row.

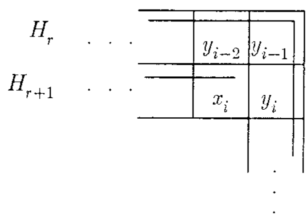


FIGURE 4.8.

But the choice of our violation means that $x_i < y_{i-2}$. Since $y_{i-2} \leqslant y_{i-1} \leqslant y_i$, this forces $x_i \leqslant y_i$, a contradiction.

Consider now the leftmost and the topmost violation of column strictness as in Figure 4.7 and assume that the top label y_i belongs to the special rim hook H_r.

From T, we construct T' as follows. First of all, the special rim hooks H_r and H_{r+1} are switched to obtain H'_r and H'_{r+1}. Note that the only extension of the switch described in Section 3 that is required for skew shapes arises when H_{r+1} starts in row $i+1$. The switch here takes the form of gluing together H_r and H_{r+1} to obtain H'_r. In pictures

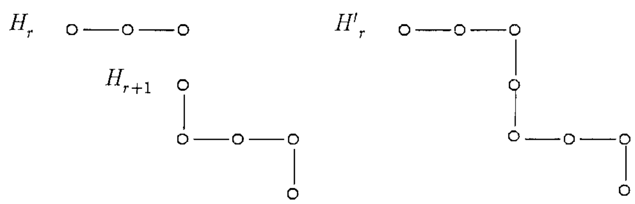


FIGURE 4.9.

Thus here, H'_r just results from H_r by adding the cells of H_{r+1} to those of H_r. Technically, we think of H'_{r+1} as the empty rim hook. Note that for $\mu = \varnothing$, this situation can arise only when H_r consists of a single cell.

After the switch, both H'_r and H'_{r+1} are relabeled by redistributing the labels in H_r and H_{r+1}. For the relabeling phase, there are two cases to consider:

(1) x_i lies on the Eastern outer rim of the skew shape $H_r \cup H_{r+1}$,
(2) x_i lies on the Southern outer rim of the skew shape $H_r \cup H_{r+1}$.

In Case (1), the involution does not change the old labels in $H_r \cup H_{r+1}$. Note that the two vertical cells in which the violation in question occurs are still on the Eastern outer rim of the skew shape $H_r \cup H_{r+1}$.

Example

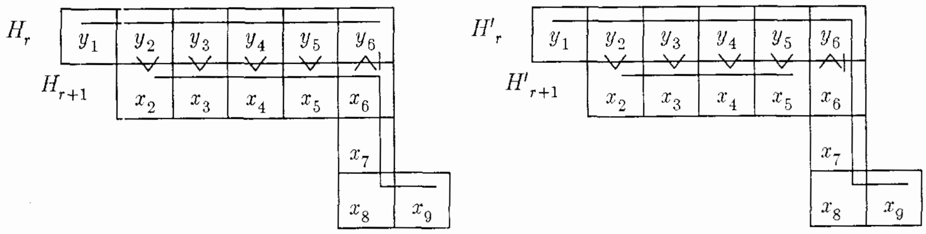


FIGURE 4.10.

In Case (2), assume that the labels in H_r from y_i on are $y_i \leqslant y_{i+1} \leqslant \cdots \leqslant y_r$, and the labels in H_{r+1} from x_i on are $x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_s$. After switching the pair of special rim hooks H_r, H_{r+1} to the pair H'_r, H'_{r+1}, the cells starting with y_i in the new special rim hook H'_r are given the labels

$$y_i \leqslant x_i \leqslant x_{i+1} \leqslant \cdots \leqslant x_s \tag{4.5}$$

and the cells starting with x_i in H'_{r+1} the labels

$$y_{i+1} \leqslant y_{i+2} \leqslant \cdots \leqslant y_r. \tag{4.6}$$

Example

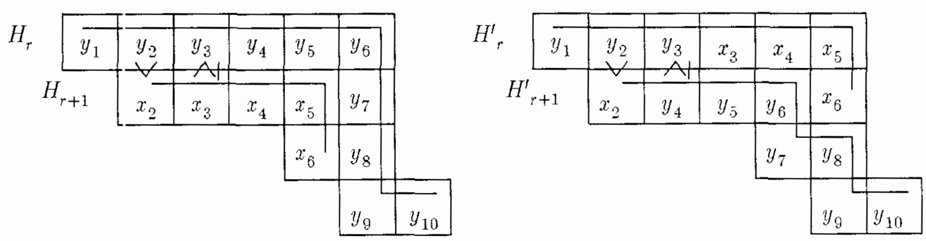


FIGURE 4.11.

Note that the location of the leftmost and then the topmost violation of row strictness is preserved and the mapping θ defined by this procedure is an involution.

If H_{r+1} happens to be void, then we are in a special subcase of Case (1) and the switch takes the special form

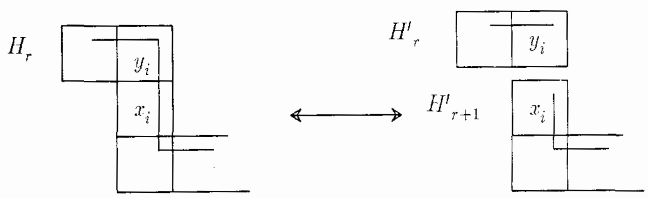


FIGURE 4.12.

without altering the labels.

As an example for this latter possibility, θ matches T and T' given below:

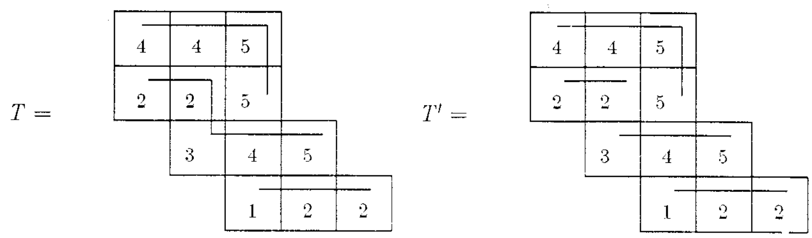


FIGURE 4.13.

Note that switching the rth and the $r+1$st special rim hooks in a f.s.r.h.-tabloid T changes the sign of T just as in Section 2. Thus $\mathrm{sign}(T') = -\mathrm{sign}(T)$. Clearly, $W(T') = W(T)$. Therefore the fixed points of θ correspond to f.s.r.h.-tabloids in which the entries are strictly increasing along the columns: i.e. column strict tableaux of shape λ/μ. ■

5. APPLICATIONS AND REMARKS

The involution θ constructed above preserves the sum of the entries in a given diagonal of a f.s.r.h.-tabloid T. Similar to the technique used in [5] to derive a multivariate generating function for reverse plane partitions from Giambelli's expansion formula, this may facilitate the closed form expansion of $S_{\lambda/\mu}(q, q^2, \ldots)$ via the determinant on the right-hand side of (4.1), at least for special classes of skew Schur functions.

In Section 2, we gave a combinatorial interpretation in terms of special rim hook tabloids of shape μ and type λ, for the coefficient of the homogeneous symmetric function h_λ in the expansion of the Schur function S_μ. Note that our argument in Section 4 shows that a similar combinatorial interpretation exists for the coefficient of h_λ in the expansion of $S_{\mu/\nu}$. In other words, if λ, μ, and ν are partitions such that $|\lambda| = |\mu/\nu|$, define

$$K^{-1}_{\lambda,\mu/\nu} = \sum_T \mathrm{sign}(T), \tag{5.1}$$

where the sum runs over all special rim hook tabloids of type λ and shape μ/ν. Then our analysis in Section 4 shows that

$$S_{\mu/\nu} = \sum_{\lambda \vdash |\mu/\nu|} K^{-1}_{\lambda,\mu/\nu} h_\lambda. \tag{5.2}$$

There is also a determinantal expression for the Schur functions in terms of the elementary symmetric functions e_λ. Here, if $\lambda = (0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k)$, then

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} \tag{5.3}$$

where $e_0 = 1$ and for each $r > 0$,

$$e_r = \sum_{0 < i_1 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Then if $\mu = (0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k)$ is such that $\mu \subseteq \lambda$ and we let v' denote the conjugate of the partition v, we have

$$S_{\lambda'/\mu'} = \det \| e_{\lambda_j - \mu_i + j - i} \| . \tag{5.4}$$

Obviously we can apply the same argument to the determinant on the right-hand side of (5.4) that we did to the determinant in (4.1) to conclude that

$$S_{\mu'/v'} = \sum_{\lambda \vdash |\mu'/v'|} K^{-1}_{\lambda, \mu/v} e_\lambda . \tag{5.5}$$

We end this section with an interesting application of (5.5). Let ω be a primitive pth root of unity. We wish to evaluate $S_{\mu/v}(1, \omega, \ldots, \omega^{p-1})$. Note that

$$\prod_{j=0}^{p-1} (1 - \omega^j x) = 1 - x^p = \sum_{r=0}^{p} (-x)^r e_r(1, \omega, \ldots, \omega^{p-1}). \tag{5.6}$$

Taking the coefficient of x^r on both sides of (5.6), we see that

$$e_r(1, \omega, \ldots, \omega^{p-1}) = \begin{cases} 1 & \text{if } r = 0, \\ (-1)^{p+1} & \text{if } r = p, \\ 0 & \text{if } r \notin \{0, p\}. \end{cases} \tag{5.7}$$

Thus for any partition λ, we see that

$$e_\lambda(1, \omega, \ldots, \omega^{p-1}) = 0 \tag{5.8}$$

if λ is not of the form (p^k), and

$$e_{(p^k)}(1, \omega, \ldots, \omega^{p-1}) = (-1)^{(p+1)k}. \tag{5.9}$$

By (5.5), (5.8), and (5.9), we obtain

$$S_{\mu/v}(1, \omega, \ldots, \omega^{p-1}) = \begin{cases} 0 & \text{if } |\mu/v| \neq p^k \text{ for some } k, \\ (-1)^{(p+1)k} K^{-1}_{(p^k), \mu'/v'} & \text{if } |\mu/v| = p^k. \end{cases} \tag{5.10}$$

We can state (5.10) in a somewhat more perspicuous manner. First we shall say that T is a *t-special rim hook tabloid* of shape μ/v and type λ if T results by reflecting a special rim hook tabloid T' about the line $y = x$ in the plane. We shall call T the *transpose* of T'. For example, if $\mu = (1, 3, 4, 5)$ and $\mu = (1, 3)$, then the special rim hook tabloid T' of shape μ'/v' in Figure 5.1

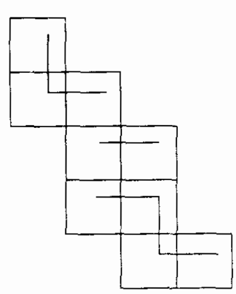
$T' =$ 

FIGURE 5.1.

gives rise to the t-special rim hook tabloid of shape μ/ν given in Figure 5.2.

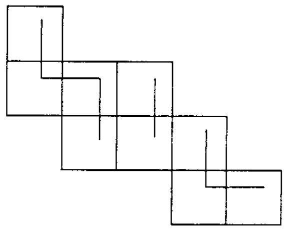
$T =$ 

FIGURE 5.2.

Note that t-special rim hook tabloids are just like special rim hook tabloids except that instead of filling the diagram μ/ν with rim hooks by recursion always starting from the North-West extreme cell, we fill the diagram μ/ν with rim hooks by recursion always starting from the South-East extreme cell. In particular, if $\nu = \varnothing$, t-special rim hook tabloids of shape $\mu = \mu/\varnothing$ have all their rim hooks starting in the first row, as opposed to special rim hook tabloids of shape μ which have all their rim hooks starting in the first column.

Now consider the term $(-1)^{(p+1)k} K^{-1}_{(p^k),\mu'/\nu'}$ which appears on the right-hand side of (5.10). When we reflect a rim hook H about the line $y = x$ to produce a rim hook H', it is easy to see that

$$\begin{aligned}\mathrm{ht}(H) &= \textit{number of rows of } H\\ &= \textit{number of columns of } H'\\ &= \textit{width of } H' = \mathrm{wd}(H').\end{aligned}$$

It is simple to prove by induction on p that if H is a rim hook of size p, then $\mathrm{ht}(H) + \mathrm{wd}(H) = p + 1$. Thus if H is a rim hook of size p and H' is its reflection as described above, then

$$\begin{aligned}(-1)^{p+1}\,\mathrm{sign}(H) &= (-1)^{\mathrm{ht}(H)+\mathrm{wd}(H)}(-1)^{\mathrm{ht}(H)-1}\\ &= (-1)^{\mathrm{ht}(H')+\mathrm{wd}(H')}(-1)^{\mathrm{ht}(H')-1}\\ &= (-1)^{\mathrm{ht}(H')-1}\\ &= \mathrm{sign}(H'). \end{aligned}\tag{5.11}$$

It follows from (5.11) that if T' is a special rim hook tabloid of shape μ'/ν' and type (p^k), and T is the transpose of T', then

$$(-1)^{(p+1)k}\,\mathrm{sign}(T') = \mathrm{sign}(T). \tag{5.12}$$

As in Corollary 5, it is not difficult to see that there is at most one special rim hook tabloid of type (p^k) for any shape. Thus there is at most one t-special rim hook tabloid of type (p^k). Therefore the following Theorem is a consequence of (5.10) and (5.12)

THEOREM 3

$$S_{\mu/\nu}(1, \omega, \ldots, \omega^{p-1}) = \begin{cases} \operatorname{sign}(T) & \text{if } |\mu/\nu| = p^k \text{ for some } k \text{ and } T \text{ is a } t\text{-special} \\ & \quad \text{rim hook tabloid of shape } \mu/\nu \text{ and type } (p^k), \\ 0 & \text{otherwise}. \end{cases}$$

We should also note that the special case of Theorem 3 when $\nu = \varnothing$ was proved by Chen [2] and was used by Chen, Garsia, and Remmel in [3] as a part of their algorithm to compute plethysms. Also, Macdonald in the forthcoming second edition of his book [11] provides an equivalent expression for $S_{\mu/\nu}(1, \omega, \ldots, \omega^{p-1})$ in terms of p-quotients and p-cores of μ and ν. The import of Theorem 3 is that it provides an extremely simple algorithm to compute $S_{\mu/\nu}(1, \omega, \ldots, \omega^{p-1})$. That is, we simply try to decompose the shape μ/ν into rim hooks of size p recursively by taking rim hooks which always start in the extreme South-East corner. If we are successful in decomposing the shape μ/ν in this manner, then $S_{\mu/\nu}(1, \omega, \ldots, \omega^{p-1})$ equals the sign of the resulting t-special rim hook tabloid. If we are not successful, then $S_{\mu/\nu}(1, \omega, \ldots, \omega^{p-1}) = 0$. For example, if $p = 5$ and $\mu = (1, 3, 4, 6, 6)$, $\nu = \varnothing$, then $S_\mu(1, \omega, \omega^2, \omega^3, \omega^4) = 0$ because as one can see in Figure 5.3, there is no t-special rim hook tabloid of shape μ and type (5^4).

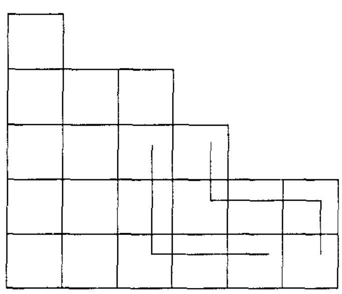


FIGURE 5.3.

On the other hand, if $\mu = (4, 5, 6, 7)$ and $\nu = (2, 2, 2)$, then $S_\mu(1, \omega, \omega^2, \omega^3, \omega^4) = 1$ since as one can see in Figure 5.4, there is a t-special rim hook tabloid T of shape μ/ν and type (5^3) with $\operatorname{sign}(T) = 1$.

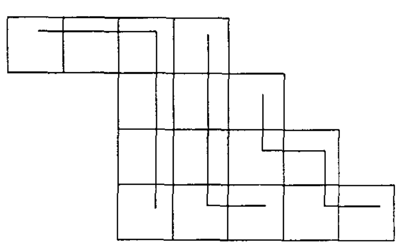


FIGURE 5.4.

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