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The Irregularity Polynomials of Fibonacci and Lucas cubes

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Abstract
Irregularity of a graph is an invariant measuring how much the graph differs from a regular graph. Albertson index is one measure of irregularity, defined as the sum of $|deg(u) - deg(v)|$ over all edges uv of the graph. Motivated by a recent result on the irregularity of Fibonacci cubes, we consider irregularity polynomials and determine them for Fibonacci and Lucas cubes. These are graph families that have been studied as alternatives for the classical hypercube topology for interconnection networks. The irregularity polynomials specialize to the Albertson index and also provide additional information about the higher moments of $|deg(u)-deg(v)|$ in these families of graphs.

Keywords Irregularity of graph · Fibonacci cube · Lucas cube

Mathematics Subject Classification 05C31 · 05C35 · 05A15 · 11B39

1 Introduction

Let $G = (V(G), E(G))$ be a simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, the degree of v is denoted by $deg_G(v)$ and defined as

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the number of edges incident with v. A graph G is regular if all its vertices have the same degree. Otherwise, it is called irregular. In the latter case, a problem of interest is the measure of how much the graph G differs from a regular graph (see, for example, [2,5]). For this purpose, several measures of graph irregularity have been proposed. These can be found in the recent papers [1,9]. A local measure of irregularity called imbalance $imb_G(e)$ of an edge $e = uv \in E(G)$ is defined as

$$imb_G(e) = |deg_G(u) - deg_G(v)|.$$

This quantity was transferred to a global irregularity measure by Albertson [3], who considered

$$irr(G) = \sum_{uv \in E(G)} |\deg_G(u) - \deg_G(v)|.$$

Using this definition, the irregularity of π-permutation graphs, Fibonacci cubes and trees are considered in [4]. Inspired by the results in [4], in this paper we consider a refinement of the irregularity of Fibonacci and Lucas cubes. We define the irregularity polynomial $I_G(x)$ of G by

$$I_G(x) = \sum_{uv \in E(G)} x^{|\deg_G(u) - \deg_G(v)|}.$$

With this definition $|E(G)| = I_G(1),\ irr(G) = I'_G(1)$, and the coefficient of x^r in $I_G(x)$ is the number of edges $e \in G$ with $imb_G(e) = r$. Using this polynomial, we extend the results for Fibonacci cubes given in [4] to Lucas cubes and additionally refine the enumeration results for both families of graphs.

The rest of the paper is organized as follows: In Sect. 2, we present preliminaries and the definition of Fibonacci cubes and Lucas cubes. In Sect. 3 and Sect. 4, we present the irregularity polynomials for Fibonacci cubes and Lucas cubes, respectively.

2 Preliminaries

Fibonacci numbers are defined by the recursion $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. Similarly, the Lucas numbers L_n are defined by $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$.

It is known that the n-dimensional hypercube Q_n is a regular graph in which each vertex has degree n. Vertices of Q_n can be represented using binary strings of length n. In [6], the n-dimensional Fibonacci cube Γ_n is defined as the subgraph of Q_n induced by the vertices that contain no consecutive 1s in their binary string representation. Let $B = \{0, 1\}$ and for all $n \geq 1$, B_n denote the set of all binary sequences of length n, that is,

$$B_n = \{b_1 b_2 \dots b_n \mid b_i \in B,\ 1 \leq i \leq n\}.$$

Then, we can write $\Gamma_n = (V(\Gamma_n), E(\Gamma_n))$ with

$$V(\Gamma_n) = \{b_1 b_2 \dots b_n \in B_n \mid b_i \cdot b_{i+1} = 0,\ 1 \le i \le n-1\},$$
$$E(\Gamma_n) = \{e = uv \mid u, v \in V(\Gamma_n) \text{ and } d_H(u, v) = 1\},$$

where d_H is the Hamming distance, that is, the number of different coordinates. In [8], the n-dimensional Lucas cube Λ_n is defined as the subgraph of Γ_n induced by the vertices that do not start and end with 1. Here, we can write,

$$V(\Lambda_n) = \{b_1 b_2 \dots b_n \in B_n \mid b_i \cdot b_{i+1} = 0,\ 1 \le i \le n-1 \text{ and } b_1 \cdot b_n = 0\}$$
$$E(\Lambda_n) = \{e = uv \mid u, v \in V(\Lambda_n) \text{ and } d_H(u, v) = 1\}.$$

Note that $|V(\Gamma_n)| = f_{n+2}$ and $|V(\Lambda_n)| = L_n$.

For any $n \ge 2$, Γ_n can be decomposed into two subgraphs induced by the vertices that start with 0 and 10, respectively. This is called the *fundamental decomposition* of Γ_n [7]. The vertices that start with 0 constitute a graph isomorphic to Γ_{n-1} and the vertices that start with 10 constitute a graph isomorphic to Γ_{n-2}. Additionally, there is a matching of cardinality $|V(\Gamma_{n-2})|$ between the subgraphs. We denote this decomposition symbolically as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}\ . \tag{1}$$

Here, the perfect matching is the set of edges between the vertices in $10\Gamma_{n-2}$ and the corresponding ones in $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$. These are also referred to as the *link* edges. Using (1), we can write

$$\begin{aligned} \Gamma_n &= 0\Gamma_{n-1} + 10\Gamma_{n-2} && (2)\\ &= (00\Gamma_{n-2} + 010\Gamma_{n-3}) + 10\Gamma_{n-2} && (3)\\ &= ((000\Gamma_{n-3} + 0010\Gamma_{n-4}) + 010\Gamma_{n-3}) + (100\Gamma_{n-3} + 1010\Gamma_{n-4}), && (4) \end{aligned}$$

where there are perfect matchings (see Fig. 1) between

- $10\Gamma_{n-2}$ and $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$ in (2),
- $10\Gamma_{n-2}$ and $00\Gamma_{n-2}$; $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$ in (3),
- $010\Gamma_{n-3}$ and $000\Gamma_{n-3}$; $100\Gamma_{n-3}$ and $000\Gamma_{n-3}$; $1010\Gamma_{n-4}$ and $0010\Gamma_{n-4}$; $0010\Gamma_{n-4}$ and $0000\Gamma_{n-4} \subset 000\Gamma_{n-3}$; $1010\Gamma_{n-4}$ and $1000\Gamma_{n-4} \subset 100\Gamma_{n-3}$ in (4).

Similar to the fundamental decomposition of Γ_n, Λ_n can be decomposed into two subgraphs induced by its vertices that start with 0 and 10, respectively. We will call this the fundamental decomposition of Λ_n even though the subgraphs in the decomposition are not Lucas cubes themselves. This decomposition in terms of Fibonacci cubes

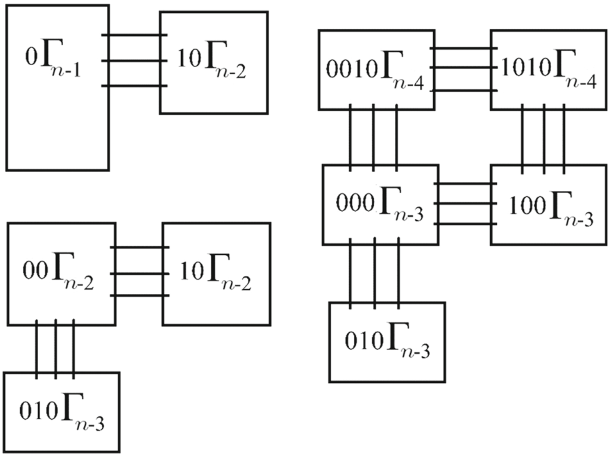

Fig. 1 Fundamental decomposition and the matchings of cardinality $|V(\Gamma_{n-i})|$, $i \in \{2, 3, 4\}$, between the subgraphs of Fibonacci cube Γ_n $(n \geq 4)$

together with the perfect matchings between the induced subgraphs of Λ_n is as follows:

$$\begin{aligned} \Lambda_n &= 0\Gamma_{n-1} + 10\Gamma_{n-3}0 && (5)\\ &= (00\Gamma_{n-2} + 010\Gamma_{n-3}) + 10\Gamma_{n-3}0 && (6)\\ &= (00\Gamma_{n-3}0 + 00\Gamma_{n-4}01) + (010\Gamma_{n-4}0 + 010\Gamma_{n-5}01) + 10\Gamma_{n-3}0\,. && (7)\end{aligned}$$

We can describe the perfect matchings shown in Fig. 2 depending on the decompositions of Λ_n as follows:

- In (5), there is a perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$.
- In (6), there are perfect matchings between $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$ and also $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subset 00\Gamma_{n-2}$.
- In (7), in addition to the perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0$, there are perfect matchings between $00\Gamma_{n-4}01$ and $00\Gamma_{n-4}00 \subset 00\Gamma_{n-3}0$; $010\Gamma_{n-5}01$ and $010\Gamma_{n-5}00 \subset 010\Gamma_{n-4}0$; $010\Gamma_{n-4}0$ and $000\Gamma_{n-4}0 \subset 00\Gamma_{n-3}0$; and also $010\Gamma_{n-5}01$ and $000\Gamma_{n-5}01 \subset 00\Gamma_{n-4}01$.

3 Irregularity Polynomial of Fibonacci Cubes

In this section, we obtain the irregularity polynomial of Γ_n. The idea is similar to the one used for the boundary enumerator polynomial of Γ_n obtained in [10].

Let $I_n(x) = I_{\Gamma_n}(x)$ denote the irregularity polynomial of Γ_n. We have the following result.

Theorem 1 *For any $n \geq 4$, the irregularity polynomial $I_n(x)$ of Γ_n is given by*

$$I_n(x) = 2I_{n-1}(x) + I_{n-2}(x) - 2I_{n-3}(x) - I_{n-4}(x) \tag{8}$$

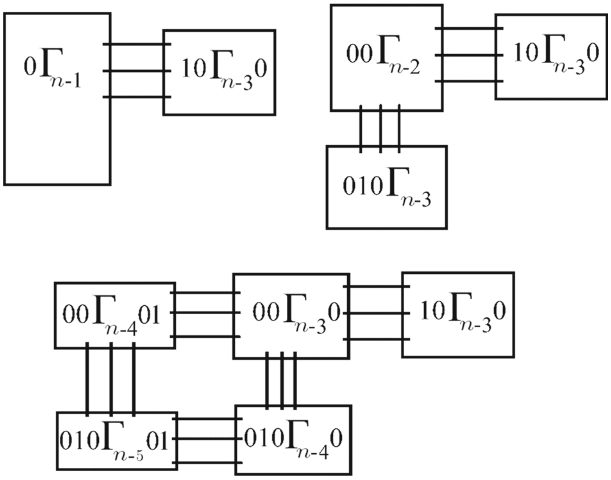

Fig. 2 Fundamental decomposition and the matchings of cardinality $|V(\Gamma_{n-i})|$, $i \in \{3, 4, 5\}$, between the subgraphs of the Lucas cube Λ_n ($n \geq 5$)

with $I_0(x) = 0$, $I_1(x) = 1$, $I_2(x) = 2x$ *and* $I_3(x) = x^2 + 2x + 2$.

Proof The values of $I_n(x)$ for $n < 4$ can be directly obtained from the definition of Γ_n. Now, assume that $n \geq 4$. Using the fundamental decomposition of Γ_n, we need to consider the following three cases:

1. Assume that $e \in 10\Gamma_{n-2}$. The irregularity polynomial of Γ_{n-2} is $I_{n-2}(x)$, and the degrees of vertices of all $e \in 10\Gamma_{n-2}$ increase by one in Γ_n. Consequently, there will be no change in the imbalance of such edges. Therefore, these edges contribute $I_{n-2}(x)$ to $I_n(x)$.
2. Assume that $e = uv \in \Gamma_n$ such that $u \in 10\Gamma_{n-2}$ and $v \in 0\Gamma_{n-1}$. (In particular, $v \in 00\Gamma_{n-2}$.) We know that there is a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, which means that the number of neighbors of u and v in $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ is the same. The only difference for the degrees of such vertices happens if there exists a neighbor of v in $010\Gamma_{n-3}$ due to the perfect matching between $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$. In total, we have f_{n-1} edges each of which contributes x to $I_n(x)$ for a total of $f_{n-1}x$, and there are $f_n - f_{n-1} = f_{n-2}$ edges each of which contributes x^0 to $I_n(x)$, for a total contribution of $f_{n-2}x^0$. Therefore, these edges together contribute $f_{n-1}x + f_{n-2}$ to $I_n(x)$.
3. Assume that $e \in 0\Gamma_{n-1}$. Since $0\Gamma_{n-1} = 00\Gamma_{n-2} + 010\Gamma_{n-3}$, we have three subcases to consider here.

   (a) Assume that $e \in 010\Gamma_{n-3}$. The degrees of vertices of all these edges increase by one in Γ_n, and therefore, they contribute $I_{n-3}(x)$ to $I_n(x)$.

   (b) Assume that $e = uv \in 0\Gamma_{n-1}$ such that $u \in 010\Gamma_{n-3}$ and $v \in 00\Gamma_{n-2}$. As in case 2 above, the contribution of these edges to $I_{n-1}(x)$ is $f_{n-2}x + f_{n-3}$. Since there is a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, the degrees of all such vertices v must increase by 1 in Γ_n. Therefore, the total contribution of these edges to $I_n(x)$ is $x \cdot (f_{n-2}x + f_{n-3}) = f_{n-2}x^2 + f_{n-3}x$.

(c) Assume that $e \in 00\Gamma_{n-2}$. These edges are the ones of $0\Gamma_{n-1}$ that are not in $010\Gamma_{n-3}$ and that are not created during the connection of $00\Gamma_{n-2}$ and $010\Gamma_{n-3}$. Furthermore, the degree of the vertices of each of these edges increases by 1 due to the perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, which does not change the contribution of these edges to $I_n(x)$. Therefore, their contribution to $I_n(x)$ is $I_{n-1}(x) - I_{n-3}(x) - (f_{n-2}x + f_{n-3})$.

For any $n \geq 4$ summing up all the above contributions, we obtain that

$$I_n(x) = I_{n-1}(x) + I_{n-2}(x) + f_{n-2}x^2 + 2f_{n-3}x + f_{n-4}\,. \tag{9}$$

To eliminate the terms which involve Fibonacci numbers in (9), we can write

$$I_{n+1}(x) = I_n(x) + I_{n-1}(x) + f_{n-1}x^2 + 2f_{n-2}x + f_{n-3} \tag{10}$$

and

$$I_{n+2}(x) = I_{n+1}(x) + I_n(x) + f_n x^2 + 2f_{n-1}x + f_{n-2}\,. \tag{11}$$

Then, by adding (9) and (10) and using the recursion of Fibonacci numbers, we have

$$I_{n+1}(x) = 2I_{n-1}(x) + I_{n-2}(x) + f_n x^2 + 2f_{n-1}x + f_{n-2}\,. \tag{12}$$

Then, using (11) and (12), we can write

$$\begin{aligned} f_n x^2 + 2f_{n-1}x + f_{n-2} &= I_{n+2}(x) - I_{n+1}(x) - I_n(x) \\ &= I_{n+1}(x) - 2I_{n-1}(x) - I_{n-2}(x) \end{aligned}$$

which gives the desired result. □

Let E_n denote the number of edges in $E(\Gamma_n)$. We know that $E_0 = 0$, $E_1 = 1$ and for any $n \geq 2$, it is shown in [8] that

$$E_n = \tfrac{1}{5}\big(nf_{n+1} + 2(n+1)f_n\big) \tag{13}$$

with generating function

$$\sum_{n\geq 0} E_n y^n = \frac{y}{\left(1 - y - y^2\right)^2}\,. \tag{14}$$

Define the generating function of the sequence $\{I_n(x)\}_{n\geq 0}$ of irregularity polynomials $I_n(x)$ of Γ_n by setting

$$I(x, y) = \sum_{n\geq 0} I_n(x)y^n = y + 2xy^2 + (x^2 + 2x + 2)y^3 + \cdots$$

Using Theorem 1, we obtain a closed form for $I(x, y)$ and consequently for the polynomials $I_n(x)$ themselves. We then use the relationship between this generating function and the generating function of the number of edges of Γ_n to obtain further results.

Corollary 1 *The generating function of the sequence* $\{I_n(x)\}_{n\geq 0}$ *of the irregularity polynomials* $I_n(x)$ *of* Γ_n *is given by*

$$I(x, y) = \sum_{n\geq 0} I_n(x)y^n = \frac{y\big(1 + (x - 1)y\big)^2}{\big(1 - y - y^2\big)^2} . \tag{15}$$

Proof We multiply identity (8) of Theorem 1 by y^n and sum for all $n \geq 4$, that is,

$$\sum_{n\geq 4} I_n(x)y^n = \sum_{n\geq 4} \Big(2I_{n-1}(x) + I_{n-2}(x) - 2I_{n-3}(x) - I_{n-4}(x)\Big)y^n \tag{16}$$

Using (16) and the definition of $I(x, y)$, we can write

$$\begin{aligned} I(x, y) - \sum_{n=0}^{3} I_n(x)y^n = 2y &\left(I(x, y) - \sum_{n=1}^{3} I_{n-1}(x)y^{n-1}\right) \\ &+y^2\left(I(x, y) - \sum_{n=2}^{3} I_{n-2}(x)y^{n-2}\right) \\ &-2y^3\Big(I(x, y) - I_0(x)\Big) - y^4 I(x, y) . \end{aligned}$$

Then, using the first few polynomials $I_0(x) = 0$, $I_1(x) = 1$, $I_2(x) = 2x$ and $I_3(x) = x^2 + 2x + 2$ as given in Theorem 1 and with some algebra, we obtain an identity satisfied by $I(x, y)$ given in (15). □

Corollary 2 *The irregularity polynomial of Fibonacci cube* Γ_n *is given by*

$$\begin{aligned} I_n(x) &= E_n + 2E_{n-1}(x - 1) + E_{n-2}(x - 1)^2 && (17) \\ &= (E_n - 2E_{n-1} + E_{n-2}) + (2E_{n-1} - 2E_{n-2})\, x + E_{n-2}x^2 && (18) \end{aligned}$$

for any $n \geq 2$ *where* E_n *is the number of edges of* Γ_n *as given in (13).*

Proof Combining the generating functions given in (14) and (15), we can write

$$\begin{aligned} I(x, y) = \sum_{n\geq 0} I_n(x)y^n &= \frac{y}{\big(1 - y - y^2\big)^2}\big(1 + (x - 1)y\big)^2 \\ &= \left(\sum_{n\geq 0} E_n y^n\right)\Big(1 + 2(x - 1)y + (x - 1)^2y^2\Big) \end{aligned}$$

and equate the coefficient of y^n on both sides. □

In [4], it is shown that the irregularity of Γ_n is

$$irr(\Gamma_n) = \tfrac{2}{5}\big((n-1)f_n + 2nf_{n-1}\big) = 2E_{n-1}\ .$$

Since $irr(\Gamma_n)$ is obtained by taking the derivative of $I_n(x)$ and then substitution $x = 1$, the first expression (17) immediately gives

$$irr(\Gamma_n) = 2E_{n-1}\ .$$

The second expression (18) provides formulas for the number of edges $uv \in E(\Gamma_n)$ for which $\left|\deg_{\Gamma_n}(u) - \deg_{\Gamma_n}(v)\right| = r$. Denoting this number by $\delta_r(\Gamma_n)$ for $n \geq 2$, we have $\delta_r(\Gamma_n) = 0$ for $r > 2$ with

$$\begin{aligned}\delta_0(\Gamma_n) &= E_n - 2E_{n-1} + E_{n-2} = \tfrac{1}{5}\,(nL_{n-3} + 2f_n)\\ \delta_1(\Gamma_n) &= 2E_{n-1} - 2E_{n-2} = \tfrac{2}{5}\,(nL_{n-2} + f_n)\\ \delta_2(\Gamma_n) &= E_{n-2} = \tfrac{1}{5}\,(nL_{n-1} - 2f_n)\ ,\end{aligned} \tag{19}$$

in terms of Fibonacci and Lucas numbers, where we have used (13) and the classical identity $L_n = f_{n+1} + f_{n-1}$. These in turn give the higher moments of $\left|\deg_{\Gamma_n}(u) - \deg_{\Gamma_n}(v)\right|$ over $uv \in E(\Gamma_n)$ as

$$\begin{aligned}\sum_{uv\in E(\Gamma_n)} |\deg_{\Gamma_n}(u) - \deg_{\Gamma_n}(v)|^m &= \delta_1(\Gamma_n) + 2^m\delta_2(\Gamma_n)\\ &= 2E_{n-1} + (2^m - 2)E_{n-2}\ .\end{aligned} \tag{20}$$

So of course, the $m = 1$ case of (20) gives $irr(\Gamma_n) = 2E_{n-1}$ as obtained in [4]. Calculating from (20) and (13), we find that the second moment is given by

$$\sum_{uv\in E(\Gamma_n)} |\deg_{\Gamma_n}(u) - \deg_{\Gamma_n}(v)|^2 = 2E_n - 2f_n\ .$$

4 Irregularity Polynomial of Lucas Cubes

Using the fundamental decomposition of Lucas cubes in terms of Fibonacci cubes and considering the cases similar to the ones that arise in the proof of Theorem 1, we obtain the following result for the irregularity polynomial of Lucas cubes.

Let $J_n(x) = I_{\Lambda_n}(x)$ denote the irregularity polynomial of Λ_n. Then

Theorem 2 *For any $n \geq 4$, the irregularity polynomial $J_n(x)$ of Λ_n is given by*

$$J_{n+2}(x) - J_{n+1}(x) - J_n(x) = I_{n+1}(x) - I_n(x) - I_{n-2}(x) - I_{n-3}(x) \tag{21}$$

where $I_n(x)$ is the irregularity polynomial of Γ_n, and $J_1(x) = 0$, $J_2(x) = 2x$, $J_3(x) = 3x^2$, $J_4(x) = 4x^2 + 4$, $J_5(x) = 5x^2 + 10x$ and $J_6(x) = 12x^2 + 12x + 6$.

Proof The proof follows along the lines of the argument given for Fibonacci cubes, but there are more cases to consider because of the peculiarities of Lucas cubes. We include the proof for completeness.

The polynomials $J_n(x)$ for $n \leq 6$ are found by direct inspection. For all $n \geq 7$, we use the fundamental decomposition (5) and consider the following three cases:

1. Assume that $e \in 10\Gamma_{n-3}0$. We know that the irregularity polynomial of Γ_{n-3} is $I_{n-3}(x)$ and the degrees of vertices of all $e \in 10\Gamma_{n-3}0$ increase by one in Λ_n since there is a perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$. Consequently, there will be no change in the imbalance of such edges of Λ_n. Therefore, all of these edges contribute $I_{n-3}(x)$ to $J_n(x)$.
2. Assume that $e = uv \in \Lambda_n$ with $u \in 10\Gamma_{n-3}0$ and $v \in 00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$. The number of neighbors of u and v in $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0$ is the same. The difference in the degrees of such vertices is possible if there exists any neighbor of v in $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$ (where "\" denotes set difference) due to the perfect matchings in (7). Here, we have the following subcases:

   (a) If $v \in 00\Gamma_{n-4}00 \cap 000\Gamma_{n-4}0 = 000\Gamma_{n-5}00$, then there are two neighbors of v in $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$, one in $000\Gamma_{n-5}01 \subset 00\Gamma_{n-4}01$ and one in $010\Gamma_{n-5}00 \subset 010\Gamma_{n-4}0$. The total contribution of this case to $J_n(x)$ is $f_{n-3}x^2$.
   (b) If $v \in 00\Gamma_{n-4}00 \setminus 000\Gamma_{n-4}0 = 0010\Gamma_{n-6}00$, then there is exactly one neighbor of v in $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$, which is the one in $0010\Gamma_{n-6}01 \subset 00\Gamma_{n-4}01$. The contribution of this case to $J_n(x)$ is $f_{n-4}x$.
   (c) If $v \in 000\Gamma_{n-4}0 \setminus 00\Gamma_{n-4}00 = 000\Gamma_{n-6}010$, then there is exactly one neighbor of v in $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$, which is the one in $010\Gamma_{n-6}010 \subset 010\Gamma_{n-4}0$. The contribution of this case to $J_n(x)$ is $f_{n-4}x$.
   (d) If $v \in 00\Gamma_{n-3}0 \setminus (00\Gamma_{n-4}00 \cup 000\Gamma_{n-4}0)$, then there is no neighbor of v in $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$. Then, the contribution of this case to $J_n(x)$ is $f_{n-1} - 2f_{n-2} + f_{n-3} = f_{n-5}$.

3. Assume that $e \in 0\Gamma_{n-1}$. In view of (6), we have three subcases.

   (a) Assume that $e \in 010\Gamma_{n-3}$. Considering all such edges, we see that the degrees of vertices of all these edges increase by one due to the perfect matching between $010\Gamma_{n-3}$ and $000\Gamma_{n-3}$ in Λ_n, and therefore, the total contribution of such edges to $J_n(x)$ is $I_{n-3}(x)$.
   (b) Assume that $e = uv \in 0\Gamma_{n-1}$ such that $u \in 010\Gamma_{n-3}$ and $v \in 00\Gamma_{n-2}$. These edges are the ones in the perfect matching between $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$. We have the following further subcases:
      i. If $u \in 010\Gamma_{n-4}0 \subset 010\Gamma_{n-3}$ and $v \in 000\Gamma_{n-4}0 \subset 000\Gamma_{n-3}$, then the number of neighbors of u and v in $010\Gamma_{n-3}$ and $000\Gamma_{n-3}$ is the same, respectively. But in Λ_n, if $v \in 0000\Gamma_{n-5}0$, then $deg(v) - deg(u) = 2$, since v has exactly one neighbor in $1000\Gamma_{n-5}0 \subset 10\Gamma_{n-3}0$ and one in $0010\Gamma_{n-5}0 \subset 0010\Gamma_{n-4}$, and if $v \in 00010\Gamma_{n-6}0$, then $deg(v) - deg(u) = 1$, since v has one neighbor in $10010\Gamma_{n-6}0 \subset 10\Gamma_{n-3}0$. The contribution of this case to $J_n(x)$ is $f_{n-3}x^2 + f_{n-4}x$.
      ii. If $u \in 010\Gamma_{n-5}01 \subset 010\Gamma_{n-3}$ and $v \in 000\Gamma_{n-5}01 \subset 000\Gamma_{n-3}$, then in Λ_n, we have $deg(v) - deg(u) \in \{0, 1\}$ and the only difference comes if v

has neighbor in $0010\Gamma_{n-6}01 \subset 0010\Gamma_{n-4}$. The contribution of this case to $J_n(x)$ is $f_{n-4}x + f_{n-5}$.

(c) Assume that $e \in 00\Gamma_{n-2}$. These edges are the ones of $0\Gamma_{n-1}$ that are not in $010\Gamma_{n-3}$ and that are not occur during the connection of $00\Gamma_{n-2}$ and $010\Gamma_{n-3}$. Using (7), we have the following subcases to consider:

i. If $e \in 00\Gamma_{n-4}01$, then considering all such edges of $0\Gamma_{n-1} \subset \Lambda_n$, assume that all of them contribute X to $I_{n-1}(x)$ of $0\Gamma_{n-1}$ and $J_n(x)$ since there is no edge between $00\Gamma_{n-4}01$ and $10\Gamma_{n-3}0$ in Λ_n.

ii. Assume that $e = uv \in 00\Gamma_{n-2}$ such that $u \in 00\Gamma_{n-4}01$ and $v \in 00\Gamma_{n-3}0$. These edges are the ones in the perfect matching between $00\Gamma_{n-4}01$ and $00\Gamma_{n-4}00 \subset 00\Gamma_{n-3}0 \subset 00\Gamma_{n-2}$. We have the following subcases:

– If $u \in 0010\Gamma_{n-6}01$ and $v \in 0010\Gamma_{n-6}00$, then in Λ_n, we have $deg(v) - deg(u) \in \{1, 2\}$ since all of such v has exactly one neighbor $1010\Gamma_{n-6}00 \subset 10\Gamma_{n-3}0$ and f_{n-5} of them has another neighbor in $00000\Gamma_{n-7}00$. The contribution of this case to $J_n(x)$ is $f_{n-5}x^2 + f_{n-6}x$.

– If $u \in 000\Gamma_{n-5}01$ and $v \in 000\Gamma_{n-5}00$, then in Λ_n, u has exactly one neighbor in $010\Gamma_{n-5}01 \subset 010\Gamma_{n-3}$ and v has exactly one neighbor in $010\Gamma_{n-5}00 \subset 010\Gamma_{n-3}$. Furthermore, each v has one more neighbor in $100\Gamma_{n-5}00 \subset 10\Gamma_{n-3}0$ and also, f_{n-4} of them has a neighbor in $000\Gamma_{n-6}010 \subset 00\Gamma_{n-5}010 \subset 00\Gamma_{n-4}00$. Therefore, the contribution of this case to $J_n(x)$ is $f_{n-4}x^2 + (f_{n-3} - f_{n-4})x = f_{n-4}x^2 + f_{n-5}x$.

iii. If $e \in 00\Gamma_{n-3}0$, then considering all such edges, we see that their contribution to $I_{n-1}(x)$ of $0\Gamma_{n-1}$ and contribution to $J_n(x)$ are the same since there is a perfect matching between $00\Gamma_{n-3}0$ and $10\Gamma_{n-3}0$. This contribution can be evaluated by subtracting the contribution of the edges of $0\Gamma_{n-1} \setminus 00\Gamma_{n-3}0$ from $I_{n-1}(x)$. Note that, since we need to consider the edges of $0\Gamma_{n-1}$, we have to discard the effect of the perfect matching between $00\Gamma_{n-3}0$ and $10\Gamma_{n-3}0$ in the above calculations in this case. Therefore, using the above subcases in the 3rd case, all of $e \in 00\Gamma_{n-3}0$ contribute $I_{n-1}(x) - I_{n-3}(x) - X - (f_{n-3}x + f_{n-4}) - (f_{n-4}x + f_{n-5}) - (f_{n-5}x + f_{n-6}) - (f_{n-4}x + f_{n-5})$ to $J_n(x)$.

Summing up all of the above contributions, we obtain that

$$J_n(x) = I_{n-1}(x) + I_{n-3}(x) + 3f_{n-3}x^2 + 2f_{n-6}x - L_{n-5}\,. \tag{22}$$

By using the recursion of Fibonacci and Lucas numbers, (22) and the values of $I_n(x)$ and $J_n(x)$ for $n \leq 6$, we obtain the desired result similar to the last part of the proof of Theorem 1. □

Define the generating function of the sequence $\{J_n(x)\}_{n\geq 0}$ of the irregularity polynomials $J_n(x)$ of Λ_n by

$$J(x, y) = \sum_{n\geq 1} J_n(x)y^n = 2xy^2 + 3x^2y^3 + (4x^2 + 4)y^4 + \cdots$$

Corollary 3 *The generating function $J(x, y)$ of the sequence $\{J_n(x)\}_{n\geq 0}$ of the irregularity polynomials $J_n(x)$ of Λ_n is given by*

$$J(x, y) = \frac{y\Big(c_1(x)y + c_2(x)y^2 + c_3(x)y^3 + c_4(x)y^4 + c_5(x)y^5 + c_6(x)y^6\Big)}{\big(1 - y - y^2\big)^2} \tag{23}$$

where

$$\begin{aligned}
c_1(x) &= 2x \\
c_2(x) &= x(3x - 4) \\
c_3(x) &= -2(x - 1)(x + 2) \\
c_4(x) &= -2(x - 1)(3x - 4) \\
c_5(x) &= 2(x - 1)(2x - 1) \\
c_6(x) &= 3(x - 1)^2.
\end{aligned}$$

Proof We multiply identity (21) of Theorem 2 by y^{n+2} and sum for $n \geq 4$. Using the expression for the generating function $I(x, y)$ already obtained in Corollary 1 and the first few polynomials as given in Theorem 2, we obtain an identity satisfied by $J(x, y)$ which is then solved and simplified to obtain the expression in (23). We omit the details. □

Using the generating function in (14) and multiplying out the right-hand side of (23), we immediately obtain for $n \geq 6$,

$$\begin{aligned}
J_n(x) = {} & E_{n-1}c_1(x) + E_{n-2}c_2(x) + E_{n-3}c_3(x) \\
& + E_{n-4}c_4(x) + E_{n-5}c_5(x) + E_{n-6}c_6(x)\ ,
\end{aligned} \tag{24}$$

where the polynomials $c_1(x), c_2(x), \ldots, c_6(x)$ are as defined in Corollary 3. Collecting powers of x in (24), we have

$$J_n(x) = \delta_0(\Lambda_n) + \delta_1(\Lambda_n)x + \delta_2(\Lambda_n)x^2 \tag{25}$$

where

$$\begin{aligned}
\delta_0(\Lambda_n) &= 4E_{n-3} - 8E_{n-4} + 2E_{n-5} + 3E_{n-6} \\
\delta_1(\Lambda_n) &= 2E_{n-1} - 4E_{n-2} - 2E_{n-3} + 14E_{n-4} - 6E_{n-5} - 6E_{n-6} \\
\delta_2(\Lambda_n) &= 3E_{n-2} - 2E_{n-3} - 6E_{n-4} + 4E_{n-5} + 3E_{n-6}\ ,
\end{aligned}$$

in terms of the number of edges E_n of Γ_n. We already know the explicit formula for E_n as given in (13). Simplifying the above expressions, we obtain

$$\begin{aligned}\delta_0(\Lambda_n) &= nf_{n-5}\\ \delta_1(\Lambda_n) &= 2nf_{n-4}\\ \delta_2(\Lambda_n) &= nf_{n-3}\end{aligned} \tag{26}$$

as the coefficients in (25).

We note that the coefficients above happen to give the correct irregularity polynomials for $n = 4, 5$ as well.

We can also expand $J_n(x)$ in powers of $x - 1$ and simplify the resulting coefficients. As a consequence, we have

Corollary 4 *The irregularity polynomial $J_n(x)$ of the Lucas cube Λ_n is given by*

$$J_n(x) = nf_{n-1} + 2nf_{n-2}(x-1) + nf_{n-3}(x-1)^2 \tag{27}$$

$$= nf_{n-5} + 2nf_{n-4}x + nf_{n-3}x^2 \tag{28}$$

for $n \geq 4$ with $J_1(x) = 0$, $J_2(x) = 2x$, and $J_3(x) = 3x^2$.

Note that the irregularity polynomial $J_n(x)$ of Lucas cube Λ_n is always a multiple of n and an integral polynomial.

Let $\ell_n = |E(\Lambda_n)|$ denote the number of edges of Λ_n. It can be shown [8] that $\ell_n = nf_{n-1}$. Since $irr(\Lambda_n)$ is obtained by taking the derivative of $J_n(x)$ and then substitution $x = 1$, the first expression (27) immediately gives the irregularity of Λ_n:

Corollary 5 *For all $n \geq 3$*

$$irr(\Lambda_n) = 2nf_{n-2} = 2\ell_{n-1} + 2f_{n-2}\ .$$

The second expression (28) provides formulas for the number of edges $uv \in E(\Lambda_n)$ for which $\left|\deg_{\Lambda_n}(u) - \deg_{\Lambda_n}(v)\right| = r$. These are the coefficients that appear in (26). The higher moments of $\left|\deg_{\Lambda_n}(u) - \deg_{\Lambda_n}(v)\right|$ over $uv \in E(\Lambda_n)$ are obtained as

$$\sum_{uv\in E(\Lambda_n)} \left|\deg_{\Lambda_n}(u) - \deg_{\Lambda_n}(v)\right|^m = \delta_1(\Lambda_n) + 2^m\delta_2(\Lambda_n) = 2nf_{n-4} + 2^m nf_{n-3} \tag{29}$$

Again, the $m = 1$ case of (29) gives $irr(\Lambda_n) = 2nf_{n-2}$. It is interesting that the second moment is given by

$$\sum_{uv\in E(\Lambda_n)} \left|\deg_{\Lambda_n}(u) - \deg_{\Lambda_n}(v)\right|^2 = 2nf_{n-4} + 4nf_{n-3} = 2\ell_n\ ,$$

which is the sum of the degrees of the vertices in Λ_n, the so-called handshaking lemma.

Remark 1 The pairwise asymptotic ratios of the quantities for Λ_n in (26) are identical to the asymptotic ratios of the corresponding ones for Γ_n in (19). In other words, the behavior of the zeros of $I_n(x)$ and $J_n(x)$ for large n is identical. Using the properties of the Fibonacci and the Lucas numbers, we find that asymptotically both polynomials have a double root at $x = -\phi^{-1} = \frac{1-\sqrt{5}}{2}$.

References

1. Abdo, H., Dimitrov, D., Gutman, I.: Graphs with maximal σ irregularity. Discrete Appl. Math. **250**, 57–64 (2018)
2. Alavi, Y., Chartrand, G., Chung, F.R.K., Erdős, P., Graham, R.L., Oellermann, O.R.: Highly irregular graphs. J. Graph Theory **11**, 235–249 (1987)
3. Albertson, M.O.: The irregularity of a graph. Ars Combin. **46**, 219–225 (1997)
4. Alizadeh, Y., Deutsch, E., Klavžar, S.: On the irregularity of π-permutation graphs, Fibonacci cubes, and trees. Bull. Malays. Math. Sci. Soc. (2020). https://doi.org/10.1007/s40840-020-00932-9
5. Chartrand, G., Erdős, P., Oellermann, O.R.: How to define an irregular graph. College Math. J. **19**, 36–42 (1988)
6. Hsu, W.-J.: Fibonacci cubes-a new interconnection technology. IEEE Trans. Parallel Distrib. Syst. **4**, 3–12 (1993)
7. Klavžar, S.: Structure of Fibonacci cubes: a survey. J. Comb. Optim. **25**, 505–522 (2013)
8. Munarini, E., Cippo, C.P., Zagaglia Salvi, N.: On the Lucas cubes. Fibonacci Quart. **39**, 12–21 (2001)
9. Réti, T.: On some properties of graph irregularity indices with a particular regard to the σ-index. Appl. Math. Comput. **344–345**, 107–115 (2019)
10. Saygı, E., Eğecioğlu, Ö.: Boundary enumerator polynomial of hypercubes in Fibonacci cubes. Discrete Appl. Math. **266**, 191–199 (2019)