International Journal of Foundations of Computer Science

THE ISOPERIMETRIC NUMBER OF d–DIMENSIONAL k–ARY ARRAYS

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Received (received date)
Revised (revised date)
Communicated by Editor's name

ABSTRACT

The d–dimensional k–ary array A_k^d is the d–fold Cartesian product graph of the path graph P_k with k vertices. We show that the (edge) isoperimetric number $i(A_k^d)$ of A_k^d is given by $i(A_k^d) = i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ and identify the cardinalities and the structure of the isoperimetric sets. For odd k, the cardinalities of isoperimetric sets in A_k^d are $\frac{1}{2}(k^d-1), \frac{1}{2}(k^d-k), \ldots, \frac{1}{2}(k^d-k^{d-1})$, whereas every isoperimetric set for k even has cardinality $\frac{1}{2}k^d$.

Keywords: Isoperimetric number, array, bisection, edge–separator, partition, extremal–set.

1. Introduction

Given a graph G and a subset X of its vertices, let ∂X denote the *edge–boundary* of X; the set of edges which connect vertices in X with vertices in $V(G) \setminus X$. The *edge–isoperimetric number*, or simply the *isoperimetric number*, of G is defined as

$$i(G) = \min_{1 \leq |X| \leq \frac{|V(G)|}{2}} \frac{|\partial X|}{|X|} . \tag{1}$$

That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized over all such partitions. As examples of isoperimetric numbers:

- $i(K_k) = \lceil \frac{k}{2} \rceil$ for the complete graph K_k with k vertices,
- $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$ for the path P_k with k vertices,
- $i(C_k) = 2/\lfloor \frac{k}{2} \rfloor$ for the cycle C_k with k vertices.

A subset X which achieves the minimum ratio in (1) is called an *isoperimetric set.* We refer the reader to Mohar [13] or Chung [8] for a discussion of basic results and various interesting properties of $i(G)$ and to Bezrukov [5] for a comprehensive survey of this and related problems.

The *d–dimensional k–ary array* A_k^d is an undirected graph with k^d nodes labeled by the integers from 0 to $k^d - 1$. Two nodes in A_k^d are connected by an edge if and only if the k–ary representations of their labels differ in exactly one digit and the absolute value of the difference in that digit is exactly one. Figure 1 illustrates a 2–dimensional 4–ary array A_4^2.

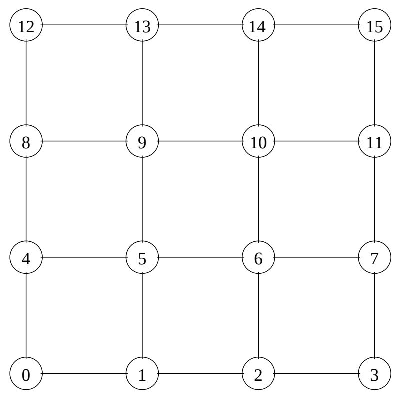

Fig. 1. The 2–dimensional 4–ary array A_4^2.

The Cartesian product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. A *multidimensional array* is the Cartesian product of paths of varying length, i.e. $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$. Thus A_k^d is a special type of multidimensional array in which $k_1 = k_2 = \cdots = k_d = k$.

In this paper, we investigate isoperimetric properties of d–dimensional k–ary arrays for arbitrary k and d. Specifically, we prove the following.

Theorem 1 *The isoperimetric number of the d–dimensional k–ary array A_k^d is given by*

$$i(A_k^d) = i(P_k) = \begin{cases} \frac{2}{k} & \text{if } k \text{ even,} \\ \frac{2}{k-1} & \text{if } k \text{ odd.} \end{cases} \tag{2}$$

As a byproduct of the proof, we also show that there are exactly d distinct cardinalities

$$\tfrac{1}{2}(k^d - 1),\ \tfrac{1}{2}(k^d - k),\ \ldots\ ,\ \tfrac{1}{2}(k^d - k^{d-1})$$

of isoperimetric sets in the d–dimensional k–ary array A_k^d when k is odd. In contrast, every isoperimetric set has cardinality $k^d/2$ when k is even. We give a description of the isoperimetric sets for both even and odd k.

1.1. Motivation

The notion of isoperimetric number of a graph G serves as a measure of connectivity of G as it quantifies the minimal interaction between a set of vertices X

and its complement $V(G) \setminus X$ in terms of the number of edges between them. This idea is also important in algorithm design. For instance, the notion of isoperimetric number is implicit in the divide-and-conquer strategy in graph algorithms. To illustrate, consider an algorithm which adopts divide-and-conquer strategy where the set of vertices of the underlying graph is split into two "fairly balanced" parts such that the algorithm can be run on the two corresponding subgraphs recursively, and the results are combined to obtain a solution for the original problem. The combining of results at the last step needs to be carried out with minimal effort if such a scheme is expected to be efficient. The idea is to split the graph in such a way as to keep the interaction between the two partitions (in terms of the number of edges in the boundary) as small as possible.

The isoperimetric number is closely related to the notion of *bisection width* $bw(G)$ of a graph G, which is the minimum number of edges that must be removed from the graph in order to split $V(G)$ into two *equal–sized* (within one) subsets. The isoperimetric number of a graph establishes a lower bound for its bisection width. For instance, one can give an alternate proof of the known lower bound

$$bw(A_k^d) \geq \begin{cases} k^{d-1} & \text{if } k \text{ even}, \\ \frac{k^d-1}{k-1} & \text{if } k \text{ odd}. \end{cases} \tag{3}$$

using the formula (2) for the isoperimetric number since,

$$\frac{bw(A_k^d)}{\lfloor \frac{k^d}{2} \rfloor} \geq i(A_k^d).$$

For even k, (3) was proved by Leighton [11] by an embedding method. Nakano [14] also used an embedding technique to prove the odd case in inequality (3). In this paper, we extend these two techniques to get tight edge–isoperimetric lower bounds which lead to the exact formula (2) for $i(A_k^d)$ for arbitrary k and d.

1.2. Outline

The outline of the rest of this paper is as follows: In Section 2 we summarize previous work on isoperimetric properties of various families of product graphs. The proof of our main result appears in Section 3. We treat the cases of even and odd k separately, and in each of these cases, we give tight upper and lower bounds that prove formula (2) for $i(A_k^d)$. In Section 4 we give the cardinalities of the feasible isoperimetric sets in A_k^d as well as describe their recursive construction. Section 5 concludes the paper with remarks and future considerations.

2. A Summary of Previous Work

There has been a significant amount of research in the area of isoperimetric bounds on various popular classes of graphs such as arrays and tori. The notion of isoperimetric number of a graph is related to the theory of *extremal sets* in graphs. An extremal set of a graph for a given m is, in a broad sense, a configuration of m vertices with

- minimum number of boundary edges, or
- maximum number of spanned edges

among all such m–vertex subsets of the given graph. The problem of finding extremal sets of the first (or, second) type is called *the minimum–boundary–edge problem* (or, *the maximum–induced–edge problem*). It can be shown that the minimum–boundary–edge and the maximum–induced–edge problems are equivalent for regular graphs [7].

The maximum–induced–edge problem for the hypercube (hence the minimum–boundary–edge problem, because of its regularity) was solved by Harper [10] and extended by Lindsey [12] to the d–dimensional k–ary clique which we shall define formally in the next section. In both instances, there is a nested structure of solutions, and the first m vertices in *lexicographical order* constitute an extremal set. The maximum–induced–edge problem for the d–dimensional k–ary array A_k^d was first solved by Bollobás and Leader [7]. Since A_k^d is not regular, this is not helpful in solving the minimum–boundary–edge problem. It was later extended to multidimensional arrays by Ahlswede and Bezrukov [1] who also gave a solution for $P_{k_1} \times P_{k_2}$ for the minimum–boundary–edge problem. The first nontrivial bounds on the minimum–boundary–edge problem for the d–dimensional k–ary arrays were given by Bollobás and Leader [7]. The bounds obtained are not tight enough to yield $i(A_k^d)$ exactly however. Similar problems have been defined in the literature for the vertex–boundary of a given configuration of vertices. For instance, Riordan [15] gave an ordering of vertices on the even discrete torus minimizing the number of vertices at shortest distance t from the vertices in the ordering. Wang and Wang [16] solved a similar problem for $P_\infty \times \cdots \times P_\infty$, i.e. the *$d$–dimensional infinite grid*, where the minimum is taken over all nonempty finite subsets of vertices. In their result, each P_∞ may be infinite in both directions or in one direction only. They also gave a simple ordering of the vertices in which the first m vertices constitute an extremal set minimizing the vertex–boundary.

A natural approach for the exact calculation of $i(A_k^d)$ is to try to exploit the recursive Cartesian product structure $A_k^d = P_k \times A_k^{d-1}$. For instance, Mohar [13] gave a proof for $i(P_{k_1} \times P_{k_2}) = \min\{i(P_{k_1}), i(P_{k_2})\}$. It can be proved in general that $i(P_k \times G) = i(P_k)$ where G is any connected graph with k vertices [2]. These results indicate that (2) holds for $d = 2$. However, it does not seem possible to extend them directly to get the desired bound for $i(P_k \times A_k^{d-1})$. It was shown in [2], for instance, that $i(P_k \times G) \neq i(P_k)$ if G has more than k vertices even when $i(G) \geq i(P_k)$ (it is well–known that $i(P_k \times G) < i(P_k)$ if $i(G) < i(P_k)$). General results on graph products based on the second smallest eigenvalue of the Laplacian [13], or the bound $\frac{1}{2}m \leq i(G_1 \times G_2 \times \cdots \times G_n) \leq m$ where $m = \min\{i(G_1), i(G_2), \cdots, i(G_n)\}$ reported by Chung and Tetali [9] do not yield the desired tight lower bound for $i(A_k^d)$, either.

3. The Isoperimetric Number of A_k^d

In this section we prove the two cases in Theorem 1 by showing that each of the two expressions on the right–hand side of equation (2) is an upper and a lower bound for $i(A_k^d)$.

3.1. The Upper Bound

To establish the upper bound, we utilize a general inequality given in [13] for the isoperimetric number of a Cartesian product graph

$$i(G \times H) \leq \min\{i(G), i(H)\}. \tag{4}$$

Thus, by taking $G = P_k$ and $H = A_k^{d-1}$ in (4), we have $i(A_k^d) \leq \min\{i(P_k), i(A_k^{d-1})\}$ which implies $i(A_k^d) \leq 1/\lfloor \frac{k}{2} \rfloor$ since $i(P_k) = 1/\lfloor \frac{k}{2} \rfloor$.

The proof of the lower bound will be handled in two cases depending on the parity of k. The case with even k is given next.

3.2. The Lower Bound for Even k

We first state our claim formally in the following proposition.

Proposition 1 *For the d–dimensional k–ary array A_k^d with k even, $i(A_k^d) \geq 2/k$.*

Proof. We prove the lower bound for even k by extending the embedding technique of Leighton [11] who used it to obtain a lower bound for $bw(A_k^d)$ for even k. Given a d–dimensional k–ary array A_k^d where k is even, we embed into A_k^d the k^d–node directed complete graph K_{k^d} with the vertex set $\{0, \ldots, k^d - 1\}$. Any two distinct vertices u and $v \in V(K_{k^d})$ are connected by the directed edges (u, v) and (v, u).

The edge from node $u = (u_1, u_2, \cdots, u_d)$ to node $v = (v_1, v_2, \cdots, v_d)$ of the complete graph is routed through the path

$$(u_1, u_2, \cdots, u_d) \rightarrow (v_1, u_2, \cdots, u_d) \rightarrow (v_1, v_2, u_3 \cdots, u_d) \rightarrow \cdots \rightarrow (v_1, v_2, \cdots, v_d)$$

in the array. That is, when routing the edge from u to v, following edges in A_k^d, we first "correct" the value of u along the dimension with the smallest index that is different from the value in v at that index until the two values become equal. Then, we correct the value of the next smallest index where they differ and so on. The process stops when all the dimensions have eventually been corrected.

When k is even, at most $k^{d+1}/2$ edges of the complete graph are routed through a given edge in the array (see [11], page 225). In other words, removal of an edge in the array is equivalent to disconnecting at most $k^{d+1}/2$ edges in the complete graph.

Now we prove that $|\partial X|/|X| \geq 2/k$ for any $X \subseteq V(A_k^d)$ where $1 \leq |X| \leq k^d/2$. Consider such a subset X. Also, let X' be the corresponding subset of vertices in $V(K_{k^d})$ under the embedding. Then, X' has boundary $\partial X'$ with $2|X'|(k^d - |X'|)$ edges in the complete graph since each vertex in X' is connected with every vertex in the remaining $k^d - |X'|$ vertices by two (directed) edges. That is, $|\partial X'| = 2|X'|(k^d - |X'|)$. We then have

$$|\partial X| \geq \frac{|\partial X'|}{\frac{k^{d+1}}{2}} \quad \Rightarrow \quad \frac{|\partial X|}{|X|} \geq \frac{|\partial X'|}{\frac{k^{d+1}}{2}|X|} = \frac{2|X'|(k^d - |X'|)}{\frac{k^{d+1}}{2}|X|}$$

But

$$\frac{2|X'|(k^d - |X'|)}{\frac{k^{d+1}}{2}|X|} \geq \frac{2}{k} \tag{5}$$

since $|X| = |X'|$ and $|X| \leq k^d/2$. □

3.3. The Lower Bound for Odd k

In this section, we prove the following proposition.

Proposition 2 *For the d–dimensional k–ary array A_k^d with $k > 1$ odd, $i(A_k^d) \geq 2/(k-1)$.*

Unfortunately, the technique used for even k does not yield a tight lower bound when k is odd. Specifically, with the same embedding method, one can at best get

$$i(A_k^d) \geq \frac{2(k^d+1)}{k^{d+1} - k^{d-1}}$$

for odd k. Note that the right hand side is a smaller than the desired lower bound $2/(k-1)$. Instead, we prove this case by extending the embedding technique of Nakano [14]. Before going into the proof, we shall first give a characterization of graphs based on *linear layouts* and develop some notation to facilitate our treatment. Our notation and terminology are similar to those used in [14].

3.4. Linear Layouts

A *linear layout* L of a graph G is a one-to-one mapping between the vertices in the graph and the numbers $\{0, \ldots, |V(G)|-1\}$, i.e., $L : V(G) \to \{0, 1, \ldots, |V(G)| - 1\}$ is a bijection. One can think of this mapping as assigning each node a position on the number line between 0 and $|V(G)|-1$ as shown in Figure 2. For a d–dimensional k–ary array A_k^d, the identity mapping I which assigns each vertex to the value of its own label is called the *label–order layout*. We refer to the region between two

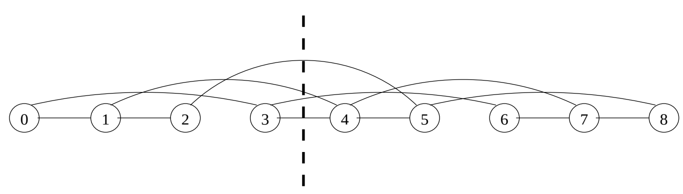

Fig. 2. The label–order layout of A_3^2 with a *cut* at *gap* 4.

nodes assigned to positions $x-1$ and x, $1 \leq x \leq |V(G)| - 1$ as *gap* x.

The *cut* of a graph G under a linear layout L at gap x, denoted by $C(G, L, x)$, is the set of edges which connect vertices assigned to positions smaller than x with those in positions greater than or equal to x. That is, $C(G, L, x) = \{(u, v) \in E(G) \mid L(u) < x \leq L(v)\}$. Evidently, $C(G, L, x)$ is equivalent to ∂X in the graph G where $X = \{L^{-1}(0), L^{-1}(1), \ldots, L^{-1}(x-1)\}$. Based on this characterization, we want to prove the following inequality,

$$\frac{|C(A_k^d, L, x)|}{x} \geq \frac{2}{k-1} \tag{6}$$

for any L and $1 \leq x \leq (k^d - 1)/2$ where $x = |X|$.

A related graph topology which is used in Nakano's technique as well as in our proof is the *d–dimensional k–ary clique* K_k^d. The topology of a K_k^d resembles A_k^d. It also has k^d nodes labeled by the integers from 0 to k^d-1. Similar to A_k^d, there is an edge between two nodes if and only if k–ary representations of their labels differ in exactly one digit. However, unlike A_k^d, the absolute value of the difference does not have to be exactly one. Another characterization is that a K_k^d is a d–fold Cartesian product of the complete graph K_k with k vertices. The label–order layout of a K_k^d is defined the same way as that of an A_k^d. Figure 3 illustrates a 2–dimensional 4–ary clique K_4^2. Nakano proved the following lemma in [14].

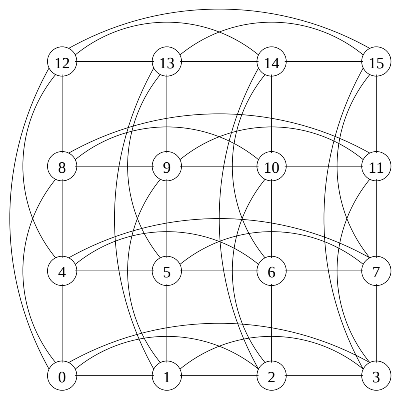

Fig. 3. The 2–dimensional 4–ary clique K_4^2.

Lemma 1 *(Nakano, 1993) For the d–dimensional k–ary clique K_k^d, the inequality*

$$|C(K_k^d, I, x)| \leq |C(K_k^d, L, x)|$$

holds for any layout L and gap x.

In other words, vertices corresponding to the first x numbers in the label–order layout (i.e. vertices with labels 0 to $x-1$) constitute an extremal set in a d–dimensional k–ary clique K_k^d minimizing the edge–boundary, among all such x–element subsets of vertices.

Intuitively, we embed a K_k^d into A_k^d and bound from below the number of boundary edges of any x–element subset of vertices in K_k^d using Nakano's lemma and the fact that, as a result of the embedding, removal of any edge in the array will result in disconnection of at most a certain number of edges in the array. To this end, we first give the following embedding lemma which characterizes the isoperimetric number problem for A_k^d in terms of a cut width problem for K_k^d when k is odd.

Lemma 2 *Given a d–dimensional k–ary array A_k^d where k is odd and an integer x with $1 \leq x \leq (k^d-1)/2$,*

$$i(A_k^d) \geq \frac{2}{k-1} \quad \text{if} \quad |C(K_k^d, I, x)| \geq \frac{k+1}{2}x.$$

Proof. The embedding of K_k^d into A_k^d is done in the obvious manner: The edge from $u = (a_1, \ldots, a_{i-1}, r, a_{i+1}, \ldots, a_d)$ to $v = (a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_d)$ of the clique with $r < s$ is embedded through the path

$$(a_1, \ldots, a_{i-1}, r, a_{i+1}, \ldots, a_d) \to (a_1, \ldots, a_{i-1}, r+1, a_{i+1}, \ldots, a_d) \to \cdots$$
$$\to (a_1, \ldots, a_{i-1}, s-1, a_{i+1}, \ldots, a_d) \to (a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_d)$$

in the array. Then, at most $(k^2-1)/4$ edges of K_k^d are routed through any edge of A_k^d [14]. Consider the set of edges in a cut of A_k^d under a linear layout L at gap x, i.e. $C(A_k^d, L, x)$. There is a set of edges in the K_k^d under the same layout L and gap x, $C(K_k^d, L, x)$, corresponding to this cut as a result of the embedding. This means

$$|C(A_k^d, L, x)| \geq \frac{|C(K_k^d, L, x)|}{\frac{(k^2-1)}{4}}.$$

Thus, we have

$$\frac{|C(A_k^d, L, x)|}{x} \geq \frac{|C(K_k^d, L, x)|}{\frac{(k^2-1)}{4}x} \geq \frac{|C(K_k^d, I, x)|}{\frac{(k^2-1)}{4}x}$$

by Nakano's lemma. Hence, it suffices to show for $1 \leq x \leq (k^d-1)/2$,

$$\frac{|C(K_k^d, I, x)|}{\frac{(k^2-1)}{4}x} \geq \frac{2}{k-1}$$

which is equivalent to showing $|C(K_k^d, I, x)| \geq x(k+1)/2$ for $1 \leq x \leq (k^d-1)/2$, and Lemma 2 follows. $\square$

At this point, we have reduced the proof of the isoperimetric number of the array into proving the following claim.

Claim 1 *For odd k, $|C(K_k^d, I, x)| \geq x(k+1)/2$ holds whenever $1 \leq x \leq (k^d-1)/2$.*

The proof of Claim 1 is by induction on d which we give next. Consider the set X of first x nodes of a K_k^d in label–order layout where $1 \leq x \leq (k^d-1)/2$. For notational convenience, let $B_d(x) = |C(K_k^d, I, x)|$. That is, $B_d(x)$ is the number of edges in the layout that connect these x nodes to the remaining $k^d - x$. We first give a recurrence relation for $B_d(x)$ which we subsequently use in our inductive argument. First note that $x = a_{d-1}k^{d-1} + \cdots + a_1 k + a_0$ with $0 \leq a_i < k$. Thus $a_{d-1}, \ldots, a_1, a_0$ are the digits of x in base k and $x = (a_{d-1} \cdots a_1 a_0)_k$. Note that X is made up of a_{d-1} copies of K_k^{d-1}, a_{d-2} copies of K_k^{d-2} and so on. An example of this is shown in Figure 4. The vertices rendered in dark constitute subset X and the boundary edges ∂X $(= C(K_3^2, I, 4))$ are shown by dashed lines. The subset X consists of one copy of K_3^1 (bottom row vertices) and one copy of K_3^0 (vertex 3). We record the following observation as a lemma, since we use it repeatedly in the proof.

Lemma 3 *If $x = (a_{d-1} \cdots a_1 a_0)_k$ and $1 \leq x \leq (k^d-1)/2$, then $a_{d-1} \leq (k-1)/2$.*

Lemma 4 *Suppose $x = (a_{d-1} \cdots a_1 a_0)_k$. Then $B_d(x)$ satisfies the recursion*

$$B_d(x) = a_{d-1}(1+a_{d-1})k^{d-1} - x(1 + 2a_{d-1} - k) + B_{d-1}(x - a_{d-1}k^{d-1}) \qquad (7)$$

with $B_1(a_0) = a_0(k - a_0)$.

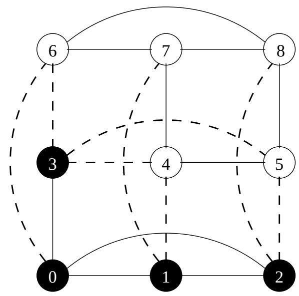

Fig. 4. The K_3^2 with set $X = \{0, 1, 2, 3\}$ and $B_2(4) = 8$.

Proof. Let $a = a_{d-1}$. We show that $B_d(x)$ satisfies the recursion

$$B_d(x) = ak^{d-1}(k-a) - (x-ak^{d-1})a + (x-ak^{d-1})(k-a-1) + B_{d-1}(x-ak^{d-1}) \quad (8)$$

which can then be simplified to (7). The first term of (8) is the number of edges on the boundary if X consisted of only a copies of K_k^{d-1}. From this we take out the number of edges that link the vertices in these copies to the remaining portion of X (which we think of as lying in the $(a+1)^{st}$ copy of K_k^{d-1} in K_k^d) and add, in turn, the number of edges from the vertices in this lower dimensional set of cardinality $x - ak^{d-1}$ to the nodes in the remaining $k - a - 1$ copies of K_k^{d-1}. At this point, we have counted all the boundary edges in one dimension, hence we can get rid of this dimension altogether and consider only the boundary edges of the remaining $x - ak^{d-1}$ vertices that reside on the $(a+1)^{st}$ subclique. But the number of boundary edges in that subclique is precisely $B_{d-1}(x - ak^{d-1})$. Hence, by adding this, we get (8). $\square$

Lemma 5 *Suppose k is odd and $x = (a_{d-1}a_{d-2}\cdots a_0)_k$ with $1 \le x \le (k^d - 1)/2$. Then $B_d(x) \ge x(k+1)/2$.*

Proof. This lemma is a restatement of Claim 1. We prove it by induction on d. For the base case $d = 1$, $B_1(a_0) = a_0(k - a_0) \ge x(k+1)/2$ since $x = a_0$. For inductive hypothesis we assume that $B_d(x) \ge x(k+1)/2$ holds whenever $1 \le x \le (k^d - 1)/2$. We are required to prove,

$$B_{d+1}(x) = a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(x - a_dk^d) \ge x\frac{k+1}{2} \quad (9)$$

for $1 \le x \le (k^{d+1} - 1)/2$ where $x = (a_da_{d-1}\cdots a_0)_k$. We prove this inequality by considering the two possible cases: $x - a_dk^d \le (k^d-1)/2$ and $x - a_dk^d \ge (k^d+1)/2$.

Case I: $x - a_dk^d \le (k^d - 1)/2$

Note that the inductive hypothesis is directly applicable and we have

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(x - a_dk^d) \ \ge$$

$$a_d(1+a_d)k^d - x(1+2a_d-k) + (x - a_dk^d)\frac{k+1}{2} \ .$$

Thus, it suffices to prove

$$a_d(1+a_d)k^d - x(1+2a_d-k) - a_dk^d\frac{k+1}{2} \ge 0.$$

After factoring and rearranging the terms, this is equivalent to

$$\left(2x - a_d k^d\right)\left(a_d - \frac{k-1}{2}\right) \geq 0. \tag{10}$$

The first factor in (10) is always strictly positive since $x \geq 1$ and $a_d k^d < x$. The second factor is nonnegative by Lemma 3. Note that equality in (10) is possible only for $a_d = (k-1)/2$.

Case II: $x - a_d k^d \geq (k^d+1)/2$

In this case, the inductive hypothesis cannot be used directly. Fortunately, however, $B_d(x) = B_d(k^d - x)$ and the inductive step can be used as $B_d(k^d - x) \geq (k^d - x)(k+1)/2$ for $x \geq (k^d+1)/2$. Now the inequality we want to prove becomes

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(k^d - (x - a_d k^d)) \geq x\frac{k+1}{2}.$$

By using the inductive hypothesis, we have

$$a_d(1+a_d)k^d - x(1+2a_d-k) + B_d(k^d - (x - a_d k^d)) \;\geq$$
$$a_d(1+a_d)k^d - x(1+2a_d-k) + (k^d - (x - a_d k^d))\frac{k+1}{2}\,.$$

Thus, after rearrangement of the terms above, it suffices to show

$$k^d\left(a_d + \frac{k+1}{2}\right) \geq 2x. \tag{11}$$

Note in this case that $a_d < (k-1)/2$ (or, equivalently $a_d \leq (k-3)/2$), as otherwise x would exceed $(k^{d+1}-1)/2$. The inequality (11) is equivalent to

$$k^d\frac{k+1}{2} \geq a_d k^d + 2a_{d-1}k^{d-1} + 2a_{d-2}k^{d-2} + \cdots + 2a_0.$$

But $a_{d-1}k^{d-1} + a_{d-2}k^{d-2} + \cdots + a_0 \leq k^d - 1$ since this is a base k expansion. Thus, it suffices to prove

$$k^d\frac{k+1}{2} \geq k^d\frac{k-3}{2} + 2k^d - 2.$$

After expansion of terms, this inequality is seen to be equivalent to

$$k^{d+1} + k^d \geq k^{d+1} + k^d - 4$$

which obviously holds. Furthermore, we also note that the inequality (9) is strict in this case. $\square$

This completes the proof of Claim 1. Therefore, Proposition 2, and consequently Theorem 1 is proved.

4. Isoperimetric Sets and Their Cardinalities

Theorem 2 *The cardinalities of the isoperimetric sets of a A_k^d are*

$$\tfrac{1}{2}(k^d-1),\ \tfrac{1}{2}(k^d-k),\ \ldots\ ,\ \tfrac{1}{2}(k^d-k^{d-1})$$

for odd k, and $k^d/2$ for even k.

Proof. For odd k, an isoperimetric set X of an A_k^d with x vertices must satisfy $B_d(x) = x(k+1)/2$ where $1 \leq x \leq |V(A_k^d)|/2$. From the proof of Lemma 5, the equation $B_d(x) = x(k+1)/2$ has exactly d roots in this interval given by

$$(m, 0, 0, \ldots, 0),\ (m, m, 0, \ldots, 0), \ldots, (m, m, \ldots, m)$$

written in base k with $m = (k-1)/2$. These can be written as

$$\tfrac{1}{2}(k^d - 1),\ \tfrac{1}{2}(k^d - k),\ \ \ldots\ ,\ \tfrac{1}{2}(k^d - k^{d-1}).$$

For even k, any isoperimetric set must make the two sides of inequality (5) equal, which occurs only for $x = k^d/2$. □

Because of the structural symmetry of arrays, there are multiple isoperimetric sets with the same cardinality. For instance, an isoperimetric set, when k is even, is the set of vertices with the $k^d/2$ smallest (or, largest) label values. For odd k the configuration of the isoperimetric sets is more interesting in that they form a nice recursive structure. For instance, the set of vertices with the first $\frac{1}{2}(k^d - k^{d-1})$ smallest labels constitute an isoperimetric set. We can obtain another isoperimetric set if we add to this set the next $\frac{k-1}{2}k^{d-2}$ vertices. Continuing this way, we can obtain an isoperimetric set corresponding to any one of the d cardinalities listed above.

5. Conclusion and Future Considerations

We used embedding based techniques to obtain an exact expression for the isoperimetric number of d–dimensional k–ary arrays for arbitrary d and k, and also gave a description of isoperimetric sets and their cardinalities. Work on extending these results to similar topologies such as tori and generalized cylinders as well as general multidimensional arrays is in progress [3]. We would like to note that a direct application of our proof for the odd case does not extend automatically to multidimensional arrays. The reason for this is that contrary to the case of A_k^d and K_k^d, an extremal set of a multidimensional clique does not correspond to an isoperimetric set of a multidimensional array through an extension of the embedding described in the proof.

Acknowledgements

We would like to thank Imrich Vrto for bringing reference [14] to our attention as well as Sergei Bezrukov and two anonymous referees for their constructive comments. The first author would like to thank İzmir Institute of Technology, İzmir, Turkey for their support in the form of a fellowship.

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