A Multilinear Operator for Almost Product Evaluation of Hankel Determinants

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Abstract

In a recent paper we have presented a method to evaluate certain Hankel determinants as *almost products*; i.e. as a sum of a small number of products. The technique to find the explicit form of the almost product relies on differential-convolution equations and trace calculations. In the trace calculations a number of intermediate nonlinear terms involving determinants occur, but only to cancel out in the end.

In this paper, we introduce a class of multilinear operators γ acting on tuples of matrices as an alternative to the trace method. These operators do not produce extraneous nonlinear terms, and can be combined easily with differentiation.

The paper is self contained. An example of an almost product evaluation using γ-operators is worked out in detail and tables of the γ-operator values on various forms of matrices are provided. We also present an explicit evaluation of a new class of Hankel determinants and conjectures.

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1 Introduction

The expansion of a determinant

$$\det[a_{i,j}]_{0\leq i,j\leq n}$$

from first principles involves calculating the signed sum of $(n+1)!$ individual products. This type of an evaluation is not of much interest, and one usually uses the multilinearity of the determinant to obtain more succinct expressions for a given family of determinants. Those determinants which may be evaluated as a single product of simple factors (such as the Vandermonde and Cauchy determinants) have a special appeal. For product form evaluations, LU decomposition, continued fractions and Dodgson condensation are some of the available methods that have been utilized with considerable success. There exists an extensive literature on this topic, going back to the treatise of Muir [10, 11]. A more recent compilation of the state of affairs of the theory of determinants appears in Krattenthaler [6, 7], in which a wide range of techniques used to study the evaluation of families of determinants are described, accompanied by an extensive bibliography on the subject.

Of particular interest are Hankel determinants, for which

$$a_{i,j} = a_{i+j} \ .$$

Certain classes of Hankel determinants with combinatorially interesting entries a_{i+j} have product representations with startling evaluations, and we mention

$$\det\left[\binom{3(i+j)+2}{i+j}\right]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{(6i+4)!(2i+1)!}{2(4i+2)!(4i+3)!}$$

and

$$\det\left[\binom{3(i+j)}{i+j}\right]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{3(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!}$$

(see [1] and [3]). The in-between case of the binomial coefficients

$$a_k = \binom{3k+1}{k} \tag{1}$$

is not amenable to standard methods since it does not have a product evaluation. In a recent paper [2] we proved that for the entries (1), the evaluation is an *almost product*; in this case a sum of $n+1$ products of simple factors:

$$\det\left[\binom{3(i+j)+1}{i+j}\right]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{(6i+4)!(2i+1)!}{2(4i+2)!(4i+3)!} \sum_{i=0}^{n} \frac{n!(4n+3)!!(3n+i+2)!}{(3n+2)!i!(n-i)!(4n+2i+3)!!} \ .$$

The technique presented in [2] to find the explicit form of the almost product for this particular Hankel determinant relies on the following steps:

(I) Using $k = i+j$, replace a_k with polynomials

$$a_k(x) = \sum_{m=0}^{k} \binom{3k+1-m}{k-m} x^m \tag{2}$$

so that $a_k(x)$ is a monic polynomial of degree k with $a_k = a_k(0)$. Consequently the associated Hankel determinant $H_n(x)$ is a polynomial, and $H_n = H_n(0)$.

(II) Establish a second order ODE satisfied by $H_n(x)$.

(III) Solve the DE in (II), and evaluate the solution at $x = 0$.

The (β,α)-*case* of this problem is the evaluation of the Hankel determinants where the entries are

$$a_k^{(\beta,\alpha)}(x) = \sum_{m=0}^{k} \binom{\beta k+\alpha-m}{k-m} x^m \ . \tag{3}$$

The bulk of the work is contained in Step (II), and this part of the argument itself relies on three essential identities. These identities are linked in the derivation of the differential equation via the application of a trace operator.

In this paper, we introduce a class of multilinear γ-operators acting on tuples of matrices which take the place of this trace operator.

If it had just been a matter of calculating the differential equation in the $(3,1)$-case as we did in [2], then which technique we used might not have mattered much. However, we wanted to try to extend the differential equation method to a larger class of (β,α)-cases, and we found that already in the $(2,2)$-case, the γ-operators simplified the calculations significantly. To be specific, in the trace approach some nonlinear terms occur in the calculations, which get canceled in the end. For example the following ratio of determinants (using the notation in [2])

$$-4(4n+3)^2\frac{K_n^2}{H_n} \tag{4}$$

appears during the course of the trace calculations (e.g. [2], p. 15), and is later cancelled.

As one goes to other cases, these nonlinear terms proliferate. In the $(2,2)$-case, there are over half a dozen of these terms that arise, which all cancel.

These nonlinear terms turn out to be an avoidable burden in a method that already involves a lot of calculation. It is easier to combine differentiation with the γ-operators than with the trace calculations of [2] and in addition the γ-operator calculations do not produce the extraneous nonlinear terms mentioned above. An added benefit is that they need not be calculated from scratch for other Hankel determinant evaluations. In Appendix III, we provide extensive table of values of γ-operators.

Let

$$a_k(x) = \sum_{m=0}^{k}\binom{2k+2-m}{k-m}x^m \tag{5}$$

and define the $(n+1)\times(n+1)$ Hankel determinants by

$$H_n(x) = \det[a_{i+j}(x)]_{0\le i,j\le n}\ . \tag{6}$$

A few of these polynomials and the Hankel determinants are as follows:

$$\begin{aligned}
a_0(x) &= 1\\
a_1(x) &= 4+x\\
a_2(x) &= 15+5x+x^2\\
a_3(x) &= 56+21x+6x^2+x^3
\end{aligned}$$

and

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= -1-3x\\
H_2(x) &= -1-x+5x^2\\
H_3(x) &= 1+6x+3x^2-7x^3\ .
\end{aligned}$$

We give the elements of the application of γ-operators by working through the proof of the following theorem.

Theorem 1 *Suppose a_k and the $H_n(x)$ are as defined in (5) and (6). Then $H_n(x)$ has the following almost product evaluations:*

$$H_n(x) = (-1)^n \sum_{k=0}^{n} \left[(2n+3)\binom{n+k}{2k+1} + (2k+1)\binom{n+k+1}{2k+1} \right] (x-2)^k \tag{7}$$

and

$$H_n(x) = \sum_{k=0}^{n} (-1)^k \left[(n+k+1)\binom{n+k}{2k} + (2n+4k+1)\binom{n+k}{2k+1} + 8(k+1)\binom{n+k+1}{2k+3} \right] (x+2)^k \,. \tag{8}$$

Alternate expressions for (7) and (8) are given in (43) and (46). The expansion of $H_n(x)$ around $x=0$ can be found in (54). The generating function of the $H_n(x)$ itself is given in (44).

It is known that [2, 5, 8]

$$\det\left[\binom{2(i+j)+2}{i+j}\right]_{0\le i,j\le n} = (-1)^{\frac{n(n+1)}{2}} \,. \tag{9}$$

Our purpose is not the derivation of this relatively simple numerical evaluation itself, but to give an exposition of the salient points of the γ-operators, which allow us to evaluate the general case of the Hankel determinants of the polynomials (5) as an almost product.

Additionally, we obtain numerical evaluations of $H_n(x)$ at special values of x. A number of these are presented in Section 8 and at the end of Section 9.

In Corollary 3 we evaluate the Hankel determinant

$$\det\left[\binom{2(i+j)+3}{i+j}\right]_{0\le i,j\le n} \,.$$

The explicit almost product evaluation of Theorem 1 is derived from the second order differential equation satisfied by these Hankel determinants. This differential equation is given in Theorem 2 in Section 7. With the definition of the polynomials in (3), the evaluation in this paper is the $(\beta,\alpha)=(2,2)$-case.

The outline of the rest of this paper is as follows: In Section 2, we define determinants H_λ for partitions λ obtained from a given Hankel matrix. This is followed by the introduction of the family of multilinear operators γ along with their basic properties and a combinatorial interpretation for their evaluation in Section 3. Section 4 presents example calculations with the γ's, and a compilation of evaluations that are used in the paper. This is followed by three identities that are typically needed for our methods, and the derivation of the equations satisfied by the various H_λ that arise in the calculations. We obtain a system of first order differential equations which results in a second order differential equation for the Hankel determinant we wish to evaluate in Section 7. Evaluation at special points are discussed in Section 8, and the general solution of the differential equation is derived in Section 9. An additional Hankel determinant evaluation is given at the end of this section in Corollary 3. In Section 10, we consider the properties of the zeros of the Hankel determinants and show that they form a Sturm sequence. Conjectures on the evaluation of similar Hankel determinants are presented in Section 11. This is followed by Appendix I - III where we give the proofs of the results stated and used in the calculations as well as tables of γ-operator evaluations. We remark that Sections 2–4 and Appendices I and III apply to general Hankel matrices, whereas Sections 5–10 and Appendix II apply to the evaluation of the case $\alpha = \beta = 2$.

2 Preliminaries

We consider general Hankel matrices $A = [a_{i+j}]_{0 \le i,j \le n}$ in the symbols a_k. In [2] and in Section 1 of the present paper we used the notation H_n for $\det(A)$. However, it is useful to have alternate notation for various determinants that arise, in which sometimes the parameter n is suppressed. Unless otherwise indicated, we assume that n has been chosen and is fixed.

A *partition* λ of an integer $m > 0$ is a weakly decreasing sequence of integers $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0)$ with $m = \lambda_1 + \lambda_2 + \cdots + \lambda_p$. Each λ_i is called a *part* of λ. For example $\lambda = (3, 2, 2)$ is a partition of $m = 7$ into $p = 3$ parts.

We use the notation $\lambda = m^{\alpha_m} \cdots 2^{\alpha_2} 1^{\alpha_1}$ for a partition λ of m, indicating that λ has α_i parts of size i. Thus for example, $\lambda = 3^2 2 1^3$ denotes the partition $3+3+2+1+1+1$ of 11. Given $n > 0$, each partition $(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0)$ with $p \le n+1$ defines a determinant of a matrix obtained from the $(n+1) \times (n+1)$ Hankel matrix $A_n = [a_{i+j}]_{0 \le i,j \le n}$ in the symbols a_k, by shifting the column indices of the entries up according to λ as follows: Let $\mu_i = \lambda_i$ for $i = 1, \ldots, p$ and $\mu_i = 0$ for $i = p+1, \ldots, n+1$. Then

$$H_\lambda = \det[a_{i+j+\mu_{n+1-j}}]_{0 \le i,j \le n} \ .$$

We use the special notation 0 to denote the sequence $\mu_i = 0$ for $i = 1, \ldots, n+1$. For example when $n = 3$,

$$H_0 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}, \ H_2 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_5 \\ a_1 & a_2 & a_3 & a_6 \\ a_2 & a_3 & a_4 & a_7 \\ a_3 & a_4 & a_5 & a_8 \end{bmatrix}, \ H_{31^2} = \det \begin{bmatrix} a_0 & a_2 & a_3 & a_6 \\ a_1 & a_3 & a_4 & a_7 \\ a_2 & a_4 & a_5 & a_8 \\ a_3 & a_5 & a_6 & a_{10} \end{bmatrix} .$$

We note that these determinants are obtained in a way similar to the expansion of Schur functions in terms of the homogeneous symmetric functions by the Jacobi-Trudi identity [9].

When the $a_k = a_k(x)$ are functions of x, then $H_\lambda = H_\lambda(x)$ is a function of x. When we need to indicate the dependence of the determinant on n as well as x, we write

$$H_\lambda(n, x)$$

for the $(n+1) \times (n+1)$ shifted Hankel determinant. As an example, with this notation (7) is written as

$$H_0(x) = H_0(n, x) = (-1)^n \sum_{k=0}^{n} \frac{2n^2 + 4n + 2k^2 + 1}{2k+1} \binom{n+k}{2k} (x-2)^k \ . \qquad (10)$$

The $(n+1) \times (n+1)$ Hankel determinant will be denoted by a number of different notations in this paper. Among these are $H_n = H_n(x)$, $H_0 = H_0(x)$, and $H_0(n, x)$. In the latter two cases it should be clear from the context that the subscript 0 refers to the partition involved and not to the dimension of the Hankel matrix.

Our aim is to obtain a first order linear system of equations

$$\begin{aligned} Q\frac{d}{dx}H_0 &= Q_0H_0 + Q_1H_1 \\ U\frac{d}{dx}H_1 &= U_0H_0 + U_1H_1 \end{aligned} \tag{11}$$

where the coefficients are polynomial functions of x and n. From this system the second order differential equation for H_0 given in Theorem 2 can be found immediately.

In the process of differentiating H_0 and H_1 the following five determinants

$$H_3, H_{21}, H_{1^3}, H_2, H_{1^2}$$

are encountered. We will express each of these in terms of the two determinants H_0, H_1.

The γ-operator that we next define allows us to do this from the three identities satisfied by the a_k, while avoiding having to deal with nonlinear expressions involving determinants. This operator has the additional advantage of simplifying differentiation of determinants, improving on the trace calculations used in [2].

3 The γ-operator

We define a multilinear operator γ on m-tuples of matrices as follows:

Definition 1 *Given $(n+1)\times(n+1)$ matrices A and $X_1, X_2, \ldots, X_m$ with $m \geq 1$, define*

$$\gamma_A(\ ) = \det(A)$$

and

$$\gamma_A(X_1, \ldots, X_m) = \partial_{t_1}\partial_{t_2}\cdots\partial_{t_m} \det(A + t_1X_1 + t_2X_2 + \cdots + t_mX_m)|_{t_1=\cdots=t_m=0}$$

where $t_1, t_2, \ldots, t_m$ are variables that do not appear in A or $X_1, X_2, \ldots, X_m$.

Next we give a computationally feasible combinatorial interpretation of $\gamma_A(X_1, \ldots, X_m)$ for small m, based on elementary properties of determinants.

Definition 2 *Suppose A and $X_1, \ldots, X_m$ are $(n+1)\times(n+1)$ matrices, $m \leq n+1$. Given a subset of column indices $S = \{j_1, j_2, \ldots, j_m\} \subseteq \{0, 1, \ldots, n\}$ and a permutation σ of $\{1, 2, \ldots, m\}$, $A_{S,\sigma}$ is defined as the matrix which is obtained from A by replacing A's j_k-th column by the j_k-th column of the matrix X_{σ_k} for $k = 1, 2, \ldots, m$.*

With this notation we have

Proposition 1 *For $m \leq n+1$,*

$$\gamma_A(X_1, \ldots, X_m) = \sum_{S,\sigma} \det(A_{S,\sigma}) \tag{12}$$

where the summation is over all subsets S of $\{0, 1, \ldots, n\}$ with $|S| = m$ and all permutation σ of $\{1, 2, \ldots, m\}$.

Note: The expansion (12) is also valid as a sum over row indices where the replacements made are rows from $X_1, \ldots, X_m$ instead of columns.

Another motivation for using the γ-operators is that they differentiate nicely; the derivative of a γ is a sum of γs.

Proposition 2 *For $m \leq n$,*

$$\frac{d}{dx}\gamma_A(X_1, \ldots, X_m) = \gamma_A(\frac{d}{dx}A, X_1, \ldots, X_m) + \sum_{j=1}^{m} \gamma_A(X_1, \ldots, X_{j-1}, \frac{d}{dx}X_j, X_{j+1}, \ldots, X_m) \ .$$

The proofs of Proposition 1 and Proposition 2 can be found in Appendix I.

Using Proposition 1, we can evaluate γ_A on matrices that are associated with A in terms of determinants H_λ for various partitions λ. Next, we give a few examples of these calculations and a compilation of the expansions needed.

4 Explicit γ_A evaluations

Let

$$A = [a_{i+j}]_{0 \leq i,j \leq n}$$

We start with a few sample calculations.

Example: In the calculation of $\gamma_A([a_{i+j}])$, the sum in (12) is over all subsets $S \subseteq \{0, 1, \ldots n\}$ with a single element and σ is the identity permutation. We are replacing a column of A with the same column, so the resulting determinant is $H_0 = \det(A)$ for each one of $n+1$ possible column selections. Thus

$$\gamma_A([a_{i+j}]) = (n+1)H_0 \ .$$

Example: In the calculation of $\gamma_A([a_{i+j+2}])$ the sum in (12) is again over all subsets $S \subseteq \{0, 1, \ldots n\}$ with one element. If $S = \{j\}$ and $j \leq n-2$, then the j-th and the $(j+2)$-nd columns are identical in $A_{S,\sigma}$ and the determinant vanishes. For $j = n$, the determinant is H_2 and for $j = n-1$ it is $-H_{1^2}$. Therefore

$$\gamma_A([a_{i+j+2}]) = H_2 - H_{1^2} \ .$$

Example: We split the calculation of $\gamma_A([(i+j)a_{i+j+2}])$ into two pieces:

$$\gamma_A([(i+j)a_{i+j+2}]) = \gamma_A([ia_{i+j+2}]) + \gamma_A([ja_{i+j+2}]).$$

In the calculation of $\gamma_A([ja_{i+j+2}])$, the determinant in (12) survives only for $S = \{n\}$ and $S = \{n-1\}$, exactly as in the case of the evaluation of $\gamma_A([a_{i+j+2}])$ above. However, now the determinant gets multiplied by the factor n of the new n-th column in the former case, and by the factor $n-1$ of the $(n-1)$-st column in the latter. Therefore

$$\gamma_A([ja_{i+j+2}]) = nH_2 - (n-1)H_{1^2}$$

$\gamma_A([ia_{i+j+2}])$ evaluates to the same expression, since now we are dealing with rows instead of columns, but otherwise the argument is the same. Therefore

$$\gamma_A([(i+j)a_{i+j+2}]) = 2nH_2 - 2(n-1)H_{1^2} \ .$$

Definition 3 *For a polynomial sequence $a_n = a_n(x)$ $(n \geq 0)$, the convolution polynomials $c_n = c_n(x)$ are defined by*

$$c_n = \sum_{k=0}^{n} a_k a_{n-k}$$

with $c_{-1} = 0$.

Example: To compute $\gamma_A([c_{i+j+1}])$ for $n = 2$, we use the expansion of the matrix $[c_{i+j+1}]$ in terms of shifted versions of A as given below. The expansion for arbitrary n can be found in Appendix I.

$$\begin{aligned}[c_{i+j+1}] \quad = \quad & a_0 \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{bmatrix} + a_1 \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} + a_2 \begin{bmatrix} 0 & a_0 & a_1 \\ 0 & a_1 & a_2 \\ 0 & a_2 & a_3 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & a_0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} \\ & + a_0 \begin{bmatrix} 0 & 0 & 0 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{bmatrix} + a_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_2 & a_3 & a_4 \end{bmatrix} \end{aligned} \tag{13}$$

A routine calculation gives

$$\begin{aligned} \gamma_A([c_{i+j+1}]_{0 \leq i,j \leq n}) &= a_0 H_1 + n a_1 H_0 + a_0 H_1 + n a_1 H_0 \\ &= 2a_0 H_1 + 2n a_1 H_0 \ . \end{aligned}$$

We provide another example of a γ calculation.

Example: In the calculation of $\gamma_A([a_{i+j+1}], [a_{i+j+2}])$ the sum in (12) is over all subsets $S \subseteq \{0, 1, \ldots n\}$ with two elements. If $S = \{j_1 < j_2\}$ with $j_2 \leq n-2$, then for $\sigma = (1)(2)$, the columns j_2 and $j_2 + 2$, and for $\sigma = (12)$, the columns j_2 and $j_2 + 1$ of $A_{S,\sigma}$ are identical. Therefore in these cases the determinant vanishes. The remaining possibilities for S, σ pairs can be enumerated as

1. $S = \{n-1, n\}$ and $\sigma = (1)(2)$,
2. $S = \{n-2, n-1\}$ and $\sigma = (1)(2)$,
3. $S = \{n-2, n\}$ and $\sigma = (12)$.

The resulting determinants are

$$H_{21}, \quad -H_{1^3}, \quad -H_{1^3},$$

respectively. Therefore

$$\gamma_A([a_{i+j+1}], [a_{i+j+2}]) = H_{21} - 2H_{1^3} \ .$$

In Tables 2-5 of Appendix III, we give a list of various γ evaluations. The ones that are needed for the computations in this paper are in Tables 2 and 3.

5 The three identities

Now we consider the $(2,2)$-case. The three identities used in the argument are given in the following three lemmas. These identities are typical of our methods. The first identity is a differential-convolution equation. The second identity involves convolutions and a_k but no derivatives. The third identity is a linear dependence among certain column vectors involving the a_k.

Lemma 1 *(First Identity (FI))*

$$\begin{aligned}(x-2)x(x+2)(3x+2)\frac{d}{dx}a_n &= 2n(x-1)a_{n+2}+(n(x-6)(x-2)+3x^2-2x+4)a_{n+1}\\ &\quad -(3x^3+18x^2-20x+24+4n(x^2+4))a_n \\ &\quad +8(x-1)^2c_n-32(x-1)^2c_{n-1}\end{aligned} \tag{14}$$

Lemma 2 *(Second Identity (SI))*

$$\begin{aligned}&(nx+3x+2)a_{n+2}-(nx(x+6)+3x^2+16x+8)a_{n+1}\\ &+2x(x+2)(2n+5)a_n+(x-1)(x-2)c_n-4(x-1)(x-2)c_{n-1}=0\end{aligned} \tag{15}$$

Lemma 3 *(Third Identity (TI))*

$$\sum_{j=0}^{n+2} w_{n,j}(x)a_{i+j}(x)=0 \tag{16}$$

for $i=0,1,\ldots,n$ *where*

$$\begin{aligned}w_{n,j}(x) &= (-1)^{n-j}\left\{\frac{2(2n+5)}{2j+1}\binom{n+j+2}{2j}+\frac{(2n+3)(2n+5)}{2j+1}\binom{n+j+2}{2j}x\right.\\ &\quad \left.+\frac{(2n+3)(2n+5)}{2j+3}\binom{n+j+2}{2j+1}x^2\right\}.\end{aligned} \tag{17}$$

The proofs can be found in Appendix II. We remark that the weights in Lemma 3 are typical of our method. Once the coefficients of the weight polynomials $w_{n,j}(x)$ are guessed, then automatic binomial identity provers can be used to prove (16).

To prove (11), we will find the expansions of both $\frac{d}{dx}H_0$ and $\frac{d}{dx}H_1$ in terms of H_0 and H_1. Since at first other determinants H_λ also appear in these derivatives, they will need to be eliminated. We do this by constructing a sufficient number of equations involving them, and then expressing each one in terms of H_0 and H_1.

6 The five equations

6.1 Equation from $\gamma_A([SI(i+j)])$

Apply

$$\gamma_A(*)$$

to the $(n+1)\times(n+1)$ matrix whose (i,j)-th entry is obtained from the second identity (15) evaluated at $i+j$ and expand using linearity. If we denote the matrix so obtained from the second identity by $[SI(i+j)]$, then the computation is the expansion of $\gamma_A([SI(i+j)])=0$. We obtain

$$\begin{aligned} 0 &= x\gamma_A([(i+j)a_{i+j+2}]) + (3x+2)\gamma_A([a_{i+j+2}]) \\ &\quad -x(x+6)\gamma_A([(i+j)a_{i+j+1}]) - (3x^2+16x+8)\gamma_A([a_{i+j+1}]) \\ &\quad +4x(x+2)\gamma_A([(i+j)a_{i+j}]) + 10x(x+2)\gamma_A([a_{i+j}]) \\ &\quad +(x-1)(x-2)\gamma_A([c_{i+j}]) - 4(x-1)(x-2)\gamma_A([c_{i+j-1}]) \end{aligned}$$

Making use of the entries in the $\gamma_A(*)$ computations from Table 2, we get

$$\begin{aligned} 0 &= x(2nH_2 - 2(n-1)H_{1^2}) + (3x+2)(H_2 - H_{1^2}) \\ &\quad -x(x+6)2nH_1 - (3x^2+16x+8)H_1 \\ &\quad +4x(x+2)n(n+1)H_0 + 10x(x+2)(n+1)H_0 \\ &\quad +(x-1)(x-2)(2n+1)H_0 \ . \end{aligned}$$

Therefore

$$\begin{aligned} &(2+3x+2nx)H_2 - (2+x+2nx)H_{1^2} - (8+16x+12nx+3x^2+2nx^2)H_1 \\ &+(2+4n+17x+22nx+8n^2x+11x^2+16nx^2+4n^2x^2)H_0 = 0 \ . \end{aligned} \tag{18}$$

6.2 Equation from $\gamma_A([SI(i+j+1)])$

Now apply γ to the matrix obtained by evaluating the second identity (15) at $i+j+1$. If we denote this matrix by $[SI(i+j+1)]$, then this computation is the expansion of $\gamma_A([SI(i+j+1)])=0$ from (15).

$$\begin{aligned} 0 &= x\gamma_A([(i+j)a_{i+j+3}]) + (4x+2)\gamma_A([a_{i+j+3}]) \\ &\quad -x(x+6)\gamma_A([(i+j)a_{i+j+2}]) - (4x^2+22x+8)\gamma_A([a_{i+j+2}]) \\ &\quad +4x(x+2)\gamma_A([(i+j)a_{i+j+1}]) + 14x(x+2)\gamma_A([a_{i+j+1}]) \\ &\quad +(x-1)(x-2)\gamma_A([c_{i+j+1}]) - 4(x-1)(x-2)\gamma_A([c_{i+j}]) \ . \end{aligned}$$

Using Table 2,

$$\begin{aligned} 0 &= x(2nH_3 - 2(n-1)H_{21} + 2(n-2)H_{1^3}) + (4x+2)(H_3 - H_{21} + H_{1^3}) \\ &\quad -x(x+6)(2nH_2 - 2(n-1)H_{1^2}) - (4x^2+22x+8)(H_2 - H_{1^2}) \\ &\quad +4x(x+2)2nH_1 + 14x(x+2)H_1 \\ &\quad +(x-1)(x-2)(2H_1 + 2n(x+4)H_0) - 4(x-1)(x-2)(2n+1)H_0 \ . \end{aligned}$$

Therefore

$$\begin{aligned} &(1+2x+nx)H_3 - (1+x+nx)H_{21} + (1+nx)H_{1^3} \\ &-(4+11x+6nx+2x^2+nx^2)H_2 + (4+5x+6nx+x^2+nx^2)H_{1^2} \\ &+(2+11x+8nx+8x^2+4nx^2)H_1 + (-2+x)(-1+x)(-2+nx)H_0 = 0 \ . \end{aligned} \tag{19}$$

6.3 Equation from $\gamma_A([a_{i+j+1}], [SI(i+j)])$

Now consider the expansion of $\gamma_A([a_{i+j+1}], [SI(i+j)]) = 0$ from (15).

$$\begin{aligned} 0 &= x\gamma_A([a_{i+j+1}], [(i+j)a_{i+j+2}]) + (3x+2)\gamma_A([a_{i+j+1}], [a_{i+j+2}]) \\ &\quad -x(x+6)\gamma_A([a_{i+j+1}], [(i+j)a_{i+j+1}]) - (3x^2+16x+8)\gamma_A([a_{i+j+1}], [a_{i+j+1}]) \\ &\quad +4x(x+2)\gamma_A([a_{i+j+1}], [(i+j)a_{i+j}]) + 10x(x+2)\gamma_A([a_{i+j+1}], [a_{i+j}]) \\ &\quad +(x-1)(x-2)\gamma_A([a_{i+j+1}], [c_{i+j}]) - 4(x-1)(x-2)\gamma_A([a_{i+j+1}], [c_{i+j-1}]) \ . \end{aligned}$$

Using the $\gamma_A([a_{i+j+1}], *)$ computations from Table 3, we get

$$\begin{aligned} 0 &= x(2nH_{21} - 2(2n-3)H_{1^3}) + (3x+2)(H_{21} - 2H_{1^3}) \\ &\quad -x(x+6)2(2n-1)H_{1^2} - (3x^2+16x+8)2H_{1^2} \\ &\quad +4x(x+2)n(n-1)H_1 + 10x(x+2)nH_1 \\ &\quad +(x-1)(x-2)((2n-1)H_1 - (2n-1)(x+4)H_0) - 4(x-1)(x-2)(-2nH_0). \end{aligned}$$

Therefore for $n \geq 2$,

$$\begin{aligned} &(2+3x+2nx)H_{21} - 4(1+nx)H_{1^3} - 4(4+5x+6nx+x^2+nx^2)H_{1^2} \\ &+(4n-2+3x+6nx+8n^2x-x^2+8nx^2+4n^2x^2)H_1 - (x-2)(x-1)(2nx-4-x)H_0 = 0. \end{aligned} \tag{20}$$

6.4 Two equations from the third identity

The third identity is as given in Lemma 3. Define the column vector

$$v_j = [a_j, a_{j+1}, \ldots, a_{j+n}]^T \ .$$

The third identity (16) says that the vectors $v_0, v_1, \ldots, v_{n+2}$ are linearly dependent with the weights given in (17), i.e.

$$\sum_{j=0}^{n+2} w_{n,j} v_j = 0 \ . \tag{21}$$

Now consider the determinant of the $(n+1)\times(n+1)$ matrix whose first n columns are the columns of A, and whose last column is the zero vector. Writing the zero vector in the form (21) and expanding the determinant by linearity, we find

$$w_{n,n+2}H_2 + w_{n,n+1}H_1 + w_{n,n}H_0 = 0 \ .$$

Substituting the weights from (17), this gives the equation

$$\begin{aligned} &(2+3x+2nx)H_2 - (10+4n+15x+16nx+4n^2x+3x^2+2nx^2)H_1 \\ &+(n+1)(2n+5)(2+3x+2nx+2x^2)H_0 = 0 \ . \end{aligned} \tag{22}$$

Next we apply the same expansion trick to the matrix whose first $n-1$ columns are those of A, i.e. $v_0, v_1, \ldots, v_{n-2}$; whose $(n-1)$-st column is v_n; and whose last column is the zero vector, written in the form (21). Expanding the determinant by linearity, this time we obtain

$$w_{n,n+2}H_{21} + w_{n,n+1}H_{1^2} - w_{n,n-1}H_0 = 0 \ .$$

Therefore another equation is

$$
\begin{aligned}
&3(2+3x+2nx)H_{21}-3\Big(10+4n+15x+16nx+4n^2x+3x^2+2nx^2\Big)H_{1^2} \\
&+\Big(2n(1+2n)(5+2n)+n(1+2n)(3+2n)(5+2n)x+3n(3+2n)(5+2n)x^2\Big)H_0=0\ .
\end{aligned} \tag{23}
$$

Equations (18), (19), (20), (22), (23), form a 5×5 linear system $Mu=b$ which expresses the determinants

$$u=\Big[H_3,H_{21},H_{1^3},H_2,H_{1^2}\Big]^T$$

in terms of the two determinants H_0,H_1. The matrix M is as follows:

$$
\begin{bmatrix}
0 & 0 & 0 & 2nx+3x+2 & -2nx-x-2 \\
nx+2x+1 & -nx-x-1 & nx+1 & -nx^2-2x^2-6nx-11x-4 & nx^2+x^2+6nx+5x+4 \\
0 & 2nx+3x+2 & -4(nx+1) & 0 & -4\left(nx^2+x^2+6nx+5x+4\right) \\
0 & 0 & 0 & 2nx+3x+2 & 0 \\
0 & 3(2nx+3x+2) & 0 & 0 & \begin{array}{l}-3(4xn^2+2x^2n+16xn\\ \quad +4n+3x^2+15x+10)\end{array}
\end{bmatrix}
$$

with

$$\det(M)=12(1+nx)(1+2x+nx)(2+x+2nx)(2+3x+2nx)^2\ .$$

Solving $Mu=b$ for u, we obtain each of $H_3,H_{21},H_{1^3},H_2,H_{1^2}$ in terms of H_0 and H_1.

$$
\begin{aligned}
3(2+3x+2nx)H_3 &= -2(n+1)\Big(8xn^3+12x^2n^2+64xn^2+8n^2+6x^3n \\
&\quad +66x^2n+162xn+52n+15x^3+90x^2+126x+84\Big)H_0 \\
&\quad +3\Big(4xn^3+4x^2n^2+32xn^2+4n^2+2x^3n+18x^2n \qquad (24)\\
&\quad +81xn+26n+3x^3+18x^2+63x+42\Big)H_1\ ,\\
3(2+x+2nx)(2+3x+2nx)H_{21} &= \Big(-64x^2n^5-48x^3n^4-416x^2n^4-128xn^4-192x^3n^3 \\
&\quad -1040x^2n^3-704xn^3-64n^3+12x^4n^2-192x^3n^2 \qquad (25)\\
&\quad -1192x^2n^2-1360xn^2-288n^2+24x^4n+24x^3n-480x^2n \\
&\quad -1120xn-416n+9x^4+63x^3+48x^2-300x-240\Big)H_0 \\
&\quad +3(4xn^2+4xn+4n-x+2)\Big(4xn^2+2x^2n+16xn \\
&\quad +4n+3x^2+15x+10\Big)H_1\ ,\\
3(2+x+2nx)H_{1^3} &= (-16xn^4-32xn^3-16n^3+28xn^2-24n^2-6x^3n-12x^2n \\
&\quad +80xn+16n-3x^3-12x^2+12x+48)H_0 \qquad (26)\\
&\quad +3\left(4xn^3+4n^2+2x^2n-9xn-2n+x^2-4\right)H_1\ ,\\
(2+3x+2nx)H_2 &= -(n+1)(2n+5)\left(2x^2+2nx+3x+2\right)H_0 \\
&\quad +(4xn^2+2x^2n+16xn+4n+3x^2+15x+10)H_1\ , \qquad (27)\\
(2+x+2nx)H_{1^2} &= (-4xn^3-12xn^2-4n^2+2x^2n-9xn-10n+x^2 \\
&\quad +2x-8)H_0+(4xn^2+4xn+4n-x+2)H_1\ . \qquad (28)
\end{aligned}
$$

Equipped with these expansions, we now proceed with the calculation of the derivatives of H_0 and H_1.

7 The derivatives of H_0 and H_1

7.1 The derivative of H_0

From Definition 1,

$$H_0 = \gamma_A(\ ) \ .$$

Therefore by Proposition 2 we have

$$\frac{d}{dx}H_0 = \gamma_A([\frac{d}{dx}a_{i+j}]) \ .$$

Using $FI(i+j)$,

$$\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([FI(i+j)]) \quad = \quad & 2(x-1)\gamma_A([(i+j)a_{i+j+2}]) \\
& +(x-6)(x-2)\gamma_A([(i+j)a_{i+j+1}]) \\
& +(3x^2-2x+4)\gamma_A([a_{i+j+1}]) \\
& -4(x^2+4)\gamma_A([(i+j)a_{i+j}]) \\
& -(3x^3+18x^2-20x+24)\gamma_A([a_{i+j}]) \\
& +8(x-1)^2\gamma_A([c_{i+j}]) \\
& -32(x-1)^2\gamma_A([c_{i+j-1}]) \ .
\end{aligned}$$

The values for $\gamma_A(*)$ from Table 2 give

$$\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([FI(i+j)]) \quad = \quad & 2(x-1)(2nH_2-2(n-1)H_{1^2}) \\
& +(x-6)(x-2)2nH_1 \\
& +(3x^2-2x+4)H_1 \\
& -4(x^2+4)n(n+1)H_0 \\
& -(3x^3+18x^2-20x+24)(n+1)H_0 \\
& +8(x-1)^2(2n+1)H_0 \ .
\end{aligned}$$

Now using the expressions in (27) and (28) for H_2 and H_{1^2} in terms of H_1, H_0, we obtain $\frac{d}{dx}H_0$ as

$$Q\frac{d}{dx}H_0 = Q_0H_0 + Q_1H_1 \tag{29}$$

where

$$\begin{aligned}
Q \quad &= \quad (x-2)(x+2)(2nx+x+2)(2nx+3x+2) \ , \\
Q_0 \quad &= \quad -(n+1)\Big(16x^2n^3+4x^3n^2+48x^2n^2+32xn^2+8x^3n \\
& \qquad +36x^2n+80xn+16n+3x^3+12x^2+12x+48\Big) \ , \\
Q_1 \quad &= \quad (2n+3)\Big(4n^2x^2+4nx^2+x^2+8nx+4\Big) \ .
\end{aligned} \tag{30}$$

7.2 The derivative of H_1

To differentiate H_1 we use the expression

$$H_1 = \gamma_A([a_{i+j+1}])$$

from Table 2. From Proposition 2 we have

$$\frac{d}{dx}H_1 = \gamma_A([a_{i+j+1}],[\frac{d}{dx}a_{i+j}]) + \gamma_A([\frac{d}{dx}a_{i+j+1}]) \ .$$

Therefore, to compute $\frac{d}{dx}H_1$

$$\gamma_A([a_{i+j+1}],[FI(i+j)]) \ \text{ and } \ \gamma_A([FI(i+j+1)])$$

are needed. For the first one of these

$$\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([a_{i+j+1}],[FI(i+j)]) \quad = \quad & 2(x-1)\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+2}]) \\
& +(x-6)(x-2)\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+1}]) \\
& +(3x^2-2x+4)\gamma_A([a_{i+j+1}],[a_{i+j+1}]) \\
& -4(x^2+4)\gamma_A([a_{i+j+1}],[(i+j)a_{i+j}]) \\
& -(3x^3+18x^2-20x+24)\gamma_A([a_{i+j+1}],[a_{i+j}]) \\
& +8(x-1)^2\gamma_A([a_{i+j+1}],[c_{i+j}]) \\
& -32(x-1)^2\gamma_A([a_{i+j+1}],[c_{i+j-1}]) \ .
\end{aligned}$$

Using the entries in Table 3 for the $\gamma_A([a_{i+j+1}],*)$ computations, we get

$$\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([a_{i+j+1}],[FI(i+j)]) \quad = \quad & 2(x-1)(2nH_{21}-2(2n-3)H_{1^3}) \\
& +(x-6)(x-2)2(2n-1)H_{1^2} \\
& +(3x^2-2x+4)2H_{1^2} \\
& -4(x^2+4)n(n-1)H_1 \\
& -(3x^3+18x^2-20x+24)nH_1 \\
& +8(x-1)^2((2n-1)H_1-(2n-1)(x+4)H_0) \\
& -32(x-1)^2(-2nH_0) \ .
\end{aligned}$$

For the term $\gamma_A([FI(i+j+1)])$, we obtain

$$\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([FI(i+j+1)]) \quad = \quad & 2(x-1)\gamma_A([(i+j)a_{i+j+3}]) \\
& +2(x-1)\gamma_A([a_{i+j+3}]) \\
& +(x-6)(x-2)\gamma_A([(i+j)a_{i+j+2}]) \\
& +2(8-5x+2x^2)\gamma_A([a_{i+j+2}]) \\
& -4(x^2+4)\gamma_A([(i+j)a_{i+j+1}]) \\
& -(40-20x+22x^2+3x^3)\gamma_A([a_{i+j+1}]) \\
& +8(x-1)^2\gamma_A([c_{i+j+1}]) \\
& -32(x-1)^2\gamma_A([c_{i+j}]) \ .
\end{aligned}$$

Using Table 2 this gives

$$
\begin{aligned}
(x-2)x(x+2)(3x+2)\gamma_A([FI(i+j+1)]) \quad = \quad & 2(x-1)(2nH_3 - 2(n-1)H_{21} + 2(n-2)H_{1^3}) \\
& +2(x-1)(H_3 - H_{21} + H_{1^3}) \\
& +(x-6)(x-2)(2nH_2 - 2(n-1)H_{1^2}) \\
& +2(8-5x+2x^2)(H_2 - H_{1^2}) \\
& -4(x^2+4)2nH_1 \\
& -(40-20x+22x^2+3x^3)H_1 \\
& +8(x-1)^2(2H_1 + 2n(x+4)H_0) \\
& -32(x-1)^2(2n+1)H_0 \ .
\end{aligned}
$$

Adding, we get

$$(x-2)x(x+2)(3x+2)\frac{d}{dx}H_1$$

as a combination of $H_3, H_{21}, H_{1^3}, H_2, H_{1^2}, H_1, H_0$. After that, we use the expressions (24)-(28) for $H_3, H_{21}, H_{1^3}, H_2$ and H_{1^2} and express $\frac{d}{dx}H_1$ as a linear combination of H_0, H_1 as

$$U\frac{d}{dx}H_1 = U_0H_0 + U_1H_1 \ . \tag{31}$$

We find

$$
\begin{aligned}
U &= (x-2)(x+2)(2nx+x+2)(2nx+3x+2) \ , \\
U_0 &= -2(n+1)\Big(16x^2n^4 + 8x^3n^3 + 72x^2n^3 + 32xn^3 \\
&\quad +28x^3n^2 + 116x^2n^2 + 112xn^2 + 16n^2 + 26x^3n + 86x^2n \\
&\quad +104xn + 56n + 7x^3 + 22x^2 + 20x + 56\Big) \ , \\
U_1 &= \Big(16x^2n^4 + 4x^3n^3 + 64x^2n^3 + 32xn^3 + 12x^3n^2 \\
&\quad +92x^2n^2 + 80xn^2 + 16n^2 + 11x^3n + 56x^2n + 44xn + 32n \\
&\quad +3x^3 + 10x^2 - 4x + 24\Big) \ .
\end{aligned} \tag{32}
$$

The explicit polynomials in (30) and (32) are the coefficients of the system of differential equations (11).

Differentiating both sides of (29) and substituting the expansions of $\frac{d}{dx}H_0$ and $\frac{d}{dx}H_1$ in terms of H_0 and H_1, we obtain

$$R\frac{d^2}{dx^2}H_0 = R_0H_0 + R_1H_1 \ , \tag{33}$$

where

$$
\begin{aligned}
R &= (x-2)^2(x+2)^2(2nx+x+2)(2nx+3x+2) \ , \\
R_0 &= (n+1)\Big(4n^3x^4 + 16n^2x^4 + 19nx^4 + 6x^4 + 32n^3x^3 + 96n^2x^3 \\
&\quad +64nx^3 + 18x^3 - 48n^3x^2 + 240nx^2 + 48x^2 + 128n^3x + 288n^2x \\
&\quad +160nx + 264x + 128n^2 + 208n - 96\Big) \ , \\
R_1 &= -2(2n+3)\Big(4n^2x^3 + 4nx^3 + x^3 - 4n^2x^2 + 8nx^2 - x^2 \\
&\quad +16n^2x + 8nx + 12x + 16n - 4\Big) \ .
\end{aligned} \tag{34}
$$

From (30) and (34), we find that $Q_1 R, R_1 Q$, and $R_1 Q_0 - Q_1 R_0$ in

$$Q_1 R \frac{d^2}{dx^2} H_0 - R_1 Q \frac{d}{dx} H_0 + (R_1 Q_0 - Q_1 R_0) H_0 = 0 \tag{35}$$

have GCD

$$(2n+3)(x-2)(x+2)(2nx+x+2)(2nx+3x+2) \ .$$

Dividing through (35) by this and defining S_2, S_1, S_0 as the resulting quotients, we obtain the second order differential equation satisfied by H_0. We record this in the following theorem.

Theorem 2 *Suppose the polynomials $a_k(x)$ and the $(n+1) \times (n+1)$ Hankel determinant $H_0 = H_0(n, x)$ are as defined in (5) and (6). Then*

$$S_2 \frac{d^2}{dx^2} H_0 + S_1 \frac{d}{dx} H_0 + S_0 H_0 = 0 \ , \tag{36}$$

where

$$\begin{aligned}
S_2 &= (x-2)(x+2)(4n^2x^2 + 4nx^2 + x^2 + 8nx + 4) \ , \\
S_1 &= 2(4n^2x^3 + 4nx^3 + x^3 - 4n^2x^2 + 8nx^2 - x^2 + 16n^2x + 8nx + 12x + 16n - 4) \ , \\
S_0 &= -n(n+1)(4n^2x^2 + 4nx^2 + x^2 + 8nx - 8x + 36) \ .
\end{aligned}$$

8 Evaluation at special points

At this point we have enough information to evaluate $H_0(x)$ at special points x without making use of the differential equation (36) itself.

Using the notation that incorporates the sizes of the matrices involved, we recall the following general result on Hankel determinants proved in [2]:

Proposition 3

$$H_0(n-1, x) H_0(n+1, x) = H_0(n, x) H_2(n, x) + H_0(n, x) H_{1^2}(n, x) - H_1(n, x)^2 \ . \tag{37}$$

8.1 Specialization at $x = 2$

At $x = 2$, the derivative expression in (29) gives

$$-2(n+1)(3 + 6n + 2n^2) H_0 + (1 + 4n + 2n^2) H_1 = 0 \ .$$

From equations (27) and (28) at $x = 2$,

$$\begin{aligned}
(n+2) H_2 &= -(n+1)(n+4)(2n+5) H_0 + (13 + 11n + 2n^2) H_1 \ , \\
(n+1) H_{1^2} &= -n(n+1)(2n+5) H_0 + n(2n+3) H_1 \ .
\end{aligned}$$

Therefore at $x = 2$ we can write (37) as

$$H_0(n-1, 2) H_0(n+1, 2) = \frac{(2n^2 - 1)(7 + 8n + 2n^2)}{(1 + 4n + 2n^2)^2} H_0(n, 2)^2 \ .$$

This is a recursion in $H_0(n,2)/H_0(n-1,2)$ with $H_0(0,2)=1$, $H_0(1,2)=-7$. Solving, we find

$$H_0(n,2) = (-1)^n(2n^2+4n+1) \ .$$

At $x=2$, the entries of the determinant in (39) specialize to

$$a_k(2) = 4^{k+1} - \binom{2k+3}{k+1}. \tag{38}$$

The evaluation of the corresponding Hankel determinant is as follows:

Corollary 1 *Suppose $a_k(x)$ is as defined in (5). Then*

$$H_0(n,2) = \det\left[a_{i+j}(2)\right]_{0\le i,j\le n} = (-1)^n(2n^2+4n+1) \ . \tag{39}$$

8.2 Specialization at $x=-2$

At $x=-2$ the expression for the derivative in (29) gives

$$-2(n+1)(3-4n+6n^2+4n^3)H_0 + (2n+3)(1+2n^2)H_1 = 0 \ .$$

Again from equations (27) and (28) we obtain at $x=-2$,

$$\begin{aligned} (n+1)H_2 &= -(n-1)(n+1)(2n+5)H_0 + (n+2)(2n+1)H_1 \ , \\ nH_{1^2} &= (2-4n-5n^2-2n^3)H_0 + (n+1)(2n-1)H_1 \ . \end{aligned}$$

Therefore we can use (37) at $x=-2$ and write

$$H_0(n-1,-2)H_0(n+1,-2) = \frac{(2n+1)(2n+5)(3-4n+2n^2)(3+4n+2n^2)}{(2n+3)^2(1+2n^2)^2} H_0(n,-2)^2 \ .$$

This is a recursion in $H_0(n,-2)/H_0(n-1,-2)$ with $H_0(0,-2)=1$, $H_0(1,-2)=5$, which can be solved to give the simple product evaluation

$$H_0(n,-2) = \frac{1}{3}(2n+3)(1+2n^2) \ . \tag{40}$$

Therefore

Corollary 2 *Suppose $a_k(x)$ is as defined in (5). Then*

$$\det\left[a_{i+j}(-2)\right]_{0\le i,j\le n} = \frac{1}{3}(2n+3)(1+2n^2) \ . \tag{41}$$

The entries in (41) do not seem to have as simple an expression as the $a_k(2)$ given in (38), although from the alternate expression for the generating function of the a_k, we get the generating function of these numbers as

$$\frac{1}{1-y-4y^2-yt} = \frac{2}{1+\sqrt{1-4y}-2y(1+4y)} \ ,$$

where t is the generating function of the Catalan numbers, as in the proof of Lemma 1 in Appendix II.

9 The differential equation solution

Natural candidates for the expansion of the power series solution to the differential equation (36) are around $x = 2$ and $x = -2$.

9.1 Solution at $x = 2$

Putting

$$H_0(x) = \sum_{k=0}^{\infty} b_k (x-2)^k \ ,$$

we find that the b_k satisfy

$$\begin{aligned} 16k(2k+1)(2n^2+4n+1)b_k \quad = \quad & 8\Big(2n^4+6n^3-10k^2n^2+18kn^2-n^2-16k^2n+26kn \\ & -7n-3k^2+4k-1\Big)b_{k-1}+2\Big(8n^4+20n^3-16k^2n^2 \\ & +60kn^2-46n^2-20k^2n+68kn-58n-4k^2+15k-14\Big)b_{k-2} \\ & +(n+3-k)(k+n-2)(2n+1)^2 b_{k-3} \end{aligned}$$

for $k \geq 2$ with $b_k = 0$ for $k < 0$. From (36), we get

$$b_1 = \frac{n(n+1)(2n^2+4n+3)}{6(2n^2+4n+1)} b_0 \ , \tag{42}$$

and therefore each b_k is a multiple of b_0. It can then be proved by induction that

$$b_k = \frac{2n^2+4n+2k^2+1}{(2n^2+4n+1)(2k+1)} \binom{n+k}{2k} b_0 \ .$$

Since $b_0 = H_0(2)$,

$$H_0(x) = \frac{H_0(2)}{2n^2+4n+1} \sum_{k=0}^{n} \frac{2n^2+4n+2k^2+1}{2k+1} \binom{n+k}{2k} (x-2)^k \ .$$

The determinants at $x = 2$ have the simple evaluation we already found in (39), so that

$$H_0(x) = (-1)^n \sum_{k=0}^{n} \frac{2n^2+4n+2k^2+1}{2k+1} \binom{n+k}{2k} (x-2)^k \ . \tag{43}$$

The coefficients in (43) can be rewritten as binomial coefficients to obtain the expansion given in (7) of Theorem 1 at $x = 2$. Note that the alternate notation $H_0(x)$ in (43) (subscript indicating the zero partition), is the $(n+1) \times (n+1)$ determinant denoted by $H_n(x)$ in Theorem 1.

Using the expansion at $x = 2$, we can immediately write down the generating function of the $H_n(x)$. We omit the proof of the following result.

Theorem 3 *Suppose $a_k(x)$ is as defined in (5). Then*

$$\sum_{n=0}^{\infty} H_n(x) t^n = \frac{1-t+t^2-t^3-xt-3xt^2}{(1+xt+t^2)^2} \ . \tag{44}$$

9.2 Solution at $x = -2$

For the solution around $x = -2$, put

$$H_0(x) = \sum_{k=0}^{\infty} d_k (x+2)^k \; .$$

We find that the d_k satisfy

$$\begin{aligned} 16k(2k+3)(2n^2+1)d_k &= -8\Big(2n^4 + 2n^3 - 10k^2n^2 + 10kn^2 + 7n^2 - 4k^2n + 6kn \\ &\quad +5n - 3k^2 + 2k + 1\Big)d_{k-1} + 2\Big(8n^4 + 12n^3 - 16k^2n^2 + 52kn^2 \\ &\quad -30n^2 - 12k^2n + 44kn - 34n - 4k^2 + 13k - 10\Big)d_{k-2} \\ &\quad +(k-n-3)(k+n-2)(2n+1)^2 d_{k-3} \end{aligned}$$

for $k \geq 2$ with $d_k = 0$ for $k < 0$. From (36), we get

$$d_1 = -\frac{n(1+n)(7+2n^2)}{10(1+2n^2)} d_0 \; , \tag{45}$$

and therefore each d_k is a multiple of d_0. It can be proved by induction that

$$d_k = (-1)^k \frac{3(2n^2+2k^2+4k+1)}{(1+2n^2)(2k+1)(2k+3)} \binom{n+k}{2k} d_0 \; .$$

Since $d_0 = H_0(-2)$,

$$H_0(x) = \frac{H_0(-2)}{1+2n^2} \sum_{k=0}^{n} (-1)^k \frac{3(2n^2+2k^2+4k+1)}{(2k+1)(2k+3)} \binom{n+k}{2k} (x+2)^k \; .$$

Using the evaluation of the determinants at $x = -2$ from (40) we obtain

$$H_0(x) = (2n+3) \sum_{k=0}^{n} (-1)^k \frac{2n^2+2k^2+4k+1}{(2k+1)(2k+3)} \binom{n+k}{2k} (x+2)^k \; . \tag{46}$$

The coefficients in (46) can be rewritten in the form (7) of Theorem 1. Again, note that $H_0(x)$ in (46) is the $(n+1) \times (n+1)$ determinant $H_n(x)$ in Theorem 1.

Evaluating (7) and (8) at $x = 0$ we obtain the expressions

$$\begin{aligned} \det\left[\binom{2(i+j)+2}{i+j}\right]_{0 \le i,j \le n} &= (-1)^n \sum_{k=0}^{n} \left[(2n+3)\binom{n+k}{2k+1} + (2k+1)\binom{n+k+1}{2k+1}\right](-2)^k & (47) \\ &= \sum_{k=0}^{n} \left[(n+k+1)\binom{n+k}{2k} + (2n+4k+1)\binom{n+k}{2k+1} + 8(k+1)\binom{n+k+1}{2k+3}\right](-2)^k & (48) \end{aligned}$$

which are alternate ways of writing the known evaluation of this determinant from (9).

As another corollary of Theorem 1, we have the following Hankel determinant evaluation at $x = 1$, which depends on the residue class of n modulo 3:

Corollary 3

$$\det\left[\binom{2(i+j)+3}{i+j}\right]_{0 \le i,j \le n} = \begin{cases} \frac{1}{3}(2n+3) & \textit{if } n \equiv 0 \pmod 3 \; , \\ -\frac{4}{3}(n+2) & \textit{if } n \equiv 1 \pmod 3 \; , \\ \frac{1}{3}(2n+5) & \textit{if } n \equiv 2 \pmod 3 \; . \end{cases} \tag{49}$$

Proof Since
$$a_k(1) = \binom{2k+3}{k} \tag{50}$$
the determinant is simply $H_0(n,1)$. We use the expression for the determinant (7) of Theorem 1 evaluated at $x = 1$. Putting $n = 3m$, $n = 3m+1$ and $n = 3m+2$ for the three residue classes modulo 3, the Corollary is a consequence of the resulting binomial identities

$$\begin{aligned}
2m+1 &= (-1)^m \sum_{k=0}^{3m} \left[(6m+3)\binom{3m+k}{2k+1} + (2k+1)\binom{3m+k+1}{2k+1} \right] (-1)^k && (51)\\
4(m+1) &= (-1)^m \sum_{k=0}^{3m+1} \left[(6m+5)\binom{3m+k+1}{2k+1} + (2k+1)\binom{3m+k+2}{2k+1} \right] (-1)^k && (52)\\
2m+3 &= (-1)^m \sum_{k=0}^{3m+2} \left[(6m+7)\binom{3m+k+2}{2k+1} + (2k+1)\binom{3m+k+3}{2k+1} \right] (-1)^k && (53)
\end{aligned}$$

which can be proved by making use of the generating function given in Theorem 3 at $x = 1$. •

9.3 Solution at $x = 0$

The power series solution to (36) around $x = 0$ is more difficult to derive directly. For a_k and the $H_n(x)$ as defined in (5) and (6) this expansion is given by

$$H_n(x) = \sum_{k=0}^{n} (-1)^{n(n-1)/2+k(k-1)/2+kn} \left(2k + (-1)^{n-k}\right) \frac{(n - \lfloor \frac{n-k+1}{2} \rfloor)!}{\lfloor \frac{n-k}{2} \rfloor! k!} x^k \ . \tag{54}$$

We are grateful to the anonymous referee for pointing out the above explicit form of the determinant around $x = 0$. This expansion is an immediate consequence of the generating function for the determinants at arbitrary x that we have provided in (44).

Following the route of the proofs of the cases $x = 2$ and $x = -2$, one would put

$$H_n(x) = \sum_{k=0}^{\infty} e_k x^k \ ,$$

and show that the e_k satisfy the recursion

$$\begin{aligned}
16k(k-1)e_k &= -8(k-1)(4kn - 12n + 1)e_{k-1} \\
&\quad -4\left(4n^2k^2 + 4nk^2 - 28n^2k - 24nk - 6k + 49n^2 + 41n + 12\right) e_{k-2} \\
&\quad -2\left(4n^3 + 4kn^2 - 12n^2 - 4k^2n + 20kn - 28n + k - 3\right) e_{k-3} \\
&\quad +(k-n-4)(k+n-3)(2n+1)^2 e_{k-4}
\end{aligned}$$

for $k \geq 2$ with $e_k = 0$ for $k < 0$. In this case each e_k is a function of e_0 and e_1. We know e_0 explicitly by (9). However in this case a relationship similar to (42) and (45) of the $x = 2$ and $x = -2$ cases does not drop out of the differential equation to give a similar relation between e_0 and e_1. This is because the special values of x that kills off the second derivative term in Theorem 2 are $x = \pm 2$. An alternate approach is to show directly that the coefficient of x^k in (54) satisfies the recurrence for the e_k, but again this would fall back on the already proved expansions of Theorem 1 for the value of the derivative at $x = 0$.

10 Zeros of $H_0(n,x)$

The determinants $H_0(n,x)$ of Theorem 1 are not orthogonal polynomials. But they satisfy a recurrence relation with polynomial coefficients involving three consecutive terms of the sequence as follows:

Corollary 4

$$(2+(2n+3)x)^2 H_0(n+2,x) + x(4+4(2n+3)x+(2n+3)(2n+5)x^2)H_0(n+1,x) \\ +(2+(2n+5)x)^2 H_0(n,x) = 0 \ . \tag{55}$$

Proof The recurrence relation can be verified by making use of the explicit form of $H_0(n,x)$ from Theorem 1. •

Table 1 gives a list of the zeros of $H_0(1,x)$ through $H_0(7,x)$. The zeros are real and interlacing. It is possible that the polynomials $H_0(n,x)$ can be obtained from an orthogonal family by a suitable transformation.

−0.333						
−0.358	0.558					
−0.601	−0.194	1.224				
−1.083	−0.207	0.324	1.522			
−1.367	−0.351	−0.137	0.815	1.678		
−1.540	−0.746	−0.146	0.229	1.127	1.768	
−1.651	−1.028	−0.246	−0.107	0.608	1.333	1.825

Table 1: Zeros of the Hankel determinants $H_0(1,x)$ through $H_0(7,x)$ of Theorem 1.

A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ with $\deg P_n = n$ is called a *Sturm sequence* on an open interval (a,b) if P_n has exactly n simple real zeros in (a,b), and for every $n \geq 1$, zeros of $P_n(x)$ and $P_{n+1}(x)$ strictly interlace.

Theorem 4 *Suppose a_k and the $H_0(n,x)$ are as defined in (5) and (6). Then $\{H_0(n,x)\}_{n\geq 0}$ is a Sturm sequence on* $(-2,2)$.

Proof

Consider the two expansions of $H_0(n,x)$ in (43) and (46). The first one of these implies that $(-1)^n H_0(n,x) > 0$ for $x \geq 2$, and the second one implies that $H_0(n,x) > 0$ for $x \leq -2$. Therefore the zeros of $H_0(n,x)$ are contained in $(-2,2)$.

We next prove that like orthogonal polynomials, $H_0(n,x)$ has n distinct real zeros and the zeros of $H_0(n,x)$ lie strictly between the zeros of $H_0(n+1,x)$. This interlacing property is a consequence of the form of the recursion (55)

$$\alpha^2 H_0(n+2,x) + x\beta H_0(n+1,x) + \gamma^2 H_0(n,x) = 0 \tag{56}$$

where $\beta > 0$ for every x and n. We use induction on n. For any two consecutive zeros r_1, r_2 of $H_0(n+1, x)$ the induction hypothesis implies that $H_0(n, r_1)$ and $H_0(n, r_2)$ have opposite signs. Therefore from the recursion, $H_0(n+2, r_1)$ and $H_0(n+2, r_2)$ also have opposite signs and so $H_0(n+2, x)$ has at least one zero in the interval (r_1, r_2). This accounts for $\geq n$ zeros of $H_0(n+2, x)$. Let $\delta_2 < 2$ be the largest zero of $H_0(n+1, x)$. By the induction hypothesis, $H_0(n, x)$ has no zeros on $[\delta_2, \infty)$. Therefore its sign at $x = \delta_2$ is the same as its sign at $x = 2$, which is $(-1)^n$. But the sign of $H_0(n+2, x)$ is also $(-1)^n$ at $x = 2$, but opposite of the sign of $H_0(n, x)$ at $x = \delta_2$ by (56). This forces $H_0(n+2, x)$ to change sign and have a zero in $(\delta_2, 2)$. By a counting argument, $H_0(n+2, x)$ has to have another zero in $(-2, \delta_1)$ where δ_1 is the smallest zero of $H_0(n+1, x)$. •

11 Discussion, patterns and conjectures

We introduced a class of multilinear operators γ acting on tuples of matrices to take the place of the trace method of our earlier calculations. This approach to evaluate Hankel determinants is easier to work with: the γ-operators are easier to differentiate, and they do not produce the extraneous nonlinear terms. In the $(2,2)$-case that we have covered in detail, we have also obtained numerical evaluations at special points as a byproduct. Furthermore we saw that the resulting polynomials have intriguing properties.

Even though the application of the γ-operator reduces the calculations involved in almost product evaluations of Hankel determinants considerably, there are still stumbling blocks in the general $(2, r)$-case, and other cases that differ little from this. We consider a few of these determinants and conjecture closed forms for the evaluations.

Corollary 3 is just one example of a strange pattern that holds for Hankel determinants where the entries are the polynomials $a_k^{(2,r)}(x)$ defined in (3). Taking $x = 0$, let

$$a_k = \binom{2k+r}{k},$$

parametrized by $r \geq 0$. For notational simplicity, define

$$F(n, r) = \det [a_{i+j}]_{0 \leq i,j \leq n}.$$

Then the evaluation (49) in Corollary 3 can be written as

$$\begin{aligned} F(3m, 3) &= 2m+1 \\ F(3m+1, 3) &= -4(m+1) \\ F(3m+2, 3) &= 2m+3\,. \end{aligned}$$

As an example, consider the following evaluations for the case $r = 7$:

$$\begin{aligned} F(7m, 7) &= (2m+1)^3 \\ F(7m+1, 7) &= (m+1)(2m+1)^2(9604m^3+9604m^2-1323m-2340)/90 \\ F(7m+2, 7) &= -(m+1)^2(2m+1)(19208m^3+67228m^2+70854m+23445)/45 \\ F(7m+3, 7) &= 64(m+1)^3 \end{aligned}$$

$$
\begin{aligned}
F(7m+4,7) &= (m+1)^2(2m+3)(19208m^3+48020m^2+32438m+3015m)/45 \\
F(7m+5,7) &= -(m+1)(2m+3)^2(9604m^3+48020m^2+75509m+38110)/90 \\
F(7m+6,7) &= (2m+3)^3 \ .
\end{aligned}
$$

These evaluations have been verified for a significant range of m. This unusual set of formulas is typical of a complex pattern of evaluations of $F(n,r)$ that continues with several unexpected dependencies on the value of n modulo r and on r modulo 4. For example, if r is odd then there is strong experimental evidence that

$$F(rm,r) = F(rm-1,r) = (2m+1)^{(r-1)/2} \ .$$

When we consider even r there is another twist to take into account. Experimental evidence tells us that

$$F(rm,r) = F(rm-1,r) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod 4 \ , \\ (-1)^m & \text{if } r \equiv 2 \pmod 4 \ . \end{cases}$$

Another interesting pattern we observe is the following for odd r:

$$F(rm+(r-1)/2,r) = 2^{r-1}(m+1)^{(r-1)/2} \ .$$

For even r there is also a simple pattern of this type:

$$F(rm+r/2,r) = \begin{cases} (-1)^{r/4+1}(2r(m+1))^{r/2-1} & \text{if } r \equiv 0 \pmod 4 \ , \\ \\ (-1)^{(r+2)/4+m}(2r(m+1))^{r/2-1} & \text{if } r \equiv 2 \pmod 4 \ . \end{cases}$$

In addition to these nice evaluations there are many that are not so simple. For example the $F(rm+1,r)$ becomes more and more complex as r increases. For $r=5$

$$F(5m+1,5) = -(m+1)(2m+1)(50m+39)/3.$$

For $r=7$ the evaluation contains a cubic factor:

$$F(7m+1,7) = (m+1)(2m+1)^2(9604m^3+9604m^2-1323m-2340)/90$$

and when $r=9$ the evaluation contains a quartic factor:

$$F(9m+1,9) = -(m+1)(2m+1)^3(3m+2)(52488m^4+69984m^3+22518m^2+1674m+1505)/70 \ .$$

We suspect that this irreducible factor keeps gaining a degree when r is increased by 2.

These conjectures appear to be difficult to prove in their full generality using either the methods described in Krattenthaler [6, 7] or with the methods of the present paper. For any fixed r, the methods of this paper might apply but it is hard to see how to approach the problem when r is left as a parameter.

Further experimental evidence suggests that the determinants

$$\det\left[\sum_{k=0}^{i+j}\binom{2i+2j+r-2k}{i+j-k}x^k\right]_{0\le i,j\le n}$$

satisfy second order differential equations. However as r gets larger the differential equations and the first and second identities of our method become increasingly complex. We mention that there are also difficulties in evaluating the family of determinants

$$\det\left[\sum_{k=0}^{i+j}\binom{2i+2j+r-k}{i+j-k}x^k\right]_{0\le i,j\le n}. \tag{57}$$

For this family, the order of the differential equation for the determinant seems to increase with r. When $r=4$, for example, experiments suggest that (57) satisfies a fourth order differential equation.

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References

[1] Ö. Eğecioğlu, T. Redmond and C. Ryavec. From a Polynomial Riemann Hypothesis to Alternating Sign Matrices. *The Electronic Journal of Combinatorics*, Volume 8 (1), (2001), #R36.

[2] Ö. Eğecioğlu, T. Redmond and C. Ryavec. Almost Product Evaluation of Hankel Determinants, Electronic J. Combin. 15, (2008), No. 1, #R6 (58 pages).

[3] Ö. Eğecioğlu, T. Redmond and C. Ryavec. Evaluation of a Special Hankel Determinant of Binomial Coefficients, Proc. Fifth Colloquium on Mathematics and Computer Science, *Discrete Mathematics and Theoretical Computer Science*, AI, 2008, pp. 251–268.

[4] I. Gessel and G. Xin. The generating function of ternary trees and continued fractions, Electronic J. Combin. 13, (2006), no. 1, R53.

[5] I. Gessel, personal communication, 2007.

[6] C. Krattenthaler. Advanced determinant calculus, Seminaire Lotharingien Combin. 42 ("The Andrews Festschrift") (1999), Article B42q, 67 pp.

[7] C. Krattenthaler. Advanced determinant calculus: a complement, *Linear Algebra Appl.* 411 (2005), pp. 68–166.

[8] C. Krattenthaler. personal communication, 2007.

[9] I. G. Macdonald. Symmetric Functions and Hall Polynomials, 2nd edition. Clarendon Press, Oxford, 1995

[10] T. Muir, The Theory of Determinants in the Historical Order of Development. in 4 vols., London: Macmillan, 1906–1923.

[11] T. Muir, A Treatise on the Theory of Determinants. Revised and Enlarged by William H. Metzler. New York: Dover, 1960

[12] G. Pólya and G. Szegö. Problems and Theorems in Analysis, Vol. I, translated by D. Aeepli, Springer-Verlag New York 1972, Part 3, Problem 216, pp. 146, 349.

[13] K. Wegschaider. *Computer Generated Proofs of Binomial Multi-sum Identities,* Ph.D. thesis, Research Institute for Symbolic Computation, Institut für Mathematik, Johannes Kepler Universität, Linz, May 1997.

[14] D. Zeilberger. `MultiZeilberger` Maple package, or after extracting coefficients `SumTools[Hypergeometric]` with command `ZeilbergerRecurrence`, or the package `EKHAD` can be used, all available from Doron Zeilberger's website.

12 Appendix I

The results in this Appendix apply to general Hankel matrices. We let $\chi(S)$ denote the indicator of the statement S: $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false.

12.1 Properties of the γ-operator

Proposition 1

For $m \leq n+1$,

$$\gamma_A(X_1, \ldots, X_m) = \sum_{S,\sigma} \det(A_{S,\sigma}) \tag{58}$$

where the summation is over all subsets S of $\{0, 1, \ldots, n\}$ with $|S| = m$ and all permutation σ of $\{1, 2, \ldots, m\}$.

Proof Expand

$$\det(A + t_1 X_1 + t_2 X_2 + \cdots + t_m X_m)$$

by columns (or rows) using the linearity of the determinant to obtain

$$\det(A + t_1 X_1 + t_2 X_2 + \cdots + t_m X_m) = t_1 t_2 \cdots t_m \sum_{S,\sigma} \det(A_{S,\sigma}) \tag{59}$$

where $A_{S,\sigma}$ is as defined in Definition 2. The proof follows by applying $\partial_{t_1} \partial_{t_2} \cdots \partial_{t_m}$ and putting $t_1 = \cdots = t_m = 0$. •

Proposition 2

For $m \leq n$,

$$\frac{d}{dx}\gamma_A(X_1, \ldots, X_m) = \gamma_A(\frac{d}{dx}A, X_1, \ldots, X_m) + \sum_{j=1}^{m} \gamma_A(X_1, \ldots, X_{j-1}, \frac{d}{dx}X_j, X_{j+1}, \ldots, X_m)$$

Proof By Proposition 1 and the expression in (59),

$$\begin{aligned} \frac{d}{dx}\gamma_A(X_1, \ldots, X_m) &= \sum_{S,\sigma} \frac{d}{dx} \det(A_{S,\sigma}) \\ &= \sum_{S,\sigma} \det(A_{S,\sigma}) \mathrm{Tr}(A_{S,\sigma}^{-1} \frac{d}{dx} A_{S,\sigma}) \end{aligned}$$

Let $B = A_{S,\sigma}$. By Cramer's rule,

$$\mathrm{Tr}(B^{-1}\frac{d}{dx}B) = \frac{1}{\det(B)} \sum_{j=0}^{n} \det(B_j)$$

where B_j is obtained from B by replacing the j-th column of B by its derivative. In terms of the matrix A, let $A_{S,\sigma,j}$ denote this matrix.

Therefore

$$\begin{aligned}
\frac{d}{dx}\gamma_A(X_1,\ldots,X_m) &= \sum_{S,\sigma}\sum_{j=0}^{n}\det(A_{S,\sigma,j}) \\
&= \sum_{j=0}^{n}\sum_{S,\sigma}\chi(j\notin S)\det(A_{S,\sigma,j}) + \sum_{j=0}^{n}\sum_{S,\sigma}\chi(j\in S)\det(A_{S,\sigma,j}) \\
&= \gamma_A(\frac{d}{dx}A, X_1,\ldots,X_m) + \sum_{j=1}^{m}\gamma_A(X_1,\ldots,X_{j-1},\frac{d}{dx}X_j, X_{j+1},\ldots,X_m)
\end{aligned}$$

•

12.2 Expansion of the convolution matrices

The expansion of the convolution matrices $[c_{i+j+k}]$ for $k \geq -1$ are as follows:

Proposition 4 *Suppose the convolution polynomial c_n is as defined in Definition 3. Then*

$$\begin{aligned}
[c_{i+j+k}]_{0\leq i,j\leq n} &= \sum_{p=0}^{n+k} a_p[a_{i+j+k-p}\chi(j\geq p-k)]_{0\leq i,j\leq n} \\
&\quad + \sum_{p=0}^{n-1} a_p[a_{i+j+k-p}\chi(i>p)]_{0\leq i,j\leq n}
\end{aligned} \tag{60}$$

Proof The (i,j)-th entry of the matrix on the right-hand side of (60) is

$$\sum_{p=0}^{n+k} a_p a_{i+j+k-p}\chi(j\geq p-k) + \sum_{p=0}^{n-1} a_p a_{i+j+k-p}\chi(i>p)$$

The upper limit of the sums need not go past $i+j+k$. In the second sum, replace p by $i+j+k-p$ and rearrange the indices. We get

$$\begin{aligned}
\sum_{p=0}^{i+j+k} a_p a_{i+j+k-p}\chi(j\geq p-k) + \sum_{p=0}^{i+j+k} a_p a_{i+j+k-p}\chi(j<p-k) &= \sum_{p=0}^{i+j+k} a_p a_{i+j+k-p} \\
&= c_{i+j+k}
\end{aligned}$$

•

Below are a few examples of the expansion of the convolution matrices obtained from (60). For $k=-1$,

$$\begin{aligned}
[c_{i+j-1}]_{0\leq i,j\leq 2} &= a_0\begin{bmatrix} 0 & a_0 & a_1 \\ 0 & a_1 & a_2 \\ 0 & a_2 & a_3 \end{bmatrix} + a_1\begin{bmatrix} 0 & 0 & a_0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} \\
&\quad + a_0\begin{bmatrix} 0 & 0 & 0 \\ a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{bmatrix} + a_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_0 & a_1 & a_2 \end{bmatrix}
\end{aligned}$$

For $k = 0$,

$$\begin{aligned}[c_{i+j}]_{0\le i,j\le 2} \quad &= \quad a_0 \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} + a_1 \begin{bmatrix} 0 & a_0 & a_1 \\ 0 & a_1 & a_2 \\ 0 & a_2 & a_3 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 & a_0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} \\ &+ a_0 \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} + a_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix}\end{aligned}$$

For $k = 1$, the expansion is as given in (13).

13 Appendix II

In this Appendix we give the proofs of the statements needed for the $(2,2)$-case.

The proofs of Lemma 1, Lemma 2 are based on generating function manipulations, as given below. The first identity for the $(2,2)$-case is:

Lemma 1

$$
\begin{aligned}
&(x-2)x(x+2)(3x+2)\frac{d}{dx}a_n + \left(3x^3+18x^2-20x+24+4n\left(x^2+4\right)\right)a_n \\
&-\left(n(x-6)(x-2)+3x^2-2x+4\right)a_{n+1} - 2n(x-1)a_{n+2} \\
&-8(x-1)^2c_n + 32(x-1)^2c_{n-1} = 0
\end{aligned}
\tag{61}
$$

Proof From [12, 2], we have

$$f = f(x,y) = \sum_{k\geq 0} a_k(x)y^k = \frac{t^3}{(2-t)(1-xyt)} \ .$$

Here

$$t = \sum_{k\geq 0}\frac{(2k)!}{(k+1)!k!}y^k = 1+y+2y^2+5y^3+\cdots$$

satisfies

$$yt^2 = t-1 \ . \tag{62}$$

Using $\frac{d}{dy}t = t^2/(1-2yt)$ in the computation of $\frac{d}{dy}f$ and using the resulting expressions for $\frac{d}{dx}f$ and $f' = \frac{d}{dy}f$, we make the substitutions

$$
\begin{aligned}
\frac{d}{dx}a_n &\rightarrow \frac{d}{dx}f \\
a_n &\rightarrow f \\
na_n &\rightarrow yf' \\
a_{n+1} &\rightarrow (f-1)/y \\
na_{n+1} &\rightarrow y((f-1)/y)' \\
na_{n+2} &\rightarrow y((f-1-(4+x)y)/y^2)' \\
c_n &\rightarrow f^2 \\
c_{n-1} &\rightarrow yf^2
\end{aligned}
$$

in the left-hand side of (61). The resulting expression factors as

$$\frac{(t-1-yt^2)}{(t-2)^2y^2(1-2ty)(1-txy)^2}\Big(64x^2y^3t^5 - 128xy^3t^5 + 64y^3t^5 - 16x^2y^2t^5 + 32xy^2t^5 - 16y^2t^5$$

$$+20x^3y^3t^4-16x^2y^3t^4+16xy^3t^4-x^3y^2t^4-4x^2y^2t^4-12xy^2t^4+32y^2t^4+6x^2yt^4$$
$$+2xyt^4-8yt^4-32x^3y^3t^3+32x^2y^3t^3-6x^3y^2t^3-4x^2y^2t^3+40xy^2t^3-80y^2t^3$$
$$-4xt^3-4x^2yt^3-4xyt^3+8yt^3+4t^3+16x^3y^2t^2+48x^2y^2t^2-64xy^2t^2+4xt^2$$
$$-16xyt^2+16yt^2-4t^2-32x^2yt+32yt+16x-16\Big)$$

and therefore vanishes by (62). •

The second identity is:

Lemma 2

$$(nx+3x+2)a_{n+2}-(nx(x+6)+3x^2+16x+8)a_{n+1}+2x(x+2)(2n+5)a_n$$
$$+(x-1)(x-2)c_n-4(x-1)(x-2)c_{n-1}=0 \tag{63}$$

Proof Again passing to the generating functions, we find that the generating function of the left-hand side of (63) factors as

$$\frac{(t-1-yt^2)}{(t-2)^2y^2(1-2ty)(1-txy)^2}\Big(24xy^3t^5-8x^2y^3t^5-16y^3t^5+2x^2y^2t^5-6xy^2t^5+4y^2t^5$$
$$+8x^3y^3t^4+16x^2y^3t^4-x^3y^2t^4-6x^2y^2t^4-4xy^2t^4-8y^2t^4+xyt^4+2yt^4-8x^3y^3t^3$$
$$-16x^2y^3t^3-12x^2y^2t^3-16xy^2t^3+16y^2t^3-xt^3+2x^2yt^3+10xyt^3+4yt^3-2t^3+4x^3y^2t^2$$
$$+24x^2y^2t^2+32xy^2t^2+xt^2-4xyt^2-8yt^2+2t^2-8x^2yt-24xyt-16yt+4x+8\Big)$$

which again vanishes by (62). •

The third identity is:

Lemma 3

$$\sum_{j=0}^{n+2} w_{n,j}(x)a_{i+j}(x)=0 \tag{64}$$

for $i=0,1,\ldots,n$ where

$$w_{n,j}(x) = (-1)^{n-j}\left\{\frac{2(2n+5)}{2j+1}\binom{n+j+2}{2j}+\frac{(2n+3)(2n+5)}{2j+1}\binom{n+j+2}{2j}x\right.$$
$$\left.+\frac{(2n+3)(2n+5)}{2j+3}\binom{n+j+2}{2j+1}x^2\right\} \tag{65}$$

We do not give the proof of the third identity Lemma 3 but remark that once the weights are guessed, the proofs of the identities can be left to automatic binomial identity provers such as `MultiZeilberger` supplied by Doron Zeilberger (in Maple [14]), and `MultiSum` by Wegschaider (in

Mathematica [13]). The main step in finding the coefficients is interpolation and a symbolic algebra system (Mathematica in our case).

The weights in general can be found from the relation

$$w_{n,n+2}H_{21^k} + w_{n,n+1}H_{1^{k+1}} + w_{n,n-k}H_0 = 0 \tag{66}$$

which holds for $k = 0, 1, \ldots, n$. This can be seen by computing the determinant of the matrix obtained from $A = [a_{i+j}]_{0\le i,j\le n}$ by replacing column $n-k$ by column n, and column n by the zero vector written as sum of column vectors as indicated by the third identity. In the present case this is (64). Expanding, all but three determinants vanish, giving (66).

We use (66) to guess third identities in general. For instance with offset 2 (i.e. the vectors involved in the third identity are v_0 through v_{n+2}), it is possible to first guess $w_{n,n+2}, w_{n,n+1}, w_{n,n}$ by linear algebra, then use (66) to solve for $w_{n,n-k}$ and consequently find the candidate coefficients by interpolation.

14 Appendix III: Tables of γ-operator evaluations

The tables given in this Appendix apply to general Hankel matrices.

$$
\begin{aligned}
\gamma_A([a_{i+j}]) &= (n+1)H_0 \\
\gamma_A([a_{i+j+1}]) &= H_1 \\
\gamma_A([a_{i+j+2}]) &= H_2 - H_{1^2} \\
\gamma_A([a_{i+j+3}]) &= H_3 - H_{21} + H_{1^3} \\
\gamma_A([a_{i+j+4}]) &= H_4 - H_{31} + H_{21^2} - H_{1^4} \\
\gamma_A([a_{i+j+5}]) &= H_5 - H_{41} + H_{31^2} - H_{21^3} + H_{1^5} \\
\gamma_A([(i+j)a_{i+j}]) &= n(n+1)H_0 \\
\gamma_A([(i+j)a_{i+j+1}]) &= 2nH_1 \\
\gamma_A([(i+j)a_{i+j+2}]) &= 2nH_2 - 2(n-1)H_{1^2} \\
\gamma_A([(i+j)a_{i+j+3}]) &= 2nH_3 - 2(n-1)H_{21} + 2(n-2)H_{1^3} \\
\gamma_A([(i+j)a_{i+j+4}]) &= 2nH_4 - 2(n-1)H_{31} + 2(n-2)H_{21^2} - 2(n-3)H_{1^4} \\
\gamma_A([(i+j)a_{i+j+5}]) &= 2nH_5 - 2(n-1)H_{41} + 2(n-2)H_{31^2} - 2(n-3)H_{21^3} + 2(n-4)H_{1^5} \\
\gamma_A([c_{i+j-1}]) &= 0 \\
\gamma_A([c_{i+j}]) &= (2n+1)a_0H_0 \\
\gamma_A([c_{i+j+1}]) &= 2a_0H_1 + 2na_1H_0 \\
\gamma_A([c_{i+j+2}]) &= 2a_0H_2 - 2a_0H_{1^2} + 2a_1H_1 + (2n-1)a_2H_0 \\
\gamma_A([c_{i+j+3}]) &= 2a_0H_3 - 2a_0H_{21} + 2a_0H_{1^3} + 2a_1H_2 \\
&\quad -2a_1H_{1^2} + 2a_2H_1 + (2n-2)a_3H_0
\end{aligned}
$$

Table 2: $\gamma_A(*)$ computations.

$$
\begin{aligned}
\gamma_A([a_{i+j+1}],[a_{i+j}]) &= nH_1\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}]) &= 2H_{1^2}\\
\gamma_A([a_{i+j+1}],[a_{i+j+2}]) &= H_{21}-2H_{1^3}\\
\gamma_A([a_{i+j+1}],[a_{i+j+3}]) &= H_{31}-H_{2^2}-H_{21^2}+2H_{1^4}\\
\gamma_A([a_{i+j+1}],[a_{i+j+4}]) &= H_{41}-H_{32}-H_{31^2}+H_{2^21}+H_{21^3}-2H_{1^5}\\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j}]) &= n(n-1)H_1\\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+1}]) &= 2(2n-1)H_{1^2}\\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+2}]) &= 2nH_{21}-2(2n-3)H_{1^3}\\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+3}]) &= 2nH_{31}-2(n-1)H_{2^2}\\
&\qquad -2(n-1)H_{21^2}+2(2n-5)H_{1^4}\\
\gamma_A([a_{i+j+1}],[(i+j)a_{i+j+4}]) &= 2nH_{41}-2(n-1)H_{32}-2(n-1)H_{31^2}\\
&\qquad +2(n-2)H_{2^21}+2(n-2)H_{21^3}-2(2n-7)H_{1^5}\\
\gamma_A([a_{i+j+1}],[c_{i+j-1}]) &= -2na_0H_0\\
\gamma_A([a_{i+j+1}],[c_{i+j}]) &= (2n-1)a_0H_1-(2n-1)a_1H_0\\
\gamma_A([a_{i+j+1}],[c_{i+j+1}]) &= 4a_0H_{1^2}+2(n-1)a_1H_1-2(n-1)a_2H_0\\
\gamma_A([a_{i+j+1}],[c_{i+j+2}]) &= 2a_0H_{21}-4a_0H_{1^3}+4a_1H_{1^2}+(2n-3)a_2H_1-(2n-3)a_3H_0
\end{aligned}
$$

Table 3: $\gamma_A([a_{i+j+1}],*)$ computations.

$$
\begin{aligned}
\gamma_A([a_{i+j+2}],[a_{i+j}]) &= nH_2-nH_{1^2}\\
\gamma_A([a_{i+j+2}],[a_{i+j+1}]) &= H_{21}-2H_{1^3}\\
\gamma_A([a_{i+j+2}],[a_{i+j+2}]) &= 2H_{2^2}-2H_{21^2}+2H_{1^4}\\
\gamma_A([a_{i+j+2}],[a_{i+j+3}]) &= H_{32}-H_{31^2}-H_{2^21}+2H_{21^3}-2H_{1^5}\\
\gamma_A([a_{i+j+2}],[(i+j)a_{i+j}]) &= n(n-1)H_2-(n^2-n+2)H_{1^2}\\
\gamma_A([a_{i+j+2}],[(i+j)a_{i+j+1}]) &= 2(n-1)H_{21}-4(n-1)H_{1^3}\\
\gamma_A([a_{i+j+2}],[(i+j)a_{i+j+2}]) &= 2(2n-1)H_{2^2}-2(2n-2)H_{21^2}+2(2n-4)H_{1^4}\\
\gamma_A([a_{i+j+2}],[(i+j)a_{i+j+3}]) &= 2nH_{32}-2nH_{31^2}-2(n-2)H_{2^21}+4(n-2)H_{21^3}-4(n-3)H_{1^5}\\
\gamma_A([a_{i+j+2}],[c_{i+j-1}]) &= -2a_0H_1-2(n-1)a_1H_0\\
\gamma_A([a_{i+j+2}],[c_{i+j}]) &= (2n-1)a_0H_2-(2n-1)a_0H_{1^2}-2a_1H_1-(2n-3)a_2H_0
\end{aligned}
$$

Table 4: $\gamma_A([a_{i+j+2}],*)$ computations.

$$
\begin{aligned}
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[a_{i+j}]) &= 2(n-1)H_{1^2}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[a_{i+j+1}]) &= 6H_{1^3}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[a_{i+j+2}]) &= 2H_{21^2}-6H_{1^4}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[a_{i+j+3}]) &= 2H_{31^2}-2H_{2^21}-2H_{21^3}+6H_{1^5}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[(i+j)a_{i+j}]) &= 2(n-1)(n-2)H_{1^2}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[(i+j)a_{i+j+1}]) &= 12(n-1)H_{1^3}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[(i+j)a_{i+j+2}]) &= 4nH_{21^2}-12(n-2)H_{1^4}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[(i+j)a_{i+j+3}]) &= 4nH_{31^2}-4(n-1)H_{2^21}-4(n-1)H_{21^3}+12(n-3)H_{1^5}\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[c_{i+j-1}]) &= -4(n-1)a_0H_1+4(n-1)a_1H_0\\
\gamma_A([a_{i+j+1}],[a_{i+j+1}],[c_{i+j}]) &= 2a_0(2n-3)H_{1^2}-2(2n-3)a_1H_1+2(2n-3)a_2H_0
\end{aligned}
$$

Table 5: $\gamma_A([a_{i+j+1}],[a_{i+j+1}],*)$ computations.