

Enumerating pattern-avoiding permutations by leading terms

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The number of 123-avoiding permutations on $\{1, 2, \dots, n\}$ with a fixed leading term is counted by the ballot numbers. The same holds for 132-avoiding permutations. These results were proven by Miner and Pak using the Robinson–Schensted–Knuth (RSK) correspondence to connect permutations with Dyck paths. In this paper, we first provide an alternate proof of these enumeration results via a direct counting argument. We then study the number of pattern-avoiding permutations with a fixed prefix of length $t \geq 1$, generalizing the $t = 1$ case. We find exact expressions for single and pairs of patterns of length three as well as the pair 3412 and 3421. These expressions depend on t , the extrema, and the order statistics. We also define r -Wilf equivalence for permutations with a single fixed leading term r , and classify the r -Wilf-equivalence classes for both classical and vincular patterns of length three.

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1. Introduction

Let $A \subseteq \mathbb{N} := \{1, 2, \dots\}$ be a finite set. We use $|A|$ to denote the number of elements in A . A permutation τ on A is a sequence $(\tau(1), \tau(2), \dots, \tau(|A|))$ of length $|A|$ consisting of distinct numbers in A . When $A \subseteq \{1, 2, \dots, 9\}$ or when there is no confusion, we simply write a permutation/sequence without commas or parentheses in single-line notation. The leading $t \geq 1$ terms of a sequence is a prefix of length t of the sequence. When $t = 1$, the prefix is also called the *leading term* of the sequence. We use S_A to denote the set of permutations on A . When $A = [n] := \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, we write S_n for $S_{[n]}$.

For any $\tau \in S_n$ and $\sigma \in S_k$, if there exist $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that for all $1 \leq a < b \leq k$, $\tau(i_a) < \tau(i_b)$ if and only if $\sigma(a) < \sigma(b)$, then

we say that τ *contains* σ as a pattern and that $(\tau(i_1), \tau(i_2), \dots, \tau(i_k))$ is a σ pattern. A permutation τ *avoids* σ if τ does not contain σ as a pattern. For example, the permutation $\tau = 12453 \in S_5$ contains the pattern 132 because $\tau(1)\tau(3)\tau(5) = 143$ is a 132 pattern; however, τ avoids the pattern 321. For any $m, n, k \in \mathbb{N}$ and $\sigma_1, \sigma_2, \dots, \sigma_m \in S_k$, we use $S_n(\sigma_1, \sigma_2, \dots, \sigma_m)$ to denote the set of permutations $\tau \in S_n$ such that τ avoids all of the patterns $\sigma_1, \sigma_2, \dots, \sigma_m$.

The interest in the study of pattern avoidance can be traced back to stack-sortable permutations in computer science [12, Section 2.1]. One of the earliest results is the enumeration of permutations avoiding $\sigma \in S_3$, i.e., patterns of length three. D. Knuth [13] proved that the number of permutations in S_n avoiding any given pattern of length three is enumerated by the Catalan numbers C_n (see also [3, Theorem 4.7]).

Theorem 1.1. [13, p. 238] *For all $n \geq 1$ and $\sigma \in S_3$, we have*

$$|S_n(\sigma)| = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

For $r, n \in \mathbb{N}$ with $r \leq n$, let $S_{n,r}$ denote the set of permutations $\tau \in S_n$ with $\tau(1) = r$. It is clear that $|S_{n,r}| = (n-1)!$ for all $r \in [n]$. While studying the shapes of pattern-avoiding permutations, Miner and Pak [15] proved that $S_{n,r}(123)$ and $S_{n,r}(132)$ are enumerated by the ballot numbers (see, for example Aval [1] for more details on the ballot numbers).

Theorem 1.2. [15, Lemmas 4.1 and 5.2] *For all $1 \leq r \leq n$, we have*

$$|S_{n,r}(123)| = |S_{n,r}(132)| = b_{n,r} = \frac{n-r+1}{n+r-1} \binom{n+r-1}{n}.$$

Miner and Pak [15] proved Theorem 1.2 via a bijection between $S_{n,r}(123)$ (respectively, $S_{n,r}(132)$) and certain types of Dyck paths using the Robinson–Schensted–Knuth (RSK) correspondence. In this paper, we prove Theorem 1.2 using a direct counting argument. By the classical bijection between $S_n(123)$ and $S_n(132)$ [3, Lemma 4.4] which preserves the leading term, one only needs to prove that $|S_{n,r}(123)| = b_{n,r}$. We achieve this by utilizing a result of Simion and Schmidt [20, Lemma 2] on the number of 123-avoiding permutations with a fixed decreasing prefix.

It is natural to consider the case in which more than one leading term of the permutation is fixed. Motivated by this general case, we study pattern-avoiding permutations with a fixed prefix (c_1, c_2, \dots, c_t) of length $t \geq 1$. Here the $t = 1$ instance corresponds to the case studied by Miner and Pak.

Definition 1.3. For any $n, m \in \mathbb{N}$, distinct integers $c_1, c_2, \dots, c_t \in [n]$, and permutation patterns $\sigma_1, \sigma_2, \dots, \sigma_m$, we use $S_{n, (c_1, c_2, \dots, c_t)}$ to denote the set of permutations $\tau \in S_n$ such that $(\tau(1), \tau(2), \dots, \tau(t)) = (c_1, c_2, \dots, c_t)$; and we use $S_{n, (c_1, c_2, \dots, c_t)}(\sigma_1, \sigma_2, \dots, \sigma_m)$ to denote the set of permutations $\tau \in S_{n, (c_1, c_2, \dots, c_t)}$ such that τ avoids all of the patterns $\sigma_1, \sigma_2, \dots, \sigma_m$.

Convention 1.4. Unless otherwise specified, for permutations avoiding the patterns $\sigma_1, \sigma_2, \dots, \sigma_m$, we assume that the fixed prefix (c_1, c_2, \dots, c_t) itself avoids all of the patterns $\sigma_1, \sigma_2, \dots, \sigma_m$, $n \geq 3$, and $1 \leq t < n$.

We first show that the cardinality of $S_{n, (c_1, c_2, \dots, c_t)}(\sigma)$ can be determined exactly for all $\sigma \in S_3$. For $\sigma \in \{123, 132, 321, 312\}$, if $|S_{n, (c_1, c_2, \dots, c_t)}(\sigma)| \neq 0$, then $S_{n, (c_1, c_2, \dots, c_t)}(\sigma)$ is enumerated by ballot numbers. This is because, as we will show later in the proof of Theorem 4.4, there is a natural bijection between $S_{n, (c_1, c_2, \dots, c_t)}(\sigma)$ and $S_{n-t+1, r}(\sigma)$ for some r which depends on $\{c_1, c_2, \dots, c_t\}$. For $\sigma \in \{213, 231\}$, if $|S_{n, (c_1, c_2, \dots, c_t)}(\sigma)| \neq 0$, then it is equal to a product of Catalan numbers. This is because, for all $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(\sigma)$, the order statistics of $\{c_1, c_2, \dots, c_t\}$ determine a ‘block’ structure for the suffix $(\tau(t+1), \tau(t+2), \dots, \tau(n))$.

We then show that for all pairs of patterns of length three, the number of permutations avoiding these patterns and with a fixed prefix can also be determined exactly. The expressions for pairs of patterns of length three depend on the extrema, the order statistics, and the length of the prefix. We also determine the cardinality of $S_{n, (c_1, c_2, \dots, c_t)}(3412, 3421)$. As in the case of $S_{n, (c_1, c_2, \dots, c_t)}(231)$, if $|S_{n, (c_1, c_2, \dots, c_t)}(3412, 3421)| \neq 0$, then for all $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(3412, 3421)$, the suffix $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ has a ‘block’ structure. However, since the patterns 3412 and 3421 are of length four, some overlap between these blocks are possible. The cardinality of $S_{n, (c_1, c_2, \dots, c_t)}(3412, 3421)$ is related to the large Schröder numbers, and our treatment incidentally provides a new combinatorial proof of a recurrence relation for large Schröder numbers \mathbb{S}_n [3, p. 446].

In addition to exact enumeration, the notion of Wilf equivalence classes are also an important topic in pattern avoidance. Two permutation patterns σ and σ' are said to be *Wilf equivalent*, denoted $\sigma \sim \sigma'$, if $|S_n(\sigma)| = |S_n(\sigma')|$ for all $n \in \mathbb{N}$. By Theorem 1.1, all permutation patterns of length three are Wilf equivalent: $123 \sim 132 \sim 213 \sim 231 \sim 312 \sim 321$. In other words, there is only one Wilf-equivalence class for permutation patterns of length three. For patterns of length four, it is known that there are three Wilf-equivalence classes [3, p. 158].

We consider a similar Wilf-equivalence concept for permutations with a fixed leading term. For a fixed $r \in \mathbb{N}$, two patterns σ and σ' are called

r -Wilf equivalent if $|S_{n,r}(\sigma)| = |S_{n,r}(\sigma')|$ for all $n \geq r$. We write $\sigma \overset{r}{\sim} \sigma'$ if σ and σ' are r -Wilf equivalent. As an example, two patterns σ and σ' are 2-Wilf equivalent, denoted $\sigma \overset{2}{\sim} \sigma'$, if for all $n \geq 2$, $|S_{n,2}(\sigma)| = |S_{n,2}(\sigma')|$. We show that there are two 1-Wilf-equivalence classes for patterns of length three, $123 \overset{1}{\sim} 132$ and $321 \overset{1}{\sim} 312 \overset{1}{\sim} 213 \overset{1}{\sim} 231$; and, for all $r \geq 2$, there are three r -Wilf-equivalence classes for patterns of length three, $213 \overset{r}{\sim} 231$, $123 \overset{r}{\sim} 132$, and $321 \overset{r}{\sim} 312$. We also show that for all $r \geq 5$, there are nine r -Wilf-equivalence classes for vincular patterns of length three as studied in [2, 6].

This paper is organized as follows. In Section 2, we define basic concepts and state our preliminary results. We then provide a new proof of Theorem 1.2 in Section 3. In Section 4, we present results on the number of permutations with a fixed prefix that avoid a single pattern of length three. The case for the avoidance of pairs of patterns of length three is presented in Section 5. Permutations avoiding both 3412 and 3421 are then studied in Section 6. In Section 7, we classify r -Wilf-equivalence classes for classical and vincular patterns of length three.

2. Preliminaries

Definition 2.1. For a permutation $\tau \in S_n$, the *complement* τ^c of τ is the permutation in S_n defined by setting $\tau^c(i) = n + 1 - \tau(i)$.

The following result relates permutations avoiding certain patterns with those permutations avoiding the complement of these patterns. Since the proof is elementary, we state it without proof.

Lemma 2.2. *Let t, n, m , and k be positive integers with $t, k \leq n$ and let $c_1, c_2, \dots, c_t \in [n]$. Also let $\sigma_1, \sigma_2, \dots, \sigma_m \in S_k$ be permutation patterns. Then we have*

$$|S_{n,(c_1,c_2,\dots,c_t)}(\sigma_1, \sigma_2, \dots, \sigma_m)| = |S_{n,(n+1-c_1,n+1-c_2,\dots,n+1-c_t)}(\sigma_1^c, \sigma_2^c, \dots, \sigma_m^c)|.$$

Definition 2.3. Let A and B be two finite subsets of \mathbb{N} with $A \subseteq B$, $\sigma \in S_A$, and $\tau \in S_B$. We say that σ is a *subpermutation* of τ on A if there exist indices $1 \leq i_1 < i_2 < \dots < i_{|A|} \leq |B|$ such that

$$(\tau(i_1), \tau(i_2), \dots, \tau(i_{|A|})) = (\sigma(1), \sigma(2), \dots, \sigma(|A|)).$$

For example, if $\tau = 543621 \in S_6$, then $\sigma = 462 \in S_{\{2,4,6\}}$ is a subpermutation of τ on $\{2, 4, 6\}$.

Definition 2.4. Suppose σ is a permutation on a set A and τ is a permutation on a set B with $A \cap B = \emptyset$. A *shuffle* of σ and τ is a permutation α on $A \cup B$ such that σ is a subpermutation of α on A and τ is a subpermutation of α on B .

For example, if $A = \{4, 5, 7\}$, $B = \{1, 3, 6\}$, $\sigma = 457 \in S_A$, and $\tau = 631 \in S_B$, then $\alpha = 643571 \in S_{A \cup B}$ and $\alpha' = 456317 \in S_{A \cup B}$ are shuffles of σ and τ . The following simple observation is crucial for our later derivations. We state it without proof.

Lemma 2.5. *Suppose $A, B \subseteq \mathbb{N}$ with $|A \cap B| = \emptyset$, $|A| = k$, and $|B| = \ell$. If $\sigma \in S_A$ and $\tau \in S_B$, then the number of shuffles of σ and τ is $\binom{k+\ell}{k}$.*

We will use the following terminology.

Definition 2.6. Let $A \subseteq \mathbb{N}$ be a finite set and $\tau \in S_A$. If $\tau(i) = a$, then we use $\mathcal{A}_\tau(a) = \{\tau(1), \tau(2), \dots, \tau(i-1)\}$ to denote the set of *ancestors* of a in τ and $\mathcal{D}_\tau(a) = \{\tau(i+1), \tau(i+2), \dots, \tau(|A|)\}$ to denote the set of *descendants* of a in τ .

For example, if $\tau = 2785 \in S_{\{2,5,7,8\}}$, then $\mathcal{A}_\tau(8) = \{2, 7\}$ and $\mathcal{D}_\tau(7) = \{5, 8\}$.

Definition 2.7. Let $n \in \mathbb{N}$ and $A \subseteq [n]$. The *standardization* of $\tau = (\tau(1), \tau(2), \dots, \tau(|A|)) \in S_A$ is the permutation $s(\tau) \in S_{|A|}$ obtained by replacing the i th smallest entry in τ with i for all i .

For example, the standardization of $567832 \in S_{\{2,3,5,6,7,8\}}$ is $345621 \in S_6$. We include below a simple observation concerning standardization.

Lemma 2.8. *Let $n \in \mathbb{N}$, $A \subseteq [n]$, and $\tau \in S_A$. If τ avoids a pattern σ , then $s(\tau)$ also avoids the pattern σ .*

Definition 2.9. Let $n, n' \in \mathbb{N}$ with $n \leq n'$, $A \subseteq [n']$ with $|A| = n$, and $\tau \in S_n$. Then the *matching permutation* τ' of τ on A is defined as follows: if $\tau(i) = j$ where $i, j \in \{1, 2, \dots, n\}$, then $\tau'(i)$ is the j th smallest integer in A .

For example, with $n = 3$, $n' = 7$, and $A = \{2, 4, 7\}$, the matching permutation of $231 \in S_3$ on $\{2, 4, 7\}$ is $472 \in S_{\{2,4,7\}}$.

Notice that the matching permutation of a permutation also preserves pattern avoidance.

The first few Catalan numbers C_n , Bell numbers B_n , and large Schröder numbers \mathbb{S}_n are listed in Table 1 for later reference.

Table 1: C_n , B_n , and \mathbb{S}_n for $n \leq 9$

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | OEIS [17] |
|----------------|---|---|---|----|----|-----|------|------|-------|--------|-----------|
| C_n | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | A000108 |
| B_n | 1 | 1 | 2 | 5 | 15 | 52 | 203 | 877 | 4140 | 21147 | A000110 |
| \mathbb{S}_n | 1 | 2 | 6 | 22 | 90 | 394 | 1806 | 8558 | 41586 | 206098 | A006318 |

We also need the following elementary results on the Catalan and the Bell numbers.

Lemma 2.10. *For all $n \geq 4$, we have $C_n < B_n$.*

Proof. It is well-known that C_n counts the number of noncrossing partitions of $[n]$ and B_n counts the total number of partitions of $[n]$. For these facts and the definitions of partitions and noncrossing partitions, see for example [16, Section 1.1] and [19]. For all $n \geq 4$, the two subsets $\{2, 4\}$ and $[n] \setminus \{2, 4\}$ form a crossing partition of $[n]$ and hence $C_n < B_n$. \square

Lemma 2.11. *For all $n \geq 3$, we have $B_n > 2B_{n-1}$.*

Proof. The Bell numbers B_n satisfy the following recurrence relation [7, p. 49]:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Let $n \geq 3$. Then we have

$$\begin{aligned} B_n &= B_{n-1} + \binom{n-1}{n-2} B_{n-2} + \binom{n-1}{n-3} B_{n-3} + \cdots + \binom{n-1}{0} B_0 \\ &> B_{n-1} + \binom{n-2}{n-2} B_{n-2} + \binom{n-2}{n-3} B_{n-3} + \cdots + \binom{n-2}{0} B_0 \\ &= B_{n-1} + \sum_{k=0}^{n-2} \binom{n-2}{k} B_k = 2B_{n-1}. \end{aligned} \quad \square$$

3. A new proof of Theorem 1.2

Let $1 \leq r \leq n$. The well-known bijection between $S_n(123)$ and $S_n(132)$ by Simion and Schmidt [20] preserves the leading term. We refer to [3, Lemma 4.4] for a proof of this fact. Hence, we have $|S_{n,r}(123)| = |S_{n,r}(132)|$. So we only need to show that $|S_{n,r}(123)| = b_{n,r}$.

First, consider $r = 1$. If $\tau \in S_{n,1}(123)$, then $(\tau(2), \tau(3), \dots, \tau(n))$ must be a decreasing sequence and hence $\tau = (1, n, n-1, \dots, 2)$. Therefore, we have $|S_{n,1}(123)| = 1 = b_{n,1}$.

Now suppose $r \geq 2$. We will need the following definition.

Definition 3.1. For all $n \in \mathbb{N}$, define $a_n(i) : [n] \rightarrow [n]$ as follows:

- (i) for all $i < n$, let $a_n(i)$ be the number of permutations $\tau \in S_n(123)$ such that i is the smallest index with $\tau(i) < \tau(i+1)$, and
- (ii) set $a_n(n) = 1$.

That is, if $i < n$, then $a_n(i)$ is the number of permutations $\tau \in S_n$ avoiding 123 such that $\tau(1) > \tau(2) > \dots > \tau(i-1) > \tau(i) < \tau(i+1)$. If $i = n$, then $a_n(n) = 1$ because there is exactly one decreasing sequence $(n, n-1, \dots, 2, 1) \in S_n$. Simion and Schmidt [20, Lemma 2] proved the following result for $a_n(i)$:

Lemma 3.2. For all $1 \leq i \leq n$,

$$a_n(i) = \binom{2n-i-1}{n-1} - \binom{2n-i-1}{n}.$$

Let \mathcal{P} be a subset of $S_{n,r}$ such that every $\tau \in \mathcal{P}$ has the following properties:

- (i) the subpermutation τ' of τ on $\{1, 2, \dots, r-1\}$ avoids 123;
- (ii) the subpermutation τ'' of τ on $\{r+1, r+2, \dots, n\}$ is $(n, n-1, \dots, r+1)$;
- (iii) if $r > 2$ and $i < r-1$ is the smallest index with $\tau'(1) > \tau'(2) > \dots > \tau'(i) < \tau'(i+1)$, then $\{r+1, r+2, \dots, n\} \subseteq \mathcal{A}_\tau(\tau'(i+1))$.

For Property (iii), we do not impose any extra condition on the positions of $\{r+1, r+2, \dots, n\}$ when $\tau' = (r-1, r-2, \dots, 1)$.

We first show that

$$|\mathcal{P}| = \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i).$$

Let $\tau \in \mathcal{P}$. If $r = 2$, then the subpermutation of τ on $\{2, 3, \dots, n\}$ is $(2, n, n-1, \dots, 3)$ and hence there are $n-1$ possible locations for 1. It is not hard to see that all these possible locations guarantee that τ avoids the pattern 123. It follows that, in this case, we have $|\mathcal{P}| = n-1$. It is easy to see that $a_1(1) = 1$ and hence $|\mathcal{P}| = \sum_{i=1}^{2-1} \binom{i+n-2}{i} a_1(i) = n-1$. If $r > 2$, then for all $i \in \{1, 2, \dots, r-2\}$ and a fixed τ' with $\tau'(1) > \tau'(2) > \dots > \tau'(i) < \tau'(i+1)$, the number of shuffles of $(\tau'(1), \tau'(2), \dots, \tau'(i))$ and $\tau'' = (n, n-1, \dots, r+1)$ is $\binom{i+n-r}{i}$. If $\tau' = (r-1, r-2, \dots, 1)$, then the number of shuffles of τ' and τ''

is $\binom{r-1+n-r}{r-1}$. As the number of such τ' is $a_{r-1}(i)$ for all $i \in \{1, 2, \dots, r-1\}$, we have $|\mathcal{P}| = \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i)$.

It remains to show that $S_{n,r}(123) = \mathcal{P}$. Let $\tau \in S_{n,r}(123)$, and let τ' be the subpermutation of τ on $\{1, 2, \dots, r-1\}$ and τ'' be the subpermutation of τ on $\{r+1, r+2, \dots, n\}$. Since τ avoids 123, τ' avoids 123 as well. We now show that τ'' avoids 12. If this is not the case then there exist $a < b \leq n-r$ such that $\tau''(a) < \tau''(b)$. Since $\tau''(a) > r$, $r\tau''(a)\tau''(b)$ is a 123 pattern, and this is a contradiction. Therefore $\tau'' = (n, n-1, \dots, r+1)$. Now suppose $r > 2$ and let $i < r-1$ be the smallest index such that $\tau'(1) > \tau'(2) > \dots > \tau'(i) < \tau'(i+1)$. We still need to show that $\{r+1, r+2, \dots, n\} \subseteq \mathcal{A}_\tau(\tau'(i+1))$. Suppose, by way of contradiction, $\{r+1, r+2, \dots, n\} \not\subseteq \mathcal{A}_\tau(\tau'(i+1))$. Then there exists $a \in \{r+1, r+2, \dots, n\} \cap \mathcal{D}_\tau(\tau'(i+1))$. Since $\tau'(i)\tau'(i+1)a$ is a 123 pattern, this would be a contradiction. Hence, we have $S_{n,r}(123) \subseteq \mathcal{P}$.

Now let $\tau \in \mathcal{P}$. We need to show that $\tau \in S_{n,r}(123)$. Let τ' be the subpermutation of τ on $\{1, 2, \dots, r-1\}$ and τ'' be the subpermutation of τ on $\{r+1, r+2, \dots, n\}$. If $r > 2$, then let $i < r-1$ be the smallest index such that $\tau'(1) > \tau'(2) > \dots > \tau'(i) < \tau'(i+1)$; if $\tau' = (r-1, r-2, \dots, 1)$, then we set $i = r-1$. Since $\tau \in \mathcal{P}$, $\tau = (r, \alpha(1), \alpha(2), \dots, \alpha(n-r+i), \tau'(i+1), \tau'(i+2), \dots, \tau'(r-1))$ where $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n-r+i))$ is a shuffle of $(\tau'(1), \tau'(2), \dots, \tau'(i))$ and $\tau'' = (n, n-1, \dots, r+1)$. We need to show that any subpermutation abc of τ is not a 123 pattern. Suppose, by way of contradiction, there exists a subpermutation abc of τ which is a 123 pattern. Then $a < b < c$. We split into two cases:

Case 1: $a \geq r$. Then $c > b > r$. It follows that bc is an increasing subpermutation of τ'' . But this contradicts the fact that $\tau'' = (n, n-1, \dots, r+1)$.

Case 2: $a < r$. Since τ' avoids 123, either $b > r$ or $c > r$. If $b > r$, then $c > b > r$. Using a similar argument as in Case 1, we have a contradiction. So we suppose $b < r$ and $c > r$. Here $b \neq r$ because $b \in D_\tau(a)$. Since ab is an increasing subpermutation of τ' , we have $b = \tau'(j)$ for some $j \geq i+1$. Hence $c \in \mathcal{A}_\tau(\tau'(i+1)) \subseteq \mathcal{A}_\tau(b)$ which is again a contradiction.

This proves that $\mathcal{P} \subseteq S_{n,r}(123)$, and, consequently, $S_{n,r}(123) = \mathcal{P}$. Therefore, by Lemma 3.2,

$$\begin{aligned} |S_{n,r}(123)| = |\mathcal{P}| &= \sum_{i=1}^{r-1} \binom{i+n-r}{i} a_{r-1}(i) \\ &= \sum_{i=1}^{r-1} \binom{i+n-r}{i} \left[\binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} \right]. \end{aligned}$$

It remains to show that $\sum_{i=1}^{r-1} \binom{i+n-r}{i} \left[\binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} \right] = b_{n,r}$. First, by the definition of binomial coefficients, we have

$$\begin{aligned} \binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} &= \binom{2r-i-3}{r-2} - \frac{r-i-1}{r-1} \binom{2r-i-3}{r-2} \\ &= \frac{i}{r-1} \binom{2r-i-3}{r-2}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{i=1}^{r-1} \binom{i+n-r}{i} \left[\binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} \right] \\ &= \sum_{i=1}^{r-1} \binom{i+n-r}{i} \frac{i}{r-1} \binom{2r-i-3}{r-2} \\ &= \sum_{i=1}^{r-1} \frac{n-r+1}{i} \binom{i+n-r}{i-1} \frac{i}{r-1} \binom{2r-i-3}{r-2} \\ &= \frac{n-r+1}{r-1} \sum_{i=1}^{r-1} \binom{i+n-r}{i-1} \binom{2r-i-3}{r-2}. \end{aligned}$$

By a variation of Vandermonde's identity,

$$\sum_{i=1}^{r-1} \binom{i+n-r}{i-1} \binom{2r-i-3}{r-2} = \binom{n+r-2}{r-2}.$$

(One way to see this is by comparing the coefficients of x^{r-2} on the two sides of the identity $(1-x)^{-(n-r+2)}(1-x)^{-(r-1)} = (1-x)^{-(n+1)}$.) Putting it all together,

$$\begin{aligned} &\sum_{i=1}^{r-1} \binom{i+n-r}{i} \left[\binom{2r-i-3}{r-2} - \binom{2r-i-3}{r-1} \right] \\ &= \frac{n-r+1}{r-1} \binom{n+r-2}{r-2} = \frac{n-r+1}{n+r-1} \binom{n+r-1}{r-1} = b_{n,r}. \end{aligned}$$

This completes the proof of Theorem 1.2.

4. Single patterns of length three

In this section, we enumerate $S_{n,(c_1,c_2,\dots,c_t)}(\sigma)$ for $\sigma \in S_3$. By Lemma 2.2, it suffices to enumerate permutations avoiding the patterns 123, 132, and 231. We start with the pattern 231. The key features about $S_{n,(c_1,c_2,\dots,c_t)}(231)$ are that the enumeration is related to the order statistics of $\{c_1, c_2, \dots, c_t\}$ and there is a ‘block’ structure for all $\tau \in S_{n,(c_1,c_2,\dots,c_t)}(231)$.

Theorem 4.1. *If $c_i < c_j$ for some $1 \leq i < j \leq t$ and there exists $\alpha < c_i$ such that $\alpha \notin \{c_1, c_2, \dots, c_t\}$, then $|S_{n,(c_1,c_2,\dots,c_t)}(231)| = 0$; otherwise, we have*

$$(4.1) \quad |S_{n,(c_1,c_2,\dots,c_t)}(231)| = \prod_{k=1}^{t+1} C_{c_{(k)} - c_{(k-1)} - 1},$$

where C_i is the i th Catalan number, $c_{(0)} = 0$, $c_{(t+1)} = n + 1$, and $c_{(1)} < c_{(2)} < \dots < c_{(t)}$ are the order statistics of $\{c_1, c_2, \dots, c_t\}$.

Proof. If $c_i < c_j$ for some $1 \leq i < j \leq t$ and there exists $\alpha < c_i$ such that $\alpha \notin \{c_1, c_2, \dots, c_t\}$, then $c_i c_j \alpha$ is a 231 pattern. Therefore, $|S_{n,(c_1,c_2,\dots,c_t)}(231)| = 0$.

Now suppose otherwise. We will build a set \mathcal{Q} whose cardinality is given by the right hand side of (4.1) and then show that $\mathcal{Q} = S_{n,(c_1,c_2,\dots,c_t)}(231)$. Let \mathcal{Q} be the subset of $S_{n,(c_1,c_2,\dots,c_t)}$ such that every $\tau \in \mathcal{Q}$ has the following properties:

- (i) for all $k, \ell \in [t + 1]$ with $k < \ell$, if

$$x \in \{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}$$

and $y \in \{c_{(\ell-1)} + 1, c_{(\ell-1)} + 2, \dots, c_{(\ell)} - 1\}$, then $x \in \mathcal{A}_\tau(y)$, and

- (ii) for all $k \in [t + 1]$, the subpermutation on

$$\{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}$$

avoids 231.

That is, for each $\tau \in \mathcal{Q}$, $(\tau(t + 1), \tau(t + 2), \dots, \tau(n))$ is the concatenation of $t + 1$ (some possibly empty) 231-avoiding permutations, which we call 231-avoiding blocks. For all $k \in [t + 1]$, the k th block is a 231-avoiding permutation on $\{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}$. Since for all $k \in [t + 1]$, $|\{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}| = c_{(k)} - c_{(k-1)} - 1$, by Theorem 1.1, the

number of 231-avoiding permutations on $\{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}$ is $C_{c_{(k)} - c_{(k-1)} - 1}$. Hence we have $|\mathcal{Q}| = \prod_{k=1}^{t+1} C_{c_{(k)} - c_{(k-1)} - 1}$.

Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(231)$. Then every subpermutation of τ avoids 231. So, τ satisfies Property (ii). Now we show that τ satisfies Property (i). Suppose not. Then there exist $k, \ell \in [t]$ with $k < \ell$,

$$x \in \{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\},$$

and $y \in \{c_{(\ell-1)} + 1, c_{(\ell-1)} + 2, \dots, c_{(\ell)} - 1\}$, such that $x \in \mathcal{D}_\tau(y)$. It follows that $c_{(\ell-1)}yx$ is a 231 pattern, which is a contradiction. Hence $\tau \in \mathcal{Q}$. This proves that $S_{n, (c_1, c_2, \dots, c_t)}(231) \subseteq \mathcal{Q}$.

Now let $\tau \in \mathcal{Q}$. To show that $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(231)$, it suffices to show that any subpermutation abc of τ is not a 231 pattern. Suppose, by way of contradiction, that a subpermutation abc of τ is a 231 pattern. Then we must have $c < a < b$ and $c \in D_\tau(b)$. We split into four cases:

Case 1: $a, b, c \notin \{c_1, c_2, \dots, c_t\}$. Since $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is a concatenation of $t+1$ (some possibly empty) 231-avoiding blocks, c must be in a block after the block a is in. Since $c < a$, this contradicts Property (i).

Case 2: $a \in \{c_1, c_2, \dots, c_t\}$ and $b, c \notin \{c_1, c_2, \dots, c_t\}$. Then we have $a = c_{(k)}$ for some $k \in [t]$. Since $c < a < b$, b and c must be in two different blocks. Since $c \in D_\tau(b)$, c is in a block after the block b is in which contradicts Property (i).

Case 3: $a, b \in \{c_1, c_2, \dots, c_t\}$ and $c \notin \{c_1, c_2, \dots, c_t\}$. Then $a = c_i$ and $b = c_j$ with $1 \leq i < j \leq t$, and $c < a$. This is a contradiction.

Case 4: $a, b, c \in \{c_1, c_2, \dots, c_t\}$. Then abc is a subpermutation of the prefix (c_1, c_2, \dots, c_t) which contradicts our convention that (c_1, c_2, \dots, c_t) avoids 231.

Hence, we have $\mathcal{Q} \subseteq S_{n, (c_1, c_2, \dots, c_t)}(231)$. \square

Remark 4.2. For all $n \geq 3$, by Theorem 1.1 and Theorem 4.1 with $t = 1$, we have

$$C_n = |S_n(231)| = \sum_{r=1}^n |S_{n,r}(231)| = \sum_{r=1}^n C_{n-r} C_{r-1}.$$

This offers an alternative interpretation for the well-known recurrence relation for the Catalan numbers, see [7, Section 3.2] and [21, Section 1.2].

In contrast to the 231 pattern, the expressions for the 123 and 132 patterns are related to the minimum of $\{c_1, c_2, \dots, c_t\}$. Recall that for a subset $A \subseteq [n]$, the *standardization* of a permutation $\tau = (\tau(1), \tau(2), \dots, \tau(|A|))$ is the permutation $s(\tau) \in S_{|A|}$ obtained by replacing the i th smallest entry in τ with i for all i . We will need the following result:

Lemma 4.3. *Suppose $A \subseteq [n]$ such that there exists $r \in [n]$ with $[r] \subseteq A$, and $\sigma \in S_k$ with $k \leq |A|$. Then $s(S_{A,r}(\sigma)) = S_{|A|,r}(\sigma)$, where $s(S_{A,r}(\sigma)) = \{s(\tau) : \tau \in S_{A,r}(\sigma)\}$.*

Proof. Let $\tau \in S_{A,r}(\sigma)$. Since $[r] \subseteq A$, $\tau(1)$ is the r th smallest number in τ . So $s(\tau) \in S_{|A|,r}$. By Lemma 2.8, we have $s(S_{A,r}(\sigma)) \subseteq S_{|A|,r}(\sigma)$.

Now let $\tau \in S_{|A|,r}(\sigma)$. Let $a_1 < a_2 < \dots < a_{|A|}$ be the elements in A . Since $[r] \subseteq A$, we have $a_r = r$. Let τ' be the matching permutation of τ on A . Since $\tau(1) = r$, we have $\tau'(1) = a_r = r$. It is not hard to see that τ' avoids σ . Hence, $\tau' \in S_{A,r}(\sigma)$. By our construction, we also have $s(\tau') = \tau$. So $\tau \in s(S_{A,r}(\sigma))$. Hence, $S_{|A|,r}(\sigma) \subseteq s(S_{A,r}(\sigma))$. \square

Theorem 4.4. *If $c_i < c_j$ for some $1 \leq i < j \leq t$ and there exists $\alpha > c_j$ such that $\alpha \notin \{c_1, c_2, \dots, c_t\}$, then $|S_{n,(c_1, c_2, \dots, c_t)}(123)| = 0$; otherwise, we have*

$$|S_{n,(c_1, c_2, \dots, c_t)}(123)| = |S_{n-t+1, \min\{c_1, c_2, \dots, c_t\}}(123)| = b_{n-t+1, \min\{c_1, c_2, \dots, c_t\}}.$$

Proof. If $c_i < c_j$ for some $i < j$ and there exists $\alpha > c_j$ such that $\alpha \notin \{c_1, c_2, \dots, c_t\}$, then $c_i c_j \alpha$ is a 123 pattern. Therefore, $|S_{n,(c_1, c_2, \dots, c_t)}(123)| = 0$.

Now suppose otherwise. For simplicity, we write $x = \min\{c_1, c_2, \dots, c_t\}$ and

$$A = ([n] \setminus \{c_1, c_2, \dots, c_t\}) \cup \{x\}.$$

Consider the map $f : S_{n,(c_1, c_2, \dots, c_t)}(123) \rightarrow S_{A,x}(123)$ such that for all $\tau \in S_{n,(c_1, c_2, \dots, c_t)}(123)$,

$$f(\tau) = (x, \tau(t+1), \tau(t+2), \dots, \tau(n)).$$

This is well defined since $(x, \tau(t+1), \tau(t+2), \dots, \tau(n))$ is a subpermutation of τ . If $\tau, \tau' \in S_{n,(c_1, c_2, \dots, c_t)}(123)$ with $\tau \neq \tau'$, then $\tau(i) \neq \tau'(i)$ for some $i \in \{t+1, t+2, \dots, n\}$ and hence $(x, \tau(t+1), \tau(t+2), \dots, \tau(n)) \neq (x, \tau'(t+1), \tau'(t+2), \dots, \tau'(n))$. So f is injective. f is also surjective because for all $\tau = (x, \tau(2), \tau(3), \dots, \tau(|A|)) \in S_{A,x}(123)$, we have

$$\tau' = (c_1, c_2, \dots, c_t, \tau(2), \tau(3), \dots, \tau(|A|)) \in S_{n,(c_1, c_2, \dots, c_t)}(123)$$

and $f(\tau') = \tau$. Therefore, f is a bijection and hence $|S_{n,(c_1, c_2, \dots, c_t)}(123)| = |S_{A,x}(123)|$.

Since $x = \min\{c_1, c_2, \dots, c_t\}$, we have $[x] \subseteq A$. Now by Lemma 4.3, we have

$$\begin{aligned} |S_{n,(c_1,c_2,\dots,c_t)}(123)| &= |S_{A,x}(123)| = |s(S_{A,x}(123))| \\ &= |S_{|A|,x}(123)| = |S_{n-t+1,x}(123)|. \end{aligned} \quad \square$$

The result for the 132 pattern is similar to the 123 pattern and hence we state the following result without proof.

Theorem 4.5. *If $c_i < c_j$ for some $i < j$ and there exists α such that $c_i < \alpha < c_j$ and $\alpha \notin \{c_1, c_2, \dots, c_t\}$, then $|S_{n,(c_1,c_2,\dots,c_t)}(132)| = 0$. Otherwise, we have*

$$|S_{n,(c_1,c_2,\dots,c_t)}(132)| = |S_{n-t+1,\min\{c_1,c_2,\dots,c_t\}}(132)| = b_{n-t+1,\min\{c_1,c_2,\dots,c_t\}}.$$

5. Pairs of patterns of length three

In this section, we enumerate permutations with fixed prefix (c_1, c_2, \dots, c_t) which avoid a pair of patterns $\{\sigma_1, \sigma_2\}$ of length three. Recall that we use $S_n(\sigma_1, \sigma_2)$ to denote the set of permutations $\tau \in S_n$ such that τ avoids both σ_1 and σ_2 . We need the following results by Simion and Schmidt [20, Section 3]:

Theorem 5.1. [20, Section 3] *For all $n \geq 1$,*

$$\begin{aligned} |S_n(123, 132)| &= |S_n(321, 312)| = |S_n(123, 213)| = |S_n(321, 231)| \\ &= |S_n(132, 213)| = |S_n(312, 231)| = |S_n(132, 231)| \\ &= |S_n(312, 213)| = |S_n(132, 312)| = |S_n(213, 231)| = 2^{n-1}, \\ |S_n(123, 312)| &= |S_n(321, 132)| = |S_n(123, 231)| = |S_n(321, 213)| = \binom{n}{2} + 1, \end{aligned}$$

and

$$|S_n(123, 321)| = \begin{cases} 0 & \text{if } n \geq 5, \\ n & \text{if } n = 1 \text{ or } n = 2, \\ 4 & \text{if } n = 3 \text{ or } n = 4. \end{cases}$$

Out of the 15 pairs of patterns of length 3, three of them are self-complementary: $\{123^c, 321^c\} = \{123, 321\}$, $\{132^c, 312^c\} = \{132, 312\}$, and $\{213^c, 231^c\} = \{213, 231\}$. By the Erdős-Szekeres theorem [9, p. 467], for $n \geq 5$, every $\tau \in S_n$ has either an increasing or a decreasing subpermutation of length three. Hence $|S_{n,(c_1,c_2,\dots,c_t)}(123, 321)| = 0$ if $n \geq 5$. Since one could

routinely calculate $|S_{n,(c_1,c_2,\dots,c_t)}(123,321)|$ when $n \leq 4$, we do not include the exact results for the pair $\{123,321\}$ here. We start with $\{132,312\}$ and $\{213,231\}$.

Theorem 5.2. *If $\{c_1, c_2, \dots, c_t\}$ is a set of consecutive integers, then*

$$|S_{n,(c_1,c_2,\dots,c_t)}(132,312)| = \binom{n-t}{\min\{c_1, c_2, \dots, c_t\} - 1};$$

otherwise, $|S_{n,(c_1,c_2,\dots,c_t)}(132,312)| = 0$.

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$ and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose that $\{c_1, c_2, \dots, c_t\}$ is a set of consecutive integers. Then we have

$$\{c_1, c_2, \dots, c_t\} = \{x, x+1, \dots, x+t-1\}.$$

Let $\tau \in S_{n,(c_1,c_2,\dots,c_t)}(132,312)$. Since τ avoids 132, the subpermutation on $\{x+t, x+t+1, \dots, n\}$, if $x+t-1 \neq n$, is $(x+t, x+t+1, \dots, n)$; and since τ avoids 312, the subpermutation on $\{1, 2, \dots, x-1\}$, if $x \neq 1$, is $(x-1, x-2, \dots, 1)$. The number of shuffles of $(x+t, x+t+1, \dots, n)$ and $(x-1, x-2, \dots, 1)$ is $\binom{n-t}{x-1}$. Hence, $|S_{n,(c_1,c_2,\dots,c_t)}(132,312)| \leq \binom{n-t}{x-1}$. Now let $\tau \in S_{n,(c_1,c_2,\dots,c_t)}$ such that $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is a shuffle of $(x+t, x+t+1, \dots, n)$ and $(x-1, x-2, \dots, 1)$. It is easy to check that $\tau \in S_{n,(c_1,c_2,\dots,c_t)}(132,312)$. Hence, we have $|S_{n,(c_1,c_2,\dots,c_t)}(132,312)| = \binom{n-t}{x-1}$.

Now suppose that $\{c_1, c_2, \dots, c_t\}$ is not a set of consecutive integers. Then there exists $y \in A$ and $i, j \in [t]$ such that $i < j$, and $c_i < y < c_j$ or $c_j < y < c_i$. Then $c_i c_j y$ is either a 132 pattern or a 312 pattern. Hence, we have $|S_{n,(c_1,c_2,\dots,c_t)}(132,312)| = 0$. \square

Theorem 5.3. *If (c_1, c_2, \dots, c_t) is a shuffle of $(1, 2, \dots, t-s)$ and $(n, n-1, \dots, n-s+1)$ for some $s \in \{0, 1, \dots, t\}$, then*

$$|S_{n,(c_1,c_2,\dots,c_t)}(213,231)| = 2^{n-t-1};$$

otherwise, $|S_{n,(c_1,c_2,\dots,c_t)}(213,231)| = 0$.

Note that, in Theorem 5.3, when $s = 0$, we mean that $(c_1, c_2, \dots, c_t) = (1, 2, \dots, t)$; and when $s = t$, we mean that $(c_1, c_2, \dots, c_t) = (n, n-1, \dots, n-t+1)$.

Proof. Write $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose (c_1, c_2, \dots, c_t) is a shuffle of $(1, 2, \dots, t-s)$ and $(n, n-1, \dots, n-s+1)$ for some $s \in \{0, 1, \dots, t\}$. Consider $f : S_{n, (c_1, c_2, \dots, c_t)}(213, 231) \rightarrow S_{n-t}(213, 231)$ such that for all $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(213, 231)$,

$$f(\tau) = s(\tau(t+1), \tau(t+2), \dots, \tau(n))$$

where $s(x)$ is the standardization of x . We will show that f is a bijection. It is easy to see that f is a well-defined function. Similarly to the logic in the proof of Theorem 4.4, one can also see that f is injective. It remains to show that f is surjective.

Let $\tau \in S_{n-t}(213, 231)$ and let τ' be the matching permutation of τ on A . Then $\pi := (c_1, c_2, \dots, c_t, \tau'(1), \tau'(2), \dots, \tau'(n-t)) \in S_{n, (c_1, c_2, \dots, c_t)}$. We need to show that π avoids 213 and 231. Let xyz be a subpermutation of π . If xyz is a subpermutation of (c_1, c_2, \dots, c_t) or τ' , then xyz is neither a 213 pattern nor a 231 pattern. So we assume that $x \in \{c_1, c_2, \dots, c_t\}$ and $z \in A$. We split into two cases:

Case 1: $x > z$. Then xyz is not a 213 pattern. Since (c_1, c_2, \dots, c_t) is a shuffle of $(1, 2, \dots, t-s)$ and $(n, n-1, \dots, n-s+1)$, we have $x \geq n-s+1$. Since $y \in \mathcal{D}_\pi(x)$, we have $y < x$ and hence xyz is not a 231 pattern.

Case 2: $x < z$. Then xyz is not a 231 pattern. Since (c_1, c_2, \dots, c_t) is a shuffle of $(1, 2, \dots, t-s)$ and $(n, n-1, \dots, n-s+1)$, we have $x \leq t-s$. Since $y \in \mathcal{D}_\pi(x)$, we have $y > x$ and hence xyz is not a 213 pattern.

Hence, $\pi \in S_{n, (c_1, c_2, \dots, c_t)}(213, 231)$ and $f(\pi) = \tau$ by our construction. This shows that f is surjective. Now f is a bijection and hence, by Theorem 5.1,

$$|S_{n, (c_1, c_2, \dots, c_t)}(213, 231)| = |S_{n-t}(213, 231)| = 2^{n-t-1}.$$

Now suppose (c_1, c_2, \dots, c_t) is not a shuffle of $(1, 2, \dots, t-s)$ and $(n, n-1, \dots, n-s+1)$ for any $s \in \{0, 1, \dots, t\}$. There are two scenarios where this could happen. The first one is when $\{c_1, c_2, \dots, c_t\} \neq \{1, 2, \dots, t-s, n-s+1, n-s+2, \dots, n\}$, and the second one is when $\{c_1, c_2, \dots, c_t\} = \{1, 2, \dots, t-s, n-s+1, n-s+2, \dots, n\}$ but for any $s \in \{0, 1, \dots, t\}$, either the subpermutation of (c_1, c_2, \dots, c_t) on $\{1, 2, \dots, t-s\}$ is not $(1, 2, \dots, t-s)$ or the subpermutation of (c_1, c_2, \dots, c_t) on $\{n-s+1, n-s+2, \dots, n\}$ is not $(n, n-1, \dots, n-s+1)$. We split into two cases based on these scenarios. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}$.

Case 3: $\{c_1, c_2, \dots, c_t\} \neq \{1, 2, \dots, t-s, n-s+1, n-s+2, \dots, n\}$ for any $s \in \{0, 1, \dots, t\}$. Then there exist $x \in \{c_1, c_2, \dots, c_t\}$ and $y, z \in A$ such that $y > x$ and $z < x$. In this case, either xyz or xzy is a subpermutation of τ and hence τ contains either a 213 pattern or a 231 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(213, 231)| = 0$.

Case 4: $\{c_1, c_2, \dots, c_t\} = \{1, 2, \dots, t-s, n-s+1, n-s+2, \dots, n\}$ for some $s \in \{0, 1, \dots, t\}$.

Subcase 4.1: The subpermutation of (c_1, c_2, \dots, c_t) on $\{1, 2, \dots, t-s\}$ is not $(1, 2, \dots, t-s)$. Then there exist $x, y \in \{c_1, c_2, \dots, c_t\}$ and $z \in A$ such that $x < y < z$ and $y \in \mathcal{A}_\tau(x)$. Now yxz is a 213 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(213, 231)| = 0$.

Subcase 4.2: The subpermutation of (c_1, c_2, \dots, c_t) on $\{n-s+1, n-s+2, \dots, n\}$ is not $(n, n-1, \dots, n-s+1)$. Then there exist $x, y \in \{c_1, c_2, \dots, c_t\}$ and $z \in A$ such that $z < y < x$ and $y \in \mathcal{A}_\tau(x)$. Now yxz is a 231 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(213, 231)| = 0$. \square

We have 12 pairs left to consider. By Lemma 2.2, it suffices to look at $\{123, 132\}$, $\{123, 213\}$, $\{132, 213\}$, $\{132, 231\}$, $\{123, 312\}$, and $\{123, 231\}$.

Theorem 5.4. *Write $\alpha = \max([n] \setminus \{c_1, c_2, \dots, c_t\})$. If $\{c_1, c_2, \dots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \dots, n\}$ or $(c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t) \neq (\alpha-1, \alpha-2, \dots, n-t)$, then $|S_{n, (c_1, c_2, \dots, c_t)}(123, 132)| = 0$; otherwise,*

$$|S_{n, (c_1, c_2, \dots, c_t)}(123, 132)| = 2^{n-t-1}.$$

Note that, in Theorem 5.4, we must have $\alpha \geq n-t$, and if $\alpha = n-t$, then $\{c_1, c_2, \dots, c_t\} = \{n-t+1, n-t+2, \dots, n\}$.

Proof. Write $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

First suppose that $\{c_1, c_2, \dots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \dots, n\}$. As mentioned earlier, if $\alpha = n-t$, then $\{c_1, c_2, \dots, c_{n-\alpha}\} = \{c_1, c_2, \dots, c_t\} = \{n-t+1, n-t+2, \dots, n\}$. So we must have $\alpha > n-t$. Since $\alpha = \max A$, there exist $i \in \{1, 2, \dots, n-\alpha\}$ and $j \in \{n-\alpha+1, n-\alpha+2, \dots, t\}$ such that $c_i < \alpha < c_j$. So $c_i c_j \alpha$ is a 132 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(123, 132)| = 0$.

Next suppose that $\{c_1, c_2, \dots, c_{n-\alpha}\} = \{\alpha+1, \alpha+2, \dots, n\}$ but

$$(c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t) \neq (\alpha-1, \alpha-2, \dots, n-t).$$

Notice that this could only happen when $\alpha > n-t$ because otherwise $n-\alpha+1 = n - (n-t) + 1 = t+1$. We split into three cases:

Case 1: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t\} \neq \{\alpha-1, \alpha-2, \dots, n-t\}$ and there exists $i \in \{n-\alpha+1, n-\alpha+2, \dots, t\}$ such that $c_i > \alpha$. Then $\{c_1, c_2, \dots, c_{n-\alpha}\} \neq \{\alpha+1, \alpha+2, \dots, n\}$. This is a contradiction.

Case 2: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t\} \neq \{\alpha-1, \alpha-2, \dots, n-t\}$ and $c_i < \alpha$ for all $i \in \{n-\alpha+1, n-\alpha+2, \dots, t\}$. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}$. Then there exist $y \in A$ and $c_i \in \{c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t\}$ such that $c_i < y < \alpha$. So either $c_i y \alpha$

or $c_i\alpha y$ is a subpermutation of τ . It follows that τ has either a 123 pattern or a 132 pattern. Hence, $|S_{n,(c_1,c_2,\dots,c_t)}(123, 132)| = 0$.

Case 3: $\{c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t\} = \{\alpha - 1, \alpha - 2, \dots, n - t\}$ but

$$(c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t) \neq (\alpha - 1, \alpha - 2, \dots, x).$$

Then there exist $i, j \in \{n - \alpha + 1, n - \alpha + 2, \dots, t\}$ with $i < j$ and $c_i < c_j$. Now $c_i c_j \alpha$ is a 123 pattern. Hence, $|S_{n,(c_1,c_2,\dots,c_t)}(123, 132)| = 0$.

The proof that

$$|S_{n,(c_1,c_2,\dots,c_t)}(123, 132)| = 2^{n-t-1}$$

when $\{c_1, c_2, \dots, c_{n-\alpha}\} = \{\alpha+1, \alpha+2, \dots, n\}$ and $(c_{n-\alpha+1}, c_{n-\alpha+2}, \dots, c_t) = (\alpha - 1, \alpha - 2, \dots, n - t - 1)$ is similar to the proof of Theorem 5.3. \square

Theorem 5.5. *If there exists $\alpha \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ and $1 \leq i < j \leq t$ with $c_i, c_j < \alpha$, then $|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| = 0$; otherwise,*

$$|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| = 2^{\max\{0, \min\{c_1, c_2, \dots, c_t\} - 2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$ and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose there exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i, c_j < \alpha$. Then $c_i c_j \alpha$ is either a 123 pattern or a 213 pattern. Hence, $|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| = 0$.

Suppose there do not exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i, c_j < \alpha$. Let $\tau \in S_{n,(c_1,c_2,\dots,c_t)}(123, 213)$.

Case 1: $x = 1$. Then $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is a decreasing sequence. Otherwise, there would exist $y, z \in A$ such that $1yz$ is a 123 pattern. Hence, $|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| = 1$.

Case 2: $x \geq 2$. Since τ avoids 123, the subpermutation of τ on $A \cap \{x + 1, x + 2, \dots, n\}$ is decreasing. Since τ avoids 213, for all $y \in \{1, 2, \dots, x - 1\}$ and for all $z \in A \cap \{x + 1, x + 2, \dots, n\}$, $z \in \mathcal{A}_\tau(y)$. So by Theorem 5.1,

$$|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| \leq |S_{x-1}(123, 213)| = 2^{x-2}.$$

It is easy to check that for all $\tau' \in S_{x-1}(123, 213)$ and decreasing subpermutation τ'' on $A \setminus [x - 1]$, the permutation

$$(c_1, c_2, \dots, c_t, \tau''(1), \tau''(2), \dots, \tau''(n - t - x + 1), \tau'(1), \tau'(2), \dots, \tau'(x - 1))$$

is in $S_{n,(c_1,c_2,\dots,c_t)}(123, 213)$. Hence, we have

$$|S_{n,(c_1,c_2,\dots,c_t)}(123, 213)| = |S_{x-1}(123, 213)| = 2^{x-2}. \quad \square$$

Theorem 5.6. *If there exist $\alpha, \beta \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ and $i \in [t]$ with $\min\{c_1, c_2, \dots, c_t\} < \alpha < c_i < \beta$, then $|S_{n, (c_1, c_2, \dots, c_t)}(132, 213)| = 0$; if there exist $\alpha \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $c_j < c_i < \alpha$, then $|S_{n, (c_1, c_2, \dots, c_t)}(132, 213)| = 0$; otherwise,*

$$|S_{n, (c_1, c_2, \dots, c_t)}(132, 213)| = 2^{\max\{0, \min\{c_1, c_2, \dots, c_t\} - 2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$ and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose there exist $\alpha, \beta \in A$ and $i \in [t]$ with $x < \alpha < c_i < \beta$. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}$. If $\alpha \in \mathcal{A}_\tau(\beta)$, then $c_i\alpha\beta$ is a 213 pattern; and if $\alpha \in \mathcal{D}_\tau(\beta)$, then $x\beta\alpha$ is a 132 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(132, 213)| = 0$.

Now suppose there exist $\alpha \in A$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $c_j < c_i < \alpha$. If $c_i < \alpha < c_j$, then $c_i c_j \alpha$ is a 132 pattern; and if $c_j < c_i < \alpha$, then $c_i c_j \alpha$ is a 213 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(132, 213)| = 0$.

The rest of the proof is similar to the proof of Theorem 5.5. \square

Theorem 5.7. *If there exist $\alpha \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $\alpha < c_i < c_j$, then $|S_{n, (c_1, c_2, \dots, c_t)}(132, 231)| = 0$; otherwise,*

$$|S_{n, (c_1, c_2, \dots, c_t)}(132, 231)| = 2^{\max\{0, \min\{c_1, c_2, \dots, c_t\} - 2\}}.$$

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$ and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose there exist $\alpha \in A$ and $i, j \in [t]$ with $i < j$ and $c_i < \alpha < c_j$ or $\alpha < c_i < c_j$. If $c_i < \alpha < c_j$, then $c_i c_j \alpha$ is a 132 pattern; and if $\alpha < c_i < c_j$, then $c_i c_j \alpha$ is a 231 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(132, 231)| = 0$.

The rest of the proof is similar to the proof of Theorem 5.5. \square

Theorem 5.8. *If $\{c_1, c_2, \dots, c_t\}$ is a set of consecutive integers, then*

$$|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 0, 1, \text{ or } \min\{c_1, c_2, \dots, c_t\};$$

otherwise, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 0$ or 1.

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$, $y = \max\{c_1, c_2, \dots, c_t\}$, and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose $\{c_1, c_2, \dots, c_t\}$ is a set of consecutive integers. Then

$$\{c_1, c_2, \dots, c_t\} = \{x, x + 1, \dots, x + t - 1\}.$$

We split into three cases:

Case 1: There exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. Then $c_i c_j \alpha$ is a 123 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 0$.

Case 2: $x = n - t + 1$ and there do not exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(123, 312)$. Then $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is a subpermutation on $\{1, 2, \dots, x-1\}$. If $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is not decreasing, then we would have a 312 pattern. It is easy to check that $(c_1, c_2, \dots, c_t, x-1, x-2, \dots, 1)$ avoids both 123 and 312 patterns. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 1$.

Case 3: $x < n - t + 1$ and there do not exist $\alpha \in A$ and $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$. In this case, we have $y < n$. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(123, 312)$. Let τ' be the subpermutation on $\{1, 2, \dots, x-1\}$ and let τ'' be the subpermutation on $\{y+1, y+2, \dots, n\}$. Since τ avoids 123, $\tau'' = (n, n-1, \dots, y+1)$, and since τ avoids 312, $\tau' = (x-1, x-2, \dots, 1)$. Moreover, if $\tau(i) \in \{1, 2, \dots, x-1\}$ and $\tau(j), \tau(k) \in \{y+1, y+2, \dots, n\}$ with $j < k$, either $\tau(j), \tau(k) \in \mathcal{A}_\tau(\tau(i))$ or $\tau(j), \tau(k) \in \mathcal{D}_\tau(\tau(i))$. Otherwise, $\tau(j)\tau(i)\tau(k)$ would be a 312 pattern. Therefore, the number of shuffles of τ' and τ'' that do not create a 312 pattern is simply $\binom{x-1+1}{1} = x$. It is easy to check that none of these shuffles creates a 123 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = x$.

Now suppose $\{c_1, c_2, \dots, c_t\}$ is not a set of consecutive integers. There are three cases:

Case 4: If there exists $\alpha \in A$, $1 \leq i < j \leq t$ with $c_i < c_j < \alpha$, then $c_i c_j \alpha$ is a 123 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 0$.

Case 5: If there exists $\alpha \in A$, $1 \leq i < j \leq t$ with $c_j < \alpha < c_i$, then $c_i c_j \alpha$ is a 312 pattern and hence $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 0$.

Case 6: The conditions in Case 4 and Case 5 are not met. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(123, 312)$. Let τ' be the subpermutation on $\{1, 2, \dots, y-1\} \cap A$ and let τ'' be the subpermutation on $\{x+1, x+2, \dots, n\} \cap A$. Since τ avoids both 123 and 312, both τ' and τ'' are decreasing. Since $\{c_1, c_2, \dots, c_t\}$ is not a set of consecutive integers, there exists $\alpha \in A$ with $x < \alpha < y$. Since α is in both τ' and τ'' , $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is a decreasing permutation on A . It is easy to see that if the conditions in Case 4 and Case 5 are not met and $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ is decreasing, then τ avoids both 123 and 312 patterns. Hence, we have $|S_{n, (c_1, c_2, \dots, c_t)}(123, 312)| = 1$. \square

Theorem 5.9. *If $(c_1, c_2, \dots, c_t) = (n, n-1, \dots, n-t+1)$, then*

$$|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = \binom{n-t}{2} + 1;$$

otherwise, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = 0$ or 1.

Proof. Write $x = \min\{c_1, c_2, \dots, c_t\}$ and $A = [n] \setminus \{c_1, c_2, \dots, c_t\}$.

Suppose $(c_1, c_2, \dots, c_t) = (n, n-1, \dots, n-t+1)$. It is easy to see that $\tau \in S_{n, (c_1, c_2, \dots, c_t)}$ avoids both 123 and 231 if and only if $(\tau(t+1), \tau(t+2), \dots, \tau(n))$ avoids both 123 and 231. So by Theorem 5.1,

$$|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = |S_{n-t}(123, 231)| = \binom{n-t}{2} + 1.$$

Now suppose $(c_1, c_2, \dots, c_t) \neq (n, n-1, \dots, n-t+1)$. We split into three cases:

Case 1: $\{c_1, c_2, \dots, c_t\} = \{n-t+1, n-t+2, \dots, n\}$. Then there exist indices $1 \leq i < j \leq t$ and $\alpha \in A$ such that $\alpha < c_i < c_j$. So $c_i c_j \alpha$ is a 231 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = 0$.

Case 2: $\{c_1, c_2, \dots, c_t\} \neq \{n-t+1, n-t+2, \dots, n\}$ and there exist indices $1 \leq i < j \leq t$ and $\alpha \in A$ such that $c_i < c_j < \alpha$ or $\alpha < c_i < c_j$, then $c_i c_j \alpha$ is either a 123 pattern or a 231 pattern. Hence, $|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = 0$.

Case 3: The conditions for Case 1 and Case 2 are not met. Let $\tau \in S_{n, (c_1, c_2, \dots, c_t)}(123, 231)$. Since τ avoids 123, the subpermutation τ' of τ on $\{x+1, x+2, \dots, n\} \cap A$ is decreasing. Since τ avoids 231, $\{x+1, x+2, \dots, n\} \cap A \subseteq \mathcal{D}_\tau(i)$ for all $i < x$. Notice that since $\{c_1, c_2, \dots, c_t\} \neq \{n-t+1, n-t+2, \dots, n\}$, there exists $\alpha \in A$ with $\alpha > x$. Now if the subpermutation τ'' of τ on $\{1, 2, \dots, x-1\}$ is not decreasing, we would have a 123 pattern. So

$$\tau = (c_1, c_2, \dots, c_t, x-1, x-2, \dots, 1, \tau'(1), \tau'(2), \dots, \tau'(n-t-x+1)).$$

It is easy to check that τ avoids both 123 and 231. Hence, we have

$$|S_{n, (c_1, c_2, \dots, c_t)}(123, 231)| = 1. \quad \square$$

6. The pair 3412 and 3421

The goal of this section is to show that the counting argument used in the proof of Theorem 4.1 can be generalized to permutations avoiding both 3412 and 3421. We need the following result proved by Kremer [14, Corollary 9]:

Theorem 6.1. *For all $n \geq 1$,*

$$|S_n(3412, 3421)| = \mathbb{S}_{n-1},$$

where \mathbb{S}_{n-1} is the $(n-1)$ st large (big) Schröder number.

For the rest of this section, \mathbb{S}_n is the n th large (big) Schröder number for all $n \in \mathbb{N}$.

We first present our result on $S_{n,r}(3412, 3421)$ since it has an easier presentation but still shows the subtle difference between this pair of pattern of length four and Theorem 4.1. In addition, our result on $S_{n,r}(3412, 3421)$ also allows us to provide an alternate proof of a recurrence relation on the large Schröder numbers.

Theorem 6.2. *For all $n \geq 2$ and $r \in \{1, 2, n\}$, we have*

$$|S_{n,r}(3412, 3421)| = \mathbb{S}_{n-2};$$

and for all $n \geq 4$ and $2 < r < n$, we have

$$|S_{n,r}(3412, 3421)| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}.$$

Proof. First, suppose $n \geq 1$ and $r \in \{1, n\}$. Let $\tau \in S_{n,r}$. If $r = 1$, then $r < a$ for all $a \in \mathcal{D}_\tau(r)$. If $r = n$, then $r > a$ for all $a \in \mathcal{D}_\tau(r)$. If $r = 2$, then there is exactly one $a \in \mathcal{D}_\tau(r)$ with $a < r$. In any case, xyz is not a 3412 pattern or a 3421 pattern for any $x, y, z \in \mathcal{D}_\tau(r)$. Hence $\tau \in S_{n,r}(3412, 3421)$ if and only if $(\tau(2), \tau(3), \dots, \tau(n))$ avoids both 3412 and 3421. Therefore, by Theorem 6.1, we have $|S_{n,r}(3412, 3421)| = |S_{n-1}(3412, 3421)| = \mathbb{S}_{n-2}$.

Now suppose $n \geq 4$ and $2 < r < n$. Let \mathcal{R} be a subset of $S_{n,r}$ such that every $\tau \in \mathcal{R}$ has the following properties:

- (i) $\{\tau(2), \tau(3), \dots, \tau(r-1)\} \subseteq \{1, 2, \dots, r-1\}$;
- (ii) the subpermutation τ' of τ on $\{1, 2, \dots, r-1\}$ avoids both 3412 and 3421;
- (iii) $\tau'' = (\tau(r), \tau(r+1), \dots, \tau(n))$ avoids both 3412 and 3421.

Let $\tau \in \mathcal{R}$. By Theorem 6.1, there are \mathbb{S}_{r-2} ways for τ' to avoid both 3412 and 3421, and for each fixed τ' , there are \mathbb{S}_{n-r} ways for τ'' to avoid both 3412 and 3421. Hence, we have $|\mathcal{R}| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}$.

Now we show that $S_{n,r}(3412, 3421) = \mathcal{R}$. Let $\tau \in S_{n,r}(3412, 3421)$, τ' the subpermutation of τ on $\{1, 2, \dots, r-1\}$, and $\tau'' = (\tau(r), \tau(r+1), \dots, \tau(n))$. Since τ avoids both 3412 and 3421, τ' avoids both 3412 and 3421 as well. Similarly, τ'' avoids both 3412 and 3421.

We now show that $\tau(i) \in \{1, 2, \dots, r-1\}$ for all $i \in \{2, 3, \dots, r-1\}$. Suppose, by way of contradiction, that $\tau(i) > r$ for some $i \in \{2, 3, \dots, r-1\}$. Then, since $\tau(1) = r$, at most $r-3$ numbers in $\{\tau(1), \tau(2), \dots, \tau(r-1)\}$ are less than r . So there exist $k > j > r-1$ such that $\tau(j), \tau(k) < r$.

Now $r\tau(i)\tau(j)\tau(k)$ is either a 3412 pattern or a 3421 pattern. This is a contradiction. Hence, we have $S_{n,r}(3412, 3421) \subseteq \mathcal{R}$.

On the other hand, suppose $\tau \in \mathcal{R}$. We show that $\tau \in S_{n,r}(3412, 3421)$. Suppose, by way of contradiction, that $xyzw$ is a subpermutation of τ which is a 3412 pattern or a 3421 pattern. Then we have $z, w < x < y$ and $z, w \in D_\tau(y)$. We split into three cases:

Case 1: $x = r$. Then $y > r$. Since $\{\tau(2), \tau(3), \dots, \tau(r-1)\} \subseteq \{1, 2, \dots, r-1\}$, we must have $y = \tau(i)$ for some $i > r-1$ and at most one $j > i$ with $\tau(j) < r$. So either $z > r = x$ or $w > r = x$ which is a contradiction.

Case 2: $x < r$. Since the subpermutation on $\{1, 2, \dots, r-1\}$ avoids both 3412 and 3421, we must have $y > r$. The rest of the argument is then the same as Case 1.

Case 3: $x > r$. Since $\{\tau(2), \tau(3), \dots, \tau(r-1)\} \subseteq \{1, 2, \dots, r-1\}$, $xyzw$ is a subpermutation of $(\tau(r), \tau(r+1), \dots, \tau(n))$. Since $(\tau(r), \tau(r+1), \dots, \tau(n))$ avoids both 3412 and 3421, $xyzw$ is not a 3412 or 3421 pattern. This is a contradiction.

This completes the proof that $\mathcal{R} \subseteq S_{n,r}(3412, 3421)$.

Hence, we have $S_{n,r}(3412, 3421) = \mathcal{R}$, and therefore

$$|S_{n,r}(3412, 3421)| = |\mathcal{R}| = \mathbb{S}_{r-2}\mathbb{S}_{n-r}. \quad \square$$

Summing over r in Theorem 6.2, we have the following recurrence relation for \mathbb{S}_n :

Corollary 6.3. *For all $n \geq 1$,*

$$\mathbb{S}_{n+1} = \mathbb{S}_n + \sum_{r=0}^n \mathbb{S}_r \mathbb{S}_{n-r}.$$

Proof. Let $n \geq 1$. Note that by Table 1, we have $\mathbb{S}_0 = 1$. So by Theorem 6.2, we have $|S_{n+2,2}(3412, 3421)| = |S_{n+2,n+2}(3412, 3421)| = \mathbb{S}_n = \mathbb{S}_n \mathbb{S}_0$. Now, by Theorems 6.1 and 6.2, we have

$$\begin{aligned} \mathbb{S}_{n+1} &= |S_{n+2}(3412, 3421)| = \sum_{r=1}^{n+2} |S_{n+2,r}(3412, 3421)| \\ &= \mathbb{S}_n + \sum_{r=2}^{n+2} \mathbb{S}_{r-2} \mathbb{S}_{n+2-r} = \mathbb{S}_n + \sum_{r=0}^n \mathbb{S}_r \mathbb{S}_{n-r}. \quad \square \end{aligned}$$

Remark 6.4. Qi and Guo [18, Theorem 5] proved Corollary 6.3 using generating functions. In [3, p. 446], it is also noted that Corollary 6.3 can also

be derived from the recurrence $\mathbb{S}_n = \sum_{i=0}^n \binom{2n-i}{i} C_{n-i}$ which was proved by West [22, p. 255]. Our proof of this identity does not use the Catalan numbers and is purely combinatorial.

Now we generalize our result for the set $S_{n,r}(3412, 3421)$ to the set $S_{n,(c_1, c_2, \dots, c_t)}(3412, 3421)$. As before, we assume that (c_1, c_2, \dots, c_t) avoids both 3412 and 3421.

Theorem 6.5. *Let $U = \{c_i : i \in [t] \text{ and there exist } j, k \in [t] \text{ such that } i < j < k \text{ and } c_i c_j c_k \text{ is a 231 pattern}\}$ and $V = \{c_i : i \in [t] \text{ and there exists } j \in [t] \text{ such that } i < j \text{ and } c_i < c_j\}$. If $U \neq \emptyset$ and $|[\max U] \setminus \{c_1, c_2, \dots, c_t\}| \geq 1$ or $V \neq \emptyset$ and $|[\max V] \setminus \{c_1, c_2, \dots, c_t\}| \geq 2$, then $|S_{n,(c_1, c_2, \dots, c_t)}(3412, 3421)| = 0$; otherwise,*

$$|S_{n,(c_1, c_2, \dots, c_t)}(3412, 3421)| = \mathbb{S}_{c_{(j)} - c_{(j-1)} - 2} \prod_{i=j+1}^{t+1} \mathbb{S}_{c_{(i)} - c_{(i-1)} - 1},$$

where $c_{(0)} = 0$, $c_{(t+1)} = n + 1$, $c_{(1)} < c_{(2)} < \dots < c_{(t)}$ are the order statistics of $\{c_1, c_2, \dots, c_t\}$, and $j = \min\{i \in [t+1] : c_{(i)} - c_{(i-1)} > 1\}$.

Proof. For all $k \in [t+1]$, let $A_k = \{c_{(k-1)} + 1, c_{(k-1)} + 2, \dots, c_{(k)} - 1\}$. We note that it is possible that $A_k = \emptyset$ for some k . Also notice that

$$[n] \setminus \{c_1, c_2, \dots, c_t\} = \bigcup_{k=1}^{t+1} A_k.$$

We first suppose that $U \neq \emptyset$ and $|[\max U] \setminus \{c_1, c_2, \dots, c_t\}| \geq 1$. Let $x, y, z \in [t]$ such that $c_x = \max U$, $x < y < z$, and $c_x c_y c_z$ is a 231 pattern. Since $|[c_x] \setminus \{c_1, c_2, \dots, c_t\}| \geq 1$, there exists $\alpha \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ such that $\alpha < c_x$. If $\alpha < c_z$, then $c_x c_y c_z \alpha$ is a 3421 pattern; and if $\alpha > c_z$, then $c_x c_y c_z \alpha$ is a 3412 pattern. Hence, $|S_{n,(c_1, c_2, \dots, c_t)}(3412, 3421)| = 0$.

Next, we suppose that $V \neq \emptyset$ and $|[\max V] \setminus \{c_1, c_2, \dots, c_t\}| \geq 2$. Let $x, y \in [t]$ such that $c_x = \max V$, $x < y$, and $c_x < c_y$.

Since $|[c_x] \setminus \{c_1, c_2, \dots, c_t\}| \geq 2$, there exist $\alpha, \beta \in [n] \setminus \{c_1, c_2, \dots, c_t\}$ such that $\alpha < \beta < c_x < c_y$. If $\alpha \in \mathcal{A}_\tau(\beta)$, then $c_x c_y \alpha \beta$ is a 3412 pattern; and if $\alpha \in \mathcal{D}_\tau(\beta)$, then $c_x c_y \beta \alpha$ is a 3421 pattern. Hence, $|S_{n,(c_1, c_2, \dots, c_t)}(3412, 3421)| = 0$.

Now suppose otherwise. Let $j = \min\{i \in [t+1] : c_{(i)} - c_{(i-1)} > 1\}$. In other words, $j \in [t+1]$ is the smallest index such that $A_j \neq \emptyset$. For all $\tau \in S_{n,(c_1, c_2, \dots, c_t)}$, let τ_j be the subpermutation of τ on A_j , and for all $i \in \{j+1, j+2, \dots, t+1\}$, let x_i be the last term of the subpermutation of τ on $A_j \cup A_{j+1} \cup \dots \cup A_{i-1}$ and τ_i be the subpermutation of τ on $A_i \cup \{x_i\}$.

Let \mathcal{R}' be the subset of $S_{n,(c_1,c_2,\dots,c_t)}$ such that every $\tau \in \mathcal{R}'$ satisfies the following:

- (i) for all $i \in \{j+1, j+2, \dots, t+1\}$, $y \in A_i$, and $z \in (A_j \cup A_{j+1} \cup \dots \cup A_{i-1}) \setminus \{x_i\}$, we have $z \in \mathcal{A}_\tau(y)$;
- (ii) and for all $i \in \{j, j+1, \dots, t+1\}$, τ_i avoids both 3412 and 3421.

Let $\tau \in \mathcal{R}'$. By Theorem 6.1, since $|A_j| = c_{(j)} - c_{(j-1)} - 1$, there are $\mathbb{S}_{c_{(j)}-c_{(j-1)}-2}$ possibilities for τ_j . Now let $i \in \{j+1, j+2, \dots, t+1\}$. Inductively, we count the possibilities for τ_i when $\tau_j, \tau_{j+1}, \dots, \tau_{i-1}$ are fixed. In this case, since $|A_i \cup \{x_i\}| = c_{(i)} - c_{(i-1)}$, by Theorem 6.1, there are $\mathbb{S}_{c_{(i)}-c_{(i-1)}-1}$ possibilities for τ_i . Hence, we have $|\mathcal{R}'| = \mathbb{S}_{c_{(j)}-c_{(j-1)}-2} \prod_{i=j+1}^{t+1} \mathbb{S}_{c_{(i)}-c_{(i-1)}-1}$.

It remains to show that $\mathcal{R}' = S_{n,(c_1,c_2,\dots,c_t)}(3412, 3421)$.

Let $\tau \in S_{n,(c_1,c_2,\dots,c_t)}(3412, 3421)$. Then Property (ii) is obviously satisfied. Now we show that τ satisfies Property (i). Suppose, by way of contradiction, that τ does not satisfy Property (i). Then there exist $i \in \{j+1, j+2, \dots, t+1\}$, $y \in A_i$, and $z \in (A_j \cup A_{j+1} \cup \dots \cup A_{i-1}) \setminus \{x_i\}$ such that $z \in D_\tau(y)$. Since x_i is the last term of the subpermutation of τ on $A_j \cup A_{j+1} \cup \dots \cup A_{i-1}$, $c_{(i-1)}yzx_i$ is a subpermutation of τ . Since $y \in A_i$, we have $y > c_{(i-1)}$. Now since $z, x_i \in A_j \cup A_{j+1} \cup \dots \cup A_{i-1}$, $c_{(i-1)}yzx_i$ is either a 3412 pattern or a 3421 pattern. This is a contradiction. Hence, τ satisfies Property (i). It follows that $S_{n,(c_1,c_2,\dots,c_t)}(3412, 3421) \subseteq \mathcal{R}'$.

We still need to prove that $\mathcal{R}' \subseteq S_{n,(c_1,c_2,\dots,c_t)}(3412, 3421)$. Let $\tau \in \mathcal{R}'$. Suppose, by way of contradiction, that $abcd$ is a subpermutation of τ which is either a 3412 pattern or a 3421 pattern. Then we have $c < d < a < b$ or $d < c < a < b$. We have five cases:

Case 1: $a, b, c, d \notin \{c_1, c_2, \dots, c_t\}$. Then $a \in A_k$ for some $k \in \{j, j+1, \dots, t+1\}$. Since $c, d < a$, we have $c \in A_{i_1}$ and $d \in A_{i_2}$ for some $i_1, i_2 \leq k$. Since $b > a$, if $b \notin A_k$, then $b \in A_{i_3}$ with $i_3 > k$. Then Property (i) is violated because $c, d \in A_j \cup A_{j+1} \cup \dots \cup A_k$. So we must have $b \in A_k$. By Property (i), at most one of c and d is in $A_j \cup A_{j+1} \cup \dots \cup A_{k-1}$. In addition, if c or d is in $A_j \cup A_{j+1} \cup \dots \cup A_{k-1}$, then it must be the last term of τ_{k-1} . So $abcd$ is a subpermutation on $A_k \cup \{x_k\}$. This contradicts Property (ii).

Case 2: $a \in \{c_1, c_2, \dots, c_t\}$ but $b, c, d \notin \{c_1, c_2, \dots, c_t\}$. Then $a = c_k$ for some $k \in [t]$. Since $b > a$, we have $b \in A_i$ for some $i > k$. Since $c, d < a$, we have $c, d \in A_j \cup A_{j+1} \cup \dots \cup A_k$. Since $c, d \in D_\tau(b)$, this violates Property (i). Hence, we have a contradiction.

Case 3: $a, b \in \{c_1, c_2, \dots, c_t\}$ but $c, d \notin \{c_1, c_2, \dots, c_t\}$. Since $a < b$ and $c, d < a$, $V \neq \emptyset$ and $|[\max V] \setminus \{c_1, c_2, \dots, c_t\}| \geq 2$ which is a contradiction.

Case 4: $a, b, c \in \{c_1, c_2, \dots, c_t\}$ but $d \notin \{c_1, c_2, \dots, c_t\}$. Then abc is a 231 pattern. Since $d < a$, $U \neq \emptyset$ and $|[\max U] \setminus \{c_1, c_2, \dots, c_t\}| \geq 1$ which is a contradiction.

Case 5: $a, b, c, d \in \{c_1, c_2, \dots, c_t\}$. This contradicts our convention that (c_1, c_2, \dots, c_t) avoids both 3412 and 3421.

Hence, τ avoids both 3412 and 3421. It follows that

$$\mathcal{R}' \subseteq S_{n, (c_1, c_2, \dots, c_t)}(3412, 3421).$$

This completes our proof. \square

7. r -Wilf-equivalence classes

In this section, we classify r -Wilf-equivalence classes for patterns of length three. Recall that for a fixed $r \in \mathbb{N}$, two patterns σ and σ' are called r -Wilf equivalent if $|S_{n,r}(\sigma)| = |S_{n,r}(\sigma')|$ for all $n \geq r$.

We start with some elementary results summarized in Table 2.

Table 2: Single Patterns of Length 3 for $n \geq 2$

| | $r = n$ | $r = n - 1$ | $r = 2$ | $r = 1$ |
|------------------|-----------|-------------|-----------|-----------|
| $ S_{n,r}(123) $ | C_{n-1} | C_{n-1} | $n - 1$ | 1 |
| $ S_{n,r}(321) $ | 1 | $n - 1$ | C_{n-1} | C_{n-1} |
| $ S_{n,r}(132) $ | C_{n-1} | C_{n-1} | $n - 1$ | 1 |
| $ S_{n,r}(312) $ | 1 | $n - 1$ | C_{n-1} | C_{n-1} |
| $ S_{n,r}(213) $ | C_{n-1} | C_{n-2} | C_{n-2} | C_{n-1} |
| $ S_{n,r}(231) $ | C_{n-1} | C_{n-2} | C_{n-2} | C_{n-1} |

It is easy to check the correctness of the expressions in Table 2. As an example, we sketch the proof of the fact that $|S_{n,n-1}(123)| = C_{n-1}$ for all $n \geq 2$. For any $i, j \in \{2, 3, \dots, n\}$ with $i < j$, either $\tau(i) < n - 1$ or $\tau(j) < n - 1$. It follows that $(n - 1, \tau(i), \tau(j))$ will never form a 123 pattern for any $i, j \in \{2, 3, \dots, n\}$ with $i < j$. Hence, $\tau \in S_{n,n-1}$ avoids 123 if and only if $(\tau(2), \tau(3), \dots, \tau(n))$ avoids 123. Therefore, by Theorem 1.1, we have $|S_{n,n-1}(123)| = |S_{n-1}(123)| = C_{n-1}$.

Next we classify the r -Wilf-equivalence classes for patterns of length three for all $r \in \mathbb{N}$.

Theorem 7.1. *There are two 1-Wilf-equivalence classes for patterns of length three: $123 \overset{1}{\sim} 132$ and $321 \overset{1}{\sim} 312 \overset{1}{\sim} 213 \overset{1}{\sim} 231$. For $r \geq 2$, there are three r -Wilf-equivalence classes for patterns of length three: $213 \overset{r}{\sim} 231$, $123 \overset{r}{\sim} 132$, and $321 \overset{r}{\sim} 312$.*

Proof. The fact that there are two 1-Wilf-equivalence classes for patterns of length three follows from the last row of Table 2.

Let $r \geq 2$. By Lemma 2.2, we have $213 \stackrel{r}{\sim} 231$, $123 \stackrel{r}{\sim} 132$, and $321 \stackrel{r}{\sim} 312$. We need to show that there exist $n_1, n_2, n_3 \geq r$ such that $|S_{n_1, r}(213)| \neq |S_{n_1, r}(123)|$, $|S_{n_2, r}(213)| \neq |S_{n_2, r}(321)|$, and $|S_{n_3, r}(123)| \neq |S_{n_3, r}(321)|$. There are three cases to consider.

Case 1: $r = 2$. Set $n_1 = n_2 = n_3 = 4$. By Tables 1 and 2, we have $|S_{4,2}(123)| = 4 - 1 = 3$, $|S_{4,2}(321)| = C_3 = 5$, and $|S_{4,2}(213)| = C_2 = 2$. Hence we have the desired result.

Case 2: $r = 3$. Set $n_1 = n_2 = n_3 = 4$. By Tables 1 and 2, we have $|S_{4,3}(123)| = C_4 = 5$, $|S_{4,3}(321)| = 4 - 1 = 3$, and $|S_{4,3}(213)| = C_2 = 2$. Hence we have the desired result.

Case 3: $r \geq 3$. Set $n_1 = n_2 = n_3 = r + 1$. By Table 2, we have $|S_{r+1, r}(123)| = C_r$, $|S_{r+1, r}(321)| = r$, and $|S_{r+1, r}(213)| = C_{r-1}$. By Table 1, we have $C_r > C_{r-1} > r$ and the theorem follows. \square

In addition to the classical patterns we have studied so far in this paper, many papers studied consecutive patterns [8], bivariate patterns [4], and mesh patterns [5, 10]. Here we briefly describe, for all $r \geq 5$, the r -Wilf equivalence classes for vincular patterns of length three studied by Babson and Steingrímsson [2] and later, Claesson [6].

In vincular patterns [10, Section 2], some consecutive elements in a permutation pattern are required to be adjacent. We use overlines to indicate that the elements under the overlines are required to be adjacent. There are twelve vincular patterns of length three where one requires exactly two numbers to be adjacent. For example, a permutation $\tau \in S_n$ contains the pattern $\overline{132}$ if there exist indices $i < j$ such that $\tau(i)\tau(j)\tau(j+1)$ is a 132 pattern. Other vincular patterns are defined similarly.

Example 7.2. In the permutation $\tau = 13542 \in S_5$, $\tau(2)\tau(3)\tau(5) = 352$ is a $\overline{231}$ pattern and $\tau(1)\tau(4)\tau(5) = 142$ is a $\overline{132}$ pattern, but τ avoids the pattern $\overline{213}$.

Claesson [6, Propositions 1-3 and Lemma 2] proved that there are two Wilf-equivalence classes for the twelve vincular patterns. They are counted either by the Catalan numbers or by the Bell numbers:

Theorem 7.3. For all $n \geq 1$,

$$\begin{aligned} |S_n(\overline{123})| &= |S_n(\overline{321})| = |S_n(\overline{123})| = |S_n(\overline{321})| = |S_n(\overline{132})| \\ &= |S_n(\overline{312})| = |S_n(\overline{213})| = |S_n(\overline{231})| = B_n, \end{aligned}$$

and

$$|S_n(\overline{213})| = |S_n(\overline{231})| = |S_n(\overline{132})| = |S_n(\overline{312})| = C_n,$$

where B_n is the n th Bell number and C_n is the n th Catalan number.

We first adapt some results in Claesson [6] to show r -Wilf equivalence for several vincular patterns.

Proposition 7.4. *For all $r \in \mathbb{N}$, $2\overline{13} \stackrel{r}{\sim} 2\overline{31}$, $1\overline{23} \stackrel{r}{\sim} 1\overline{32}$, and $3\overline{21} \stackrel{r}{\sim} 3\overline{12}$.*

Proof. Let $1 \leq r \leq n$. Using a short combinatorial argument, Claesson [6, Lemma 2] showed that for all $n \geq 1$, $\tau \in S_n$ avoids $2\overline{13}$ if and only if it avoids 213. Taking complements, for all $n \geq 1$, $\tau \in S_n$ avoids $2\overline{31}$ if and only if it avoids 231. Hence, for all $\tau \in S_{n,r}$, τ avoids $2\overline{13}$ if and only if it avoids 213 and τ avoids $2\overline{31}$ if and only if it avoids 231. Then we have $|S_{n,r}(2\overline{13})| = |S_{n,r}(213)|$ and $|S_{n,r}(2\overline{31})| = |S_{n,r}(231)|$. Now by Lemma 2.2, we have $|S_{n,r}(2\overline{13})| = |S_{n,r}(213)| = |S_{n,r}(231)| = |S_{n,r}(2\overline{31})|$. Therefore, $2\overline{13} \stackrel{r}{\sim} 2\overline{31}$.

Claesson [6, Propositions 2 and 4] constructed bijections between $S_n(1\overline{23})$ and the partitions of $[n]$, and then between $S_n(1\overline{32})$ and the partitions of $[n]$. These bijections preserve the leading terms of permutations. So for all $1 \leq r \leq n$, we have $|S_{n,r}(1\overline{23})| = |S_{n,r}(1\overline{32})|$. Taking the complements, we also have $|S_{n,r}(3\overline{21})| = |S_{n,r}(3\overline{12})|$. Therefore, we have $1\overline{23} \stackrel{r}{\sim} 1\overline{32}$ and $3\overline{21} \stackrel{r}{\sim} 3\overline{12}$. \square

By Proposition 7.4, there are at most nine r -Wilf-equivalence classes for vincular patterns. Table 3 lists the results we need to classify r -Wilf equivalence classes for all twelve vincular patterns.

Table 3: Avoiding Vincular Patterns by Leading Terms for $r \geq 3$. (We leave some entries in the table blank and only include results that are needed to classify r -Wilf equivalence classes for the twelve vincular patterns.)

| | $n = r$ | $n = r + 1$ | $n = r + 2$ |
|---|-----------|-----------------|---------------------|
| $ S_{n,r}(213) = S_{n,r}(231) $ | C_{r-1} | C_{r-1} | |
| $ S_{n,r}(1\overline{32}) $ | C_{r-1} | C_r | |
| $ S_{n,r}(3\overline{21}) = S_{n,r}(3\overline{12}) $ | 1 | 2^{r-1} | |
| $ S_{n,r}(3\overline{12}) $ | 1 | r | |
| $ S_{n,r}(1\overline{23}) = S_{n,r}(1\overline{32}) $ | B_{r-1} | B_r | $B_{r+1} - B_{r-1}$ |
| $ S_{n,r}(1\overline{23}) $ | B_{r-1} | B_r | $B_{r+1} - B_r$ |
| $ S_{n,r}(213) $ | B_{r-1} | B_{r-1} | |
| $ S_{n,r}(231) $ | B_{r-1} | $B_r - B_{r-1}$ | |
| $ S_{n,r}(321) $ | B_{r-2} | | |

Most of the expressions in Table 3 can be obtained by straightforward calculation using Theorem 7.3 and Table 2. We will only sketch the proofs of a few of them.

Lemma 7.5. *For all $r \geq 3$,*

$$|S_{r,r}(\overline{321})| = B_{r-2}.$$

Proof. Let $\tau \in S_{r,r}(\overline{321})$. If $\tau(2) \neq 1$, then $\tau(1)\tau(2)1$ is a $\overline{321}$ pattern. So we must have $\tau(2) = 1$. At the same time, $(\tau(3), \tau(4), \dots, \tau(n))$ avoids the pattern $\overline{321}$. So by Theorem 7.3, we have $|S_{r,r}(\overline{321})| \leq |S_{r-2}(\overline{321})| = B_{r-2}$.

Now let $\tau \in S_{r,r}$ with $\tau(2) = 1$ and $(\tau(3), \tau(4), \dots, \tau(n))$ avoiding the pattern $\overline{321}$. Since $r1x$ and $1xy$ are never $\overline{321}$ patterns, we must have $\tau \in S_{r,r}(\overline{321})$. Hence, we have $B_{r-2} = |S_{r-2}(\overline{321})| \leq |S_{r,r}(\overline{321})|$. This completes the proof of the lemma. \square

Lemma 7.6. *For all $r \geq 1$,*

$$|S_{r+2,r}(\overline{123})| = B_{r+1} - B_{r-1} \text{ and } |S_{r+2,r}(\overline{12\overline{3}})| = B_{r+1} - B_r.$$

Proof. We first prove that $|S_{r+2,r}(\overline{123})| = B_{r+1} - B_{r-1}$. Let $\tau \in S_{r+2,r}(\overline{123})$. Then the subpermutation $(\tau(2), \tau(3), \dots, \tau(r+2))$ avoids $\overline{123}$. By Theorem 7.3, there are $|S_{r+1}(\overline{123})| = B_{r+1}$ ways for $(\tau(2), \tau(3), \dots, \tau(r+2))$ to avoid $\overline{123}$. For these permutations on $\{1, 2, \dots, r-1, r+1, r+2\}$, the only way that $r+1$ and $r+2$ are adjacent and the subpermutation on $\{r+1, r+2\}$ is $(r+1, r+2)$ is when $\tau(2) = r+1$ and $\tau(3) = r+2$ because otherwise $(\tau(2), \tau(3), \dots, \tau(r+2))$ would contain a $\overline{123}$ pattern. This is the only case that $(r, r+1, r+2)$ is a $\overline{123}$ pattern. Since $(\tau(4), \tau(5), \dots, \tau(r+2))$ also need to avoid $\overline{123}$, by Theorem 7.3, the number of permutations $(\tau(2), \tau(3), \dots, \tau(r+2))$ avoiding $\overline{123}$, with $\tau(2) = r+1$ and $\tau(3) = r+2$, is $|S_{r-1}(\overline{123})| = B_{r-1}$. Here it is easy to check that if $\tau(2) = r+1$, $\tau(3) = r+2$, and $(\tau(4), \tau(5), \dots, \tau(r+2))$ avoids $\overline{123}$, then $(\tau(2), \tau(3), \dots, \tau(r+2))$ avoids $\overline{123}$ as well. Therefore

$$|S_{r+2,r}(\overline{123})| = |S_{r+1}(\overline{123})| - |S_{r-1}(\overline{123})| = B_{r+1} - B_{r-1}.$$

Next, we prove that $|S_{r+2,r}(\overline{12\overline{3}})| = B_{r+1} - B_r$. Let $\tau \in S_{r+2,r}(\overline{12\overline{3}})$. Then $(\tau(2), \tau(3), \dots, \tau(r+2))$ avoids $\overline{12\overline{3}}$. By Theorem 7.3, there are $|S_{r+1}(\overline{12\overline{3}})| = B_{r+1}$ ways for $(\tau(2), \tau(3), \dots, \tau(r+2))$ to avoid $\overline{12\overline{3}}$. For these permutations, the only way that we have a $\overline{12\overline{3}}$ pattern starting with r is when $\tau(2) = r+1$, then $(\tau(1), \tau(2), r+2)$ is a $\overline{12\overline{3}}$ pattern. Here, it is easy to see that if $\tau(2) = r+1$, then, for all $2 < i < j \leq r+2$, $(\tau(2), \tau(i), \tau(j))$ is never a $\overline{12\overline{3}}$ pattern. Hence, by Theorem 7.3, the number of permutations $(\tau(2), \tau(3), \dots, \tau(r+2))$, with $\tau(2) = r+1$, avoiding $\overline{12\overline{3}}$ is $|S_r(\overline{12\overline{3}})| = B_r$. Using subtraction, we have

$$|S_{r+2,r}(\overline{12\overline{3}})| = |S_{r+1}(\overline{12\overline{3}})| - |S_r(\overline{12\overline{3}})| = B_{r+1} - B_r. \quad \square$$

Lemma 7.7. *For all $r \geq 1$,*

$$|S_{r+1,r}(3\overline{21})| = 2^{r-1}.$$

Proof. Let $\tau \in S_{r+1,r}(3\overline{21})$ and let $i > 1$ be such that $\tau(i) = r + 1$. Then for all $j \in \{2, 3, \dots, i - 2\}$, we must have $\tau(j) < \tau(j + 1)$. To see this, suppose that $\tau(j) > \tau(j + 1)$ for some $j \in \{2, 3, \dots, i - 2\}$. Then $r\tau(j)\tau(j + 1)$ is a $3\overline{21}$ pattern which is a contradiction. Similarly, for all $j \in \{i + 1, i + 2, \dots, n - 1\}$, we must have $\tau(j) < \tau(j + 1)$. Hence, we have $\tau(2) < \tau(3) < \dots < \tau(i - 1)$ and $\tau(i + 1) < \tau(i + 2) < \dots < \tau(r + 1)$.

On the other hand, it is easy to check that for all $\tau \in S_{r+1,r}$, if $\tau(i) = r + 1$, $\tau(2) < \tau(3) < \dots < \tau(i - 1)$, and $\tau(i + 1) < \tau(i + 2) < \dots < \tau(r + 1)$ for some $i > 1$, then τ avoids $3\overline{21}$.

So $|S_{r+1,r}(3\overline{21})|$ is equal to the number of permutations $\tau \in S_{r+1,r}$ such that for some $i \in \{2, 3, \dots, r + 1\}$, we have $\tau(i) = r + 1$, $\tau(2) < \tau(3) < \dots < \tau(i - 1)$, and $\tau(i + 1) < \tau(i + 2) < \dots < \tau(r + 1)$. Let τ be such a permutation and $i \in \{2, 3, \dots, r + 1\}$. Then there are $\binom{r-1}{i-2}$ ways to choose $i - 2$ numbers from $\{1, 2, \dots, r - 1\}$ and assign them to $\tau(2), \tau(3), \dots, \tau(i - 1)$ so that $\tau(2) < \tau(3) < \dots < \tau(i - 1)$; once $\tau(2), \tau(3), \dots, \tau(i - 1)$ are determined, $\tau(i + 1), \tau(i + 2), \dots, \tau(r + 1)$ are uniquely determined as well. Hence, we have

$$|S_{r+1,r}(3\overline{21})| = \sum_{i=2}^{r+1} \binom{r-1}{i-2} = \sum_{i=0}^{r-1} \binom{r-1}{i} = 2^{r-1}. \quad \square$$

If $r \geq 5$, by Tables 1 and 3 and Lemmas 2.10 and 2.11, there are nine r -Wilf equivalence classes. To see this, it suffices to note that for each $r \geq 5$ and for any two distinct generalized patterns σ and σ' in different rows, either $|S_{r,r}(\sigma)| \neq |S_{r,r}(\sigma')|$, or $|S_{r+1,r}(\sigma)| \neq |S_{r+1,r}(\sigma')|$, or $|S_{r+2,r}(\sigma)| \neq |S_{r+2,r}(\sigma')|$. We briefly describe several of them as examples.

Example 7.8. By Table 3 and Lemma 2.10, for all $r \geq 5$, $|S_{r,r}(\overline{132})| = C_{r-1} < B_{r-1} = |S_{r,r}(\overline{123})|$. Hence, for all $r \geq 5$, $\overline{132}$ and $\overline{123}$ are not r -Wilf equivalent.

Example 7.9. By Table 3 and Lemma 2.11, for all $r \geq 5$, $|S_{r,r}(\overline{213})| = B_{r-1} = |S_{r,r}(\overline{231})|$, but $|S_{r+1,r}(\overline{213})| = B_{r-1} < B_r - B_{r-1} = |S_{r+1,r}(\overline{231})|$. Hence, for all $r \geq 5$, $\overline{213}$ and $\overline{231}$ are not r -Wilf equivalent.

Example 7.10. By Table 3, we have $|S_{r,r}(\overline{12\overline{3}})| = B_{r-1} = |S_{r,r}(\overline{1\overline{23}})|$ and $|S_{r+1,r}(\overline{12\overline{3}})| = B_r = |S_{r+1,r}(\overline{1\overline{23}})|$, but $|S_{r+2,r}(\overline{12\overline{3}})| = B_{r+1} - B_{r-1} > B_{r+1} - B_r = |S_{r+2,r}(\overline{1\overline{23}})|$ for all $r \geq 5$. Hence $\overline{12\overline{3}}$ and $\overline{1\overline{23}}$ belong to two different equivalence classes when $r \geq 5$.

Example 7.11. By Table 3, we have $|S_{r,r}(2\overline{13})| = C_{r-1}$ and $|S_{r,r}(\overline{321})| = B_{r-2}$ for all $r \geq 5$. By Table 1 and the generating functions of the Catalan and Bell numbers [7, Sections 3.2 and 6.1], we have $B_{r-1} \neq C_r$ for all $r \geq 5$. Hence, $2\overline{13}$ and $\overline{321}$ belong to two different equivalence classes when $r \geq 5$.

The following theorem completely classifies, for all $r \geq 5$, the r -Wilf-equivalence classes for the twelve vincular patterns of length three.

Theorem 7.12. *For all $r \geq 5$, there are nine r -Wilf-equivalence classes for vincular patterns of length three: $2\overline{13} \sim 23\overline{1}$, $1\overline{23} \sim 13\overline{2}$, $3\overline{21} \sim 3\overline{12}$, and the other six classes each contains a single vincular pattern.*

8. Concluding remarks

Miner and Pak [15] used generalizations of Theorem 1.2 to study the limit shapes of random permutations avoiding a given pattern. In this section, we give some ideas about the limit shapes of random σ -avoiding, $\sigma \in S_3$, permutations with fixed prefix (c_1, c_2, \dots, c_t) . Particularly, we are interested in exploring for large n , what a uniformly random permutation from $S_{(c_1, c_2, \dots, c_t)}(\sigma)$ looks like. To do this, we follow Miner and Pak [15] and view permutations as matrices. That is, for each $\tau \in S_n$, we look at the $n \times n$ matrix $M(\tau)$ such that $(M(\tau))_{jk} = 1$ if $\tau(j) = k$ and $(M(\tau))_{jk} = 0$ if $\tau(j) \neq k$. By Lemma 2.2, complementary patterns may be studied in pairs and it suffices to examine permutations avoiding the patterns 123, 132, and 231.

In some situations, this question is easy to answer. If $1 \in \{c_1, c_2, \dots, c_t\}$, then there is a unique permutation that avoids a 123 pattern, as the later $n-t$ digits need to be decreasing to avoid a 23 pattern in the second unfixed segment. So after asymptotic scaling, the limit of the unfixed segment is just the anti-diagonal $x + y = 1$. The situation becomes more complicated when $1 \notin \{c_1, c_2, \dots, c_t\}$. As shown in Theorem 4.4, we may project our permutation from S_n where the first t coordinates are fixed down to a permutation from S_{n-t+1} where only the first coordinate is fixed via ‘standardization.’ The limiting phenomenon of these generic ‘reduced’ 123-avoiding permutations were studied in Miner and Pak [15], where the anti-diagonal again shows up. As pointed out earlier in Section 4, the result for the 132 pattern is similar to the 123 pattern and the structure of the pattern-avoiding permutation is also preserved after projection via standardization. See Theorem 4.5 for more details. The limiting phenomenon of these reduced 132-avoiding permutations was also studied in Miner and Pak [15], where the anti-diagonal as well as the lower right corner show up in the asymptotic

analysis. Unlike 123 and 132 patterns, fixing the prefix (c_1, c_2, \dots, c_t) , a uniformly random permutation avoiding a 231 pattern will instead display a block structure as hinted in our proof of Theorem 4.1. For the initial block which consists of c_1, c_2, \dots, c_t , the segment of the permutation will be a fixed curve that is asymptotically in correspondence to the prefix (c_1, c_2, \dots, c_t) ; and for all the remaining blocks, the segment of the permutation will lie on the boundary of feasible 231-density asymptotically. See Kenyon et al. [11] and the references therein for a description of the limit shapes of these feasible regions.

We have only scratched the surface of enumerating pattern-avoiding permutations by fixed prefixes, mostly concentrating on patterns of length three. It would be interesting to study permutations with fixed prefixes that avoid other patterns; for instance, all single patterns of length greater than three are open. It would also be interesting to compute limits of pattern avoiding permutations chosen uniformly under certain constraints; fixing the prefix (c_1, c_2, \dots, c_t) as we have done in this paper is only one of the many possibilities out there.

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