Polynomial Families Satisfying a Riemann Hypothesis

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Abstract

Consider a linear transformation $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined on basis elements $1, x, x^2, \ldots$ by

$$T[x^k] = \frac{(x)_k}{k!}$$

where $(x)_k = x(x+1)(x+2)\cdots(x+k-1),\ k \geq 0$. We create infinite families of polynomials of the form $T[p_n(x)]$, each member of which satisfies a Riemann hypothesis; i.e., their zeros lie on the line $[s = \frac{1}{2} + it :\ t \text{ real}]$. These families are indexed by a real parameter r, and are of the form $p_n(x) = (x+r)^n + (1-x+r)^n$ for $n \geq 2$. Our proof uses a positivity argument together with certain elements of the theory of 3–term polynomial recursions.

Keywords: Riemann hypothesis, 3–term recursion, orthogonal polynomial, positivity.

1 Introduction: The T-transform

Let $(x)_k$ denote the upper factorial polynomial

$$(x)_k = x(x+1)(x+2)\cdots(x+k-1), \quad k \geq 1$$

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with $(x)_0 = 1$. We consider a linear transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined on basis elements by

$$T \ : \ 1 \longrightarrow 1$$
$$x^k \longrightarrow \frac{(x)_k}{k!}, \qquad k \geq 1.$$

For a polynomial $p(x)$, the transformation T can be viewed in terms of the complex integral transform

$$\int_0^1 x^s(1-x)^{1-s}p(x)\frac{dx}{x(1-x)} = \frac{\pi}{\sin \pi s} \cdot (Tp)(s)$$

for $s = \sigma + it,\ 0 < \sigma < 1$. In view of this, we use the notation $Tp(x)(s)$ for $Tp(x)$ when we view the range as the polynomial ring $\mathbb{R}[s]$ of the complex variable s. In particular for $p(x) = (x - \frac{1}{2})^n$,

$$\int_0^1 x^s(1-x)^{1-s}(x-\tfrac{1}{2})^n\frac{dx}{x(1-x)} = \frac{\pi}{\sin \pi s} \cdot L_n(s),$$

where

$$L_n(s) = \sum_{k=0}^{n}(-\tfrac{1}{2})^{n-k}\binom{n}{k} \cdot \frac{(s)_k}{k!} \ = \ T[(x-\tfrac{1}{2})^n](s). \tag{1}$$

It is well-known that the family of polynomials $L_n(s)$, indexed on the natural numbers, each satisfies a Riemann hypothesis; i. e., their zeros lie on the line

$$[s = \tfrac{1}{2} + it : \ t \ \text{real}].$$

We denote this by writing

$$L_n(s) \in Rh,$$

and we ask if there are other polynomials $p(x)$ so that the T-transforms

$$Tp^n(s)$$

of all positive integer powers of p satisfy a Riemann hypothesis.

Additionally, in somewhat greater generality, we would want to describe infinite families of polynomials, $[p_n(x) : n \geq 1]$, all of whose T-transforms satisfy a Riemann hypothesis. The main result of this paper is that for any real number r, the T-images of the family of polynomials

$$(x+r)^n + (1-x+r)^n, \qquad n \geq 2,$$

satisfy a Riemann hypothesis. This is Theorem 1 of Section 3.

2 Observations about the T-transform

(i) It is not difficult to come up with some simple necessary conditions on the coefficients of $p(x)$ if $Tp(x)$ is to satisfy a Riemann hypothesis. For example a monic polynomial p of degree n such that $Tp(x) \in Rh$ must be of the form

$$p(x) = x^n - \frac{n}{2}x^{n-1} + \cdots$$

(ii) Calculating using the definition, we have

$$T(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^{n-k} \cdot \frac{(x)_k}{k!}.$$

On the other hand from the integral transform characterization we calculate also that

$$T(1-x)^n = \frac{(1-x)_n}{n!}.$$

(iii) In terms of multiplicative properties, it can be verified that

$$T[(x(x-1)+r)(x-\tfrac{1}{2})] \in Rh \ \text{ iff } r \geq \frac{1}{24},$$

and that

$$T[(x(x-1)+r)(x-\tfrac{1}{2})^2] \in Rh \ \text{ iff } r \geq \frac{1}{16}$$

by a straightforward calculation. This shows that the T-images Tp_1 and Tp_2 of a pair of polynomials may both satisfy a Riemann hypothesis, and yet the T-image $T(p_1 p_2)$ of the product may fail to do so: e.g., with $\frac{1}{24} < r < \frac{1}{16}$, take

$$\begin{aligned} p_1(x) &= (x(x-1)+r)(x-\tfrac{1}{2}) \\ p_2(x) &= x - \tfrac{1}{2}\ . \end{aligned}$$

(iv) It is also false that $Tp^n(s) \in Rh$ for some $n \geq 2$ implies $Tp(s) \in Rh$: take

$$p(x) = (x(x-1)+r)(x-\tfrac{1}{2})^2\ .$$

Then $Tp(x) \in Rh$ iff $r \geq 1/16$. On the other hand, $Tp^2(x) \in Rh$ for values of r smaller than $1/16$. For $r = 1/16 - 1/256$ for example, $Tp^2(x) \in Rh$, but $Tp(x) \notin Rh$.

(v) That the $L_n(x)$ satisfy a Riemann hypothesis may be verified in various ways. One way is to show that the polynomials

$$\tilde{L}_n(w) = L_n(w + \tfrac{1}{2}),$$

are eigenfunctions of an operator of the type

$$S(a, b) \circ P = (w + a)P(w + b) - (w - a)P(w - b), \qquad (2)$$

for some constants $a > 0$ and $b > 0$, so that

$$S(a, b)\tilde{L}_n(w) = \lambda_n \tilde{L}_n(w). \qquad (3)$$

We show in the proof of Lemma 6 that for $\tilde{L}_n(w)$ it is possible to take $a = \frac{1}{2}$, $b = 1$, and $\lambda_n = 2n + 1$. It follows that the zeros of $\tilde{L}_n(w)$ are purely imaginary, so that $L_n(s) \in Rh$. On the other hand, the polynomial

$$p(s) = [(s - \tfrac{1}{2})^2 + 1][(s - \tfrac{1}{2})^2 + 2]$$

satisfies a Riemann hypothesis, and yet

$$p(w + \tfrac{1}{2})$$

is not the eigenfunction of any operator of the type (2). In fact, Redmond [4] has shown that there is a large class of polynomials that satisfy a Riemann hypothesis, and yet fail to be eigenfunctions of any product of operators of the type

$$S(a_1, b_1, c_1) \circ S(a_2, b_2, c_2) \circ \cdots \circ S(a_n, b_n, c_n),$$

where

$$S(a, b, c)Q(w) = (w + a)Q(cw + b) - (w - a)Q(cw - b),$$

with a, b, c positive. Hence, an approach to the study of polynomials that satisfy a Riemann hypothesis will involve techniques other than this eigenfunction approach to $L_n(s)$.

(vi) If $Tp(s) \in Rh$, then the polynomial p satisfies either $p(x) = p(1 - x)$, or $p(x) = -p(1 - x)$. Up to a multiplicative constant the only linear polynomial in this class is $p(x) = x - \frac{1}{2}$. A polynomial that satisfies $p(x) = p(1 - x)$ can be written as a sum

$$p(x) = q(x) + q(1 - x)$$

for a suitable polynomial q.

Our basic result in this paper is on the T-images of the polynomial family

$$q^n(x) + q^n(1 - x)$$

where q is linear. Such a polynomial is non-constant only for $n \geq 2$.

3 The Main Theorem

Theorem 1 *For any real number r, the T-images of the family of polynomials*

$$(x+r)^n + (1-x+r)^n, \qquad n \geq 2,$$

satisfy a Riemann hypothesis; i.e., their zeros lie on the line $[s = \frac{1}{2} + it : \ t \text{ real}]$.

Outline of proof Note that if we substitute

$$r \to -(1+r)$$

then

$$(x+r)^n + (1-x+r)^n \ \to \ (-1)^n[(x+r)^n + (1-x+r)^n].$$

Hence by symmetry, it is sufficient to prove Theorem 1 for $r \geq -\frac{1}{2}$. Henceforth, we assume that $r \geq -\frac{1}{2}$. Define

$$T(x+r)^n(s) = I_n(s,r).$$

From the integral transform formulation, we also have

$$T(1-x+r)^n(s) = I_n(1-s,r).$$

Further define

$$q_n(w,r) = I_n(w+\tfrac{1}{2}, r),$$

where we have made the substitution $s = w + \frac{1}{2}$. Hence

$$\begin{aligned} T[(x+r)^n + (1-x+r)^n](s) &= I_n(w+\tfrac{1}{2},r) + I_n(-w+\tfrac{1}{2},r) \\ &= q_n(w) + q_n(-w) \ , \end{aligned}$$

and the objective is to show that this polynomial has purely imaginary zeros in w for $n \geq 2$. Although it is not necessary to do so, for clarity we break down the proof into a number of cases:

(a) $r \geq 0$,

(b) $-\frac{1}{2} < r < 0$,

(c) $r = -\frac{1}{2}$.

As for case (a), it will be shown in the corollary to Lemma 2 that for $r > 0$, the sequence of polynomials $[q_n(w)]$ satisfies a 3–term recursion. From this recursion, we deduce (Lemma 3) that the zeros of each $q_n(w) = q_n(w,r)$ are real. In Lemma 4, a positivity argument shows that the zeros of the

$q_n(w)$ are in fact negative. Thus, the zeros of the $q_n(-w)$ are positive, in which case Lemma 1 shows that the zeros of $q_n(w) + q_n(-w)$ are purely imaginary. For the $r = 0$ instance of case (a), the 3–term recursion for $[q_n(w)]$ becomes a 2–term recursion, which can be explicitly solved to show that the zeros of $q_n(w)$ are negative. Then again by Lemma 1, it follows that the zeros of $q_n(w) + q_n(-w)$ are purely imaginary.

As for case (b), the w-zeros of $q_n(w, r)$ are not necessarily real for all r in the interval $-\frac{1}{2} < r < 0$. Lemma 5 shows however, that the zeros do have negative real parts, which is sufficient to again show that the zeros of $q_n(w) + q_n(-w)$ are purely imaginary via Lemma 1.

Finally, case (c) with $r = -\frac{1}{2}$ results in the polynomials $L_n(s)$ introduced in (1). We treat this case as a separate lemma (Lemma 6). The technique presented is different from (a) and (b), and uses the operator $S(a, b)$ defined in (2).

3.1 The proof of Theorem 1

We prove a number of lemmas as we prove cases (a), (b), and (c) above. We start with a general result:

Lemma 1 *If the real parts of the zeros of a real polynomial $Q(w)$ are negative, then the zeros of $Q(w) + Q(-w)$ are purely imaginary.*

Proof We give a proof of this lemma, though it is probably known in some form in the literature. Write the m real zeros of Q as

$$0 > -r_1 > -r_2 > \cdots > -r_m,$$

and the $M/2$ conjugate pairs of non-real zeros as

$$-\beta_k \pm i\gamma_k, \qquad (1 \le k \le M/2)$$

with

$$0 > -\beta_1 > -\beta_2 > \cdots > -\beta_{M/2} \, .$$

Thus, the degree of Q is $n = m + M$, and

$$Q(w) = \prod_{j=1}^{m}(w + r_j) \cdot \prod_{k=1}^{M/2}(w + \beta_k + i\gamma_k) \prod_{k=1}^{M/2}(w + \beta_k - i\gamma_k) \, .$$

We now put $w = it$ and consider the real solutions in t of the equation,

$$\begin{aligned} \frac{Q(it)}{Q(-it)} &= \prod_{j=1}^{m}[\frac{r_j + it}{r_j - it}] \cdot \prod_{k=1}^{M/2}[\frac{\beta_k + i(t + \gamma_k)}{\beta_k - i(t + \gamma_k)}] \prod_{k=1}^{M/2}[\frac{\beta_k + i(t - \gamma_k)}{\beta_k - i(t - \gamma_k)}] \qquad (4) \\ &= -1 \end{aligned}$$

For real t, the modulus of each factor in the product above is 1. In order to show that there are n real solutions in t, we consider the argument of the product as t ranges from $-\infty$ to ∞. As t runs from $-\infty$ to ∞, each argument

$$Arg\ [\frac{r_j + it}{r_j - it}]$$

and

$$Arg\ [\frac{\beta_k + i(t \pm \gamma_k)}{\beta_k - i(t \pm \gamma_k)}]$$

runs from $-\pi$ to π. Therefore the argument

$$Arg\ [\frac{Q(it)}{Q(-it)}]$$

runs from $-n\pi$ to $n\pi$. When n is even, the number of odd integer multiples of π in the interval $(-n\pi,\ n\pi)$ is n, and therefore equation (4) has n real solutions in t. When n is odd, the polynomial $Q_n(w)+Q_n(-w)$ is of degree $n-1$. But now the number of odd integer multiples of π on the interval $(-n\pi,\ n\pi)$ is $n-1$, which is therefore the number of solutions of (4) in positive t. □

Lemma 2 *Define*

$$T(x+r)^n(s) = I_n(s,r),$$

so that

$$\begin{aligned} I_0 &= 1 \\ I_1 &= s+r. \end{aligned}$$

Then for any real number r, the I_n satisfy the recursion

$$(n+1)I_{n+1} = [s + r(n+1) + (r+1)n] \cdot I_n - nr(r+1) \cdot I_{n-1} \qquad (n \geq 1). \tag{5}$$

Proof We write $I_n(s)$ for $I_n(s,r)$ and let $g(s) = \sin(\pi s)/\pi$. Then

$$g(s) \cdot \int_0^1 y^s (1-y)^{1-s} (y+r)^n \frac{dy}{y(1-y)} = I_n(s) \tag{6}$$

Consider the semigroup of transformations

$$\frac{y}{1-y} = \lambda \cdot \frac{x}{1-x} \qquad (\lambda > 1) \tag{7}$$

of the unit interval to itself. Thus, we have explicitly

$$\begin{aligned} y &= \frac{\lambda x}{1+(\lambda-1)x} \\ 1-y &= \frac{1-x}{1+(\lambda-1)x} \end{aligned}$$

and

$$d^*y = \frac{dy}{y(1-y)} = \frac{dx}{x(1-x)} = d^*x.$$

Making the substitution (7) in (6), we obtain

$$I_n(s) = g(s) \cdot \lambda^s \int_0^1 x^s (1-x)^{1-s} F_n(x,\lambda) d^*x,$$

with

$$F_n(x,\lambda) = \frac{1}{1+(\lambda-1)x} \cdot [\frac{\lambda x}{1+(\lambda-1)x} + r]^n$$

Hence

$$\frac{1-\lambda^{-s}}{1-\lambda} \cdot I_n(s) = g(s) \cdot \int_0^1 x^s (1-x)^{1-s} G_n(x,\lambda) d^*x \tag{8}$$

with

$$\begin{aligned}
G_n(x,\lambda) &= \frac{1}{1-\lambda} \cdot [(x+r)^n - \frac{(\frac{\lambda x}{1+(\lambda-1)x} + r)^n}{1+(\lambda-1)x}] \\
&= \frac{1}{1-\lambda} \cdot [(x+r)^n - \frac{(x+r+(r+1)(\lambda-1)x)^n}{(1+(\lambda-1)x)^{n+1}}] \\
&= \frac{(x+r)^n}{1-\lambda} \cdot [1 - \frac{(1+\frac{(r+1)(\lambda-1)x}{x+r})^n}{(1+(\lambda-1)x)^{n+1}}] \\
&= \frac{(x+r)^n}{(1+(\lambda-1)x)^{n+1}} \cdot [(n+1)(x+r) - (2rn+r+n) \\
&\qquad + \frac{r(r+1)n}{x+r} + (\lambda-1)H_n(x,\lambda)]
\end{aligned}$$

where $H_n(x,\lambda)$ is finite at $\lambda = 1$. Therefore

$$G_n(x,1) = (n+1)\cdot(x+r)^{n+1} - (2rn+r+n)\cdot(x+r)^n + r(r+1)n\cdot(x+r)^{n-1},$$

and on taking the limit in (8) as $\lambda \longrightarrow 1$, the recursion stated in the lemma follows immediately. □

Corollary 1 *With $s = w + \frac{1}{2}$, we define a sequence of polynomials*

$$q_n(w) = I_n(s,r).$$

Then for $n \geq 2$ these satisfy the recursion

$$nq_n(w) = [w + (2n-1)(r+\tfrac{1}{2})]q_{n-1}(w) - (n-1)r(r+1)q_{n-2}(w)$$

with

$$\begin{aligned} q_0(w) &= 1 \\ q_1(w) &= w + r + \tfrac{1}{2}. \end{aligned}$$

Moreover, if we define monic polynomials by

$$p_n(w) = n! \cdot q_n(w), \tag{9}$$

then the recursion for the $[p_n]$ *is*

$$p_n(w) = [w + (2n-1)(r+\tfrac{1}{2})] \cdot p_{n-1}(w) - (n-1)^2 r(r+1) \cdot p_{n-2}(w) \tag{10}$$

with

$$\begin{aligned} p_0(w) &= 1 \\ p_1(w) &= w + r + \tfrac{1}{2}. \end{aligned}$$

We start the proof of case (a) of Theorem 1 by first considering the special case of $r = 0$. In this case the recursion (10) for the $[p_n(w)]$ only has 2 terms:

$$p_n(w) = [w + n - \tfrac{1}{2}] \cdot p_{n-1}(w)$$

for $n \geq 2$ with

$$\begin{aligned} p_0(w) &= 1 \\ p_1(w) &= w + \tfrac{1}{2}. \end{aligned}$$

Therefore for $n \geq 1$

$$q_n(w) = \frac{1}{n!} \cdot (w + n - \tfrac{1}{2})(w + n - 1 - \tfrac{1}{2}) \cdots (w + \tfrac{1}{2}) \tag{11}$$

has negative real roots. By Lemma 1, $q_n(w) + q_n(-w)$ has purely imaginary roots. Therefore $T[x^n + (1-x)^n] \in Rh$ for $n \geq 2$, and the theorem is proved in this subcase of (a).

To prove (a) when $r > 0$, we first argue that the zeros of q_n are real for $r > 0$, and then show that the roots are actually negative. By (9), p_n is a constant multiple of q_n, so we consider the sequence $[p_n]$ instead, and write (10) as

$$p_n(w) = (w - \alpha_n) \cdot p_{n-1}(w) - \beta_n \cdot p_{n-2}(w), \tag{12}$$

where

$$\begin{aligned} \alpha_n &= (1-2n)(r+\tfrac{1}{2}) \\ \beta_n &= (n-1)^2 r(r+1)\ . \end{aligned}$$

We quote a result of Favard [2] (see [3], Chapter II, Theorem 1.5).

Lemma 3 *Suppose* $[\ p_n(w) : n = 0, 1, \cdots;\ degree(p_n) = n\ ]$ *is a sequence of monic polynomials with real coefficients, satisfying a recursion formula of the type (12) with positive* β_n *and real* α_n*. Then for* $n \geq 1$*,* $p_n(w)$ *has* n *real, simple zeros.*

Proof Theorem 1.5, Chapter II of [3] states that with the hypotheses of Lemma 3, the sequence $[p_n]$ is an orthogonal sequence on the real line with respect to an m-distribution. Theorem 2.2, Chapter I [3] states that each polynomial in such an orthogonal sequence has n simple, real zeros. The sequence β_n is a positive sequence if $r > 0$, and in this case the zeros of the p_n, and therefore of the q_n, are real. □

We next see that the zeros of q_n are negative. For notational convenience, we make one more normalization and define

$$\begin{aligned} P_n(w) &= 2^n \cdot p_n(w) \\ &= 2^n \cdot n! \cdot q_n(w). \end{aligned}$$

We work with the polynomials P_n which now satisfy the recursion

$$P_n(w) = A_n \cdot P_{n-1}(w) - B_n \cdot P_{n-2}(w), \tag{13}$$

with

$$\begin{aligned} A_n &= A_n(w, r) = 2w + (2r + 1)(2n - 1) \\ B_n &= B_n(w, r) = 4(n - 1)^2 r(r + 1), \end{aligned}$$

and initial conditions

$$\begin{aligned} P_0(w) &= 1 \\ P_1(w) &= 2w + 2r + 1. \end{aligned}$$

Evidently, the $P_n = P_n(w, r)$ are polynomials in w and r with integer coefficients. For any polynomial

$$Q(w, r) = \sum_{i,j \geq 0} c_{i,j} w^i r^j$$

with real coefficients, we put

$$Q \succeq 0$$

iff the coefficients of Q are non-negative. Additionally, we say that $Q \succeq R$ iff the coefficients of $Q - R$ are non-negative. It is clear that if both

$$\begin{aligned} R &\succeq S \\ Q &\succeq 0, \end{aligned}$$

hold, then both

$$\begin{aligned} QR &\succeq QS, \\ Q+R &\succeq S \end{aligned}$$

hold.

Lemma 4 *For $n \geq 1$ and $r > 0$, the zeros of the polynomials $P_n(w,r)$ are negative.*

Proof We show by induction that for $n \geq 1$,

$$P_n \succeq n(2r+1)P_{n-1} \succeq 0.$$

For $n = 1$, we have

$$P_1 = 2w + 2r + 1 \succeq 2r + 1.$$

For $n = 2$,

$$\begin{aligned} P_2 &= [2w + 3(2r+1)] \cdot (2w + 2r + 1) - 4r(r+1) \\ &= 4w^2 + 16wr + 8w + 8r^2 + 8r + 3, \end{aligned}$$

whereas

$$2(2r+1)P_1 = 8wr + 4w + 8r^2 + 8r + 2,$$

and the difference is

$$4w^2 + 8wr + 4w + 1 \succeq 0.$$

For $n > 2$, subtract $n(2r+1)P_{n-1}$ from both sides of (13) to obtain

$$\begin{aligned} &P_n - n(2r+1)P_{n-1} \\ &= 2wP_{n-1} + (n-1)(2r+1)P_{n-1} - 4(n-1)^2 r(r+1)P_{n-2} \\ &= 2wP_{n-1} + (n-1)(2r+1) \cdot [P_{n-1} - (n-1)(2r+1)P_{n-2}] + (n-1)^2 P_{n-2} \end{aligned}$$

and the claim follows by induction on n. Setting $w = r = 0$ in (13) gives the constant term of P_n as $1 \cdot 3 \cdots (2n-1)$, so the zeros are negative. □

As a consequence of Lemma 4, the roots of $q_n(w) = q_n(w,r)$ are negative real numbers for $r > 0$. This completes the proof of Theorem 1 in case (a).

Now we consider case (b) in which $-\frac{1}{2} < r < 0$. It is no longer true that the zeros of the $q_n(w,r)$ are real. For instance, from the recursion (5) for the T-images $I_n(s,r)$, we find that

$$I_2(s,r) = \tfrac{1}{2}s^2 + s(2r + \tfrac{1}{2}) + r^2$$

and therefore

$$\begin{aligned} q_2(w,r) &= \tfrac{1}{2}(w+\tfrac{1}{2})^2 + (w+\tfrac{1}{2})(2r+\tfrac{1}{2}) + r^2 \\ &= \tfrac{1}{2}w^2 + (2r+1)w + r^2 + r + \frac{3}{8}. \end{aligned}$$

The w-zeros of this polynomial are

$$-(1+2r) \pm \tfrac{1}{2}\sqrt{1+8r(1+r)}\ ,$$

and these are not necessarily real. For example for $r = -\frac{1}{4}$ the roots of $q_2(w,r)$ are

$$-\frac{1}{2} \pm i\frac{\sqrt{2}}{2}\ .$$

To prove the theorem in case (b), we make use of the following lemma:

Lemma 5 *For $-\frac{1}{2} < r < 0$, the roots of $q_n(w) = q_n(w,r)$ have negative real parts for $n \geq 1$.*

Proof Note that

$$\begin{aligned} q_0(w) &= 1, \\ q_1(w) &= w + r + \tfrac{1}{2}, \end{aligned}$$

and for $n \geq 2$

$$nq_n(w) = [w + (2n-1)(r+\tfrac{1}{2})] \cdot q_{n-1}(w) - (n-1)r(r+1) \cdot q_{n-2}(w)\ . \quad (14)$$

From Lemma 4 and the explicit formula (11) for $q_n(w,r)$ of $r = 0$ case, we know that the roots of $q_n(w,r)$ are negative for $r \geq 0$ and $n \geq 1$. If for some r^+ in the range $-\frac{1}{2} < r^+ < 0$ and for some $n \geq 2$, a w-root of $q_n(w,r)$ has positive real part, then by continuity we could find an r' with $r^+ < r' < 0$ so that $q_n(w,r')$ has a pair of purely imaginary roots:

$$w = \pm i\gamma\ .$$

From (14), we see that for $n \geq 2$, the $q_n(w)$ satisfy a recursion

$$nq_n(w) = [w + A_n(r)] \cdot q_{n-1}(w) + B_n(r) \cdot q_{n-2}(w),$$

where we need only note that the $A_n(r)$ and $B_n(r)$ are positive for the range of r in question. Consider the case $n = 3$. From the recursion for q_3 we obtain in the obvious way, the continued fraction expansion

$$\frac{3q_3(w)}{q_2(w)} = w + A_3 + \frac{B_3}{w + A_2 + \frac{B_2}{w+A_1}}\ .$$

For $w = i\gamma$, the left side of this expression is 0. We now observe that if a complex number has positive real part then its reciprocal has positive real part. Thus, beginning with $i\gamma + A_1$, which has positive real part, we see that the right hand side of the above expression also has positive real part. This contradiction proves the theorem in the $n = 3$ case. The general case can be proved similarly. □

By Lemma 4 and Lemma 1, case (b) of Theorem 1 follows.

The only case left to prove to complete the proof of Theorem 1 is (c) in which $r = -\frac{1}{2}$. The resulting polynomials are

$$q_n(w, -\tfrac{1}{2}) = T(x - \tfrac{1}{2})^n(s) = L_n(s),$$

where $L_n(s)$ are the polynomials defined in (1) of the introduction. We treat this case by a standard eigenfunction argument.

Lemma 6 $\quad$ *$T(x - \frac{1}{2})^n(s) = L_n(s) \in Rh$ for $n \geq 1$.*

Proof Write

$$L_n(s) = (-\tfrac{1}{2})^n \cdot \sum_{k=0}^{n} b_{n,k}(s)_k$$

where

$$b_{n,k} = \binom{n}{k} \frac{(-2)^k}{k!}. \tag{15}$$

Let $P_n(s) = (-2)^n L_n(s)$. To prove the lemma we show that $\tilde{L}_n(w) = L_n(w + \frac{1}{2})$ (where we put $s = w + \frac{1}{2}$), satisfies (3) with $a = \frac{1}{2}, b = 1$, and $\lambda_n = 2n + 1$. This is equivalent to showing that

$$(w + \tfrac{1}{2})L_n(w + \tfrac{3}{2}) - (w - \tfrac{1}{2})L_n(w - \tfrac{1}{2}) = (2n + 1)L_n(w + \tfrac{1}{2}). \tag{16}$$

We show that for every $n \geq 1$,

$$(z - 1)P_n(z) - (z - 2)P_n(z - 2) - (2n + 1)P_n(z - 1) \equiv 0. \tag{17}$$

from which (16) follows by taking $z = w + \frac{3}{2}$.

We use the following expansions for $k \geq 0$, whose proofs are straightforward:

$$\begin{aligned} x(x)_k &= (x)_{k+1} - k(x)_k, \\ x^2(x)_k &= (x)_{k+2} - (2k + 1)(x)_{k+1} + k^2(x)_k. \end{aligned} \tag{18}$$

To prove (17), we expand each term in (17) in the basis $\{(z - 1)_k\}_{k \geq 0}$ using the expansions in (18), and then show that coefficient $c_k = c_{n,k}$ of

$(z-1)_k$ on the left hand side of (17) is identically zero for $k = 0, 1, \ldots, n$. The first and the third terms are given by

$$\begin{aligned} (z-1)P_n(z) &= \sum_{k=0}^{n} b_{n,k}(z-1)_{k+1} \qquad (19) \\ (2n+1)P_n(z-1) &= \sum_{k=0}^{n}(2n+1)b_{n,k}(z-1)_k, \end{aligned}$$

respectively. For the middle term,

$$\begin{aligned} (z-2)P_n(z-2) &= (z-2)\left[1 + (z-2)\sum_{k=1}^{n} b_{n,k}(z-1)_{k-1}\right] \\ &= (z-2) + (z-2)^2 \sum_{k=1}^{n} b_{n,k}(z-1)_{k-1}. \end{aligned}$$

Writing $(z-2)^2 = (z-1)^2 - 2(z-1) + 1$ and using (18) with $x = z-1$, and then simplifying, we obtain

$$\begin{aligned} (z-2)P_n(z-2) &= -1 + (z-1) + \sum_{k=1}^{n} b_{n,k}(z-1)_{k+1} \\ &\quad - \sum_{k=1}^{n}(2k+1)b_{n,k}(z-1)_k \\ &\quad + \sum_{k=1}^{n} k^2 b_{n,k}(z-1)_{k-1}. \qquad (20) \end{aligned}$$

From (19) and (20),

$$\begin{aligned} c_0 &= 1 - b_{n,1} - (2n+1)b_{n,0} \\ &= 1 - (-2)\binom{n}{1} - (2n+1) = 0, \end{aligned}$$

$$\begin{aligned} c_1 &= b_{n,0} - (1 - 3b_{n,1} + 4b_{n,2}) - (2n+1)b_{n,1} \\ &= 1 - (1 + 6n + 4n(n-1)) + (2n+1)(2n) = 0 \end{aligned}$$

and for $k > 1$,

$$\begin{aligned} c_k &= b_{n,k-1} - (b_{n,k-1} - (2k+1)b_{n,k} + (k+1)^2 b_{n,k+1}) - (2n+1)b_{n,k} \\ &= (2k-2n)b_{n,k} - (k+1)^2 b_{n,k+1} \end{aligned}$$

Using (15), the above sum simplifies to

$$(-2)^{k+1}\left[\frac{(n-k)}{k!}\binom{n}{k}-\frac{k+1}{k!}\binom{n}{k+1}\right]=0.$$

This proves Lemma 6, and finishes the proof of Theorem 1. □

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