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STATISTICS ON RESTRICTED FIBONACCI WORDS

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Abstract. We study two foremost Mahonian statistics, the major index and the inversion number for a class of binary words called restricted Fibonacci words. The language of restricted Fibonacci words satisfies recurrences which allow for the calculation of the generating functions in two different ways. These yield identities involving the q-binomial coefficients and provide non-standard q-analogues of the Fibonacci numbers. The major index generating function for restricted Fibonacci words turns out to be a q-power multiple of the inversion generating function.

1. Introduction and preliminaries

The major index $\mathrm{maj}\,(w)$ and the inversion number $\mathrm{inv}\,(w)$ of a word $w = w_1 w_2 \cdots w_n$ over a linearly ordered alphabet $\Sigma = \{a_1, a_2, \ldots, a_k\}$ are defined as

$$
\begin{aligned}
\mathrm{maj}\,(w) &= \sum_{i=1}^{n-1} i\,\chi(w_i > w_{i+1}), \\
\mathrm{inv}\,(w) &= \sum_{1 \le i < j \le n} \chi(w_i > w_j)\ .
\end{aligned}
\tag{1.1}
$$

Here for any statement S, $\chi(S)$ is 1 or 0 according to whether the S is true or false. The statistics maj and inv are two of the four permutation statistics that MacMahon considered on words at the turn of the 20th century. These were descents, excedances, inversions and the major index. MacMahon showed algebraically that maj is equidistributed with inv on words over an alphabet with a given

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frequency of letters [13]. This is in the sense that restricted to the words with n_i occurrences of the letter a_i for $1 \leq i \leq k$, both are enumerated by q-multinomial coefficients:

$$\sum_{w} q^{\mathrm{maj}\,(w)} = \begin{bmatrix} n_1 + n_2 \cdots + n_k \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q = \sum_{w} q^{\mathrm{inv}\,(w)} \, , \tag{1.2}$$

where

$$\begin{bmatrix} n_1 + n_2 + \cdots + n_k \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q = \frac{(q)_{n_1+n_2+\cdots+n_k}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}}$$

with $(q)_n = \prod_{i=1}^{n}(1-q^i)$ for $n > 0$ and $(q)_0 = 1$.

In his honor, statistics on permutations, or more generally on words, that are equidistributed with inv are said to be *Mahonian.* Some of the bibliography on the rich existing literature and continuing work on extensions of various Mahonian statistics can be found in [7, 14] and the references therein. Such statistics interpolating between maj and inv appear in Kadell [12]. The connections with hypergeometric series statistics on binary words can be found in [2].

Foata gave combinatorial proofs of MacMahon's equidistribution results [8, 9]. Another combinatorial proof due to Carlitz, called the insertion method gives a recursive bijection that proves that inv and maj are equidistributed [5]. We refer to Andrews [1] and Stanley [16] for more on these statistics, and to Sagan and Savage [14] for a very readable account of Foata's fundamental bijection.

In this paper we are particularly interested in the two Mahonian statistics for a class of restricted words w over the binary alphabet $\{0, 1\}$. For the class of binary words that are a restricted version of Fibonacci words, we find the generating functions of maj and inv in two different ways and obtain a number of identities in the spirit of [2].

For unrestricted binary words w with n occurrences of 1 and m occurrences of 0, the common generating function of these two Mahonian statistics is

$$\sum_{w} q^{\mathrm{maj}\,(w)} = \sum_{w} q^{\mathrm{inv}\,(w)} = \begin{bmatrix} n+m \\ m \end{bmatrix}_q . \tag{1.3}$$

The q-binomial coefficients above satisfy the recursion

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \tag{1.4}$$

and the q-binomial theorem is [1, 16]

$$(1+z)(1+qz)\cdots(1+q^{n-1}z) = \sum_{k=0}^{n} z^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

We also note that q-binomial coefficient in (1.3) is also the generating function by weight of integer partitions whose Ferrers diagrams are included in the $n \times m$ rectangle.

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2. Fibonacci words

A Fibonacci word is a binary word with no two consecutive 1's. The number of Fibonacci words of length n is the Fibonacci number f_{n+2} where $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The null word λ is a Fibonacci word of length zero. Fibonacci words are freely generated by the letters $\Sigma = \{s_0, s_1, s_2, \ldots\}$ where $s_i = (10)^i 0$ for $i \geq 0$. In other words every nonnull Fibonacci word has a unique factorization as a juxtaposition of letters from this alphabet. For example, using 0 for s_0, we have

$$00\,10100\,101010100\,000\,100\,0\,1010100\,0 = 00\,s_2 s_4\,000\,s_1\,0\,s_3\,0\ .$$

The generating function of Σ by length is

$$t + t^3 + t^5 + \cdots = \frac{t}{1-t^2}$$

so by unique factorization the generating function of the words over Σ (including the null word) by length is

$$\frac{1}{1 - \frac{t}{1-t^2}} = \frac{1-t^2}{1-t-t^2} = 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + \cdots$$

This gives the familiar generating function of the Fibonacci numbers as

$$\frac{1-t^2}{1-t-t^2} - 1 = \frac{t}{1-t-t^2}\ .$$

For a binary word w, $\operatorname{inv}(w)$ is the number of pairs of indices $1 \leq i < j \leq n$ with $w_i = 1$ and $w_j = 0$. So for each 0 from left to right, we count the number of 1's before it in w, and add these numbers. For example for the Fibonacci word $w = 10100$ we have $\operatorname{maj}(w) = 1 + 3 = 4$ and $\operatorname{inv}(w) = 1 + 2 + 2 = 5$, and for $w = 10010100$, we have $\operatorname{maj}(w) = 1+4+6 = 11$ and $\operatorname{inv}(w) = 1+1+2+3+3 = 10$. Further examples can be found in Table 1.

3. Restricted Fibonacci words

Consider the alphabet

$$\mathcal{P} = \{s_1, s_2, \ldots\}\ . \tag{3.1}$$

Let $\mathcal{P}^*$ be the language of binary words (including the null word λ of length zero) that are generated by $\mathcal{P}$ and let $\mathcal{P}^+ = \mathcal{P}^* - \{\lambda\}$. Note that

i) each word in $\mathcal{P}^+$ starts with 10,
ii) each 1 in $\mathcal{P}^+$ is followed by a 0 or a 00,
iii) the shortest word in $\mathcal{P}^+$ is $s_1 = 100$.

Each $w \in \mathcal{P}^*$ has a unique factorization as a word in the s_i, for example

$$10100\,101010100\,100\,1010100 = s_2 s_4 s_1 s_3\ .$$

We can call $\mathcal{P}^*$ "0-free" Fibonacci words or "positive" Fibonacci words, though neither term is satisfactory. Instead, we will use the following terminology:

Definition 3.1. *The language $\mathcal{P}^*$ of binary words generated by the alphabet $\mathcal{P}$ of (3.1) is called* restricted Fibonacci words*.*

It is clear that we have the identities

$$\textit{First identity:} \quad \mathcal{P}^+ = s_1 + 10\mathcal{P}^+ + s_1\mathcal{P}^+ \tag{3.2}$$

$$\textit{Second identity:} \quad \mathcal{P}^* = \lambda + s_1\mathcal{P}^* + s_2\mathcal{P}^* + \cdots \tag{3.3}$$

where "+" denotes disjoint union. (3.2) expresses the fact that words in $\mathcal{P}^+$ start with either 10 or 100 whereas (3.3) expresses the fact that a word in $\mathcal{P}^*$ is either λ or begins with one of the s_i.

Denoting by $|w|_0$ and $|w|_1$ the number of 0's and 1's in a binary word w, we first define

$$f(x,y) = \sum_{w\in\mathcal{P}^*} x^{|w|_1} y^{|w|_0} \tag{3.4}$$

and put $\widetilde{f}(x,y) = f(x,y) - 1$. $\widetilde{f}$ is the generating function of words in $\mathcal{P}^+$. From (3.2) above we obtain

$$\widetilde{f}(x,y) = xy^2 + xy\widetilde{f}(x,y) + xy^2\widetilde{f}(x,y)$$

and from (3.3) the equivalent identity

$$f(x,y) = 1 + xy^2 f(x,y) + x^2y^3 f(x,y) + \cdots = 1 + \frac{xy^2 f(x,y)}{1 - x^2y^2}$$

so that

$$f(x,y) = \frac{1 - xy}{1 - xy - xy^2}\ . \tag{3.5}$$

In particular the generating function $\mathcal{P}^*$ by length is

$$f(t,t) = \frac{1 - t^2}{1 - t^2 - t^3} = 1 + \frac{t^3}{1 - t^2 - t^3}\ . \tag{3.6}$$

Lemma 3.2. *Let $\mathcal{P}^*_{n,m}$ denote the number of restricted Fibonacci words w with $|w|_1 = n$ and $|w|_0 = m$. Its cardinality is given by*

$$\left|\mathcal{P}^*_{n,m}\right| = \binom{n-1}{m-n-1}\ .$$

Proof. We can directly use the recursion (3.2) and induction, or use the series expansion

$$\frac{1}{1 - xy - xy^2} = \sum_{n,k\geq 0} x^n y^{n+k} \binom{n}{k}$$

together with (3.5) and an elementary binomial identity. □

In particular $\left|\mathcal{P}^*_{n,m}\right| = 0$ unless $n + 1 \leq m \leq 2n$.

Next we define the generating functions of inv and maj for $\mathcal{P}^*_{n,m}$.

Definition 3.3.

$$\mathcal{M}_{n,m}(q) = \sum_{w\in\mathcal{P}^*_{n,m}} q^{\operatorname{maj}(w)} \quad \text{and} \quad \mathcal{I}_{n,m}(q) = \sum_{w\in\mathcal{P}^*_{n,m}} q^{\operatorname{inv}(w)}\ .$$

4. Consequences of the First identity

4.1. **Major index.** In restricted Fibonacci words there are no adjacent 1s, and therefore the major index of $w \in \mathcal{P}^*$ is simply the sum of the indices of the 1s in w. We augment $f(x, y)$ of (3.4) to $f(x, y, q)$ by setting

$$f(x,y,q) = \sum_{w \in \mathcal{P}^*} x^{|w|_1} y^{|w|_0} q^{\mathrm{maj}\,(w)} \ . \tag{4.1}$$

We can consider f as a power series in x with coefficients that are polynomials in y and q, or as a power series in y with coefficients that are polynomials in x and q. With this view we write (4.1) as

$$f(x) = 1 + \sum_{n \geq 1} a_n(y,q) x^n = 1 + \sum_{m \geq 1} b_m(x,q) y^m = f(y) \ . \tag{4.2}$$

As before, we set $\widetilde{f}(x, y, q) = f(x, y, q) - 1$. Next we determine the polynomials a_n and b_m.

Theorem 4.1.

(1) $a_0 = 1$ *and for* $n \geq 1$,

$$a_n(y,q) = y^{n+1} q^{n^2} \prod_{k=1}^{n-1} (1 + q^k y) \ , \tag{4.3}$$

(2) $b_0 = 1$, $b_1 = 0$ *and for* $m \geq 2$,

$$b_m(x,q) = \sum_{n=\lceil \frac{m}{2} \rceil}^{m-1} x^n q^{n^2 + \frac{1}{2}(m-n)(m-n-1)} \begin{bmatrix} n-1 \\ m-n-1 \end{bmatrix}_q \ . \tag{4.4}$$

Proof. We make use of the First identity (3.2). The contribution of s_1 to $\widetilde{f}$ is xy^2q. If the contribution of $u \in \mathcal{P}^+$ is $x^r y^s q^t$, then the contribution of $10u$ is $xyq\,(x^r y^s q^{t+2r})$, as the index of each 1 in u goes up by 2. Similarly the contribution of $100u$ is $xy^2q\,(x^r y^s q^{3r+t})$. It follows that

$$\widetilde{f}(x,y,q) = xy^2q + xyq\,\widetilde{f}(xq^2, y, q) + xy^2q\,\widetilde{f}(xq^3, y, q) \ , \tag{4.5}$$

in other words

$$\widetilde{f}(x) = xy^2q + xyq\,\widetilde{f}(xq^2) + xy^2q\,\widetilde{f}(xq^3) \ .$$

Therefore

$$a_n(y,q) = y\,a_{n-1}(y,q) \cdot q \cdot q^{2(n-2)} + y^2\,a_{n-1}(y,q) \cdot q \cdot q^{3(n-1)} \ .$$

Consequently

$$a_n = y\,q^{2n-1}(1 + yq^{n-1})\,a_{n-1} \ . \tag{4.6}$$

and (4.3) follows by induction on n in (4.6).

To prove (4.4), we use the q-binomial theorem with $z = yq$ to obtain the expansion

$$a_n(y,q) = \sum_{k=0}^{n-1} y^{n+1+k} q^{n^2 + \frac{1}{2}k(k-1)+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q ,$$

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and the coefficient of y^m here is

$$\mathcal{M}_{n,m}(q) = q^{n^2+\frac{1}{2}(m-n)(m-n-1)} \begin{bmatrix} n-1 \\ m-n-1 \end{bmatrix}_q . \tag{4.7}$$

Adding up the contributions according to the number of 1's proves (4.4). □

4.2. **Inversions.** To enumerate the inversions in restricted Fibonacci words, we set

$$g(x,y,q) = \sum_{w \in \mathcal{P}^*} x^{|w|_1} y^{|w|_0} q^{\mathrm{inv}\,(w)} \tag{4.8}$$

with $\widetilde{g}(x,y,q) = g(x,y,q) - 1$. In analogy with $f(x,y,q)$, we can consider g as a power series in x with coefficients that are polynomials in y and q, or as a power series in y with coefficients that are polynomials in x and q. We write (4.8) as

$$g(x) = 1 + \sum_{n \geq 1} c_n(y,q) x^n = 1 + \sum_{m \geq 1} d_m(x,q) y^m = g(y) . \tag{4.9}$$

In other words the polynomials $c_n(y,q)$ and $d_m(x,q)$ are to the inversion statistic what $a_n(y,q)$ and $b_m(x,q)$ are to the major index statistic.

Theorem 4.2.

(1) $c_0 = 1$, *and for* $n \geq 1$,

$$c_n(y,q) = y^{n+1} q^{\frac{1}{2}n(n+3)} \prod_{k=1}^{n-1} (1 + q^k y) , \tag{4.10}$$

(2) $d_0 = 1$, $d_1 = 0$ *and for* $m \geq 2$,

$$d_m(x,q) = \sum_{n=\lceil \frac{m}{2} \rceil}^{m-1} x^n q^{\frac{1}{2}n(n+3)+\frac{1}{2}(m-n)(m-n-1)} \begin{bmatrix} n-1 \\ m-n-1 \end{bmatrix}_q . \tag{4.11}$$

Proof. We again make use of the First identity (3.2). The contribution of s_1 to $\widetilde{g}$ is xy^2q^2. If the contribution of $u \in \mathcal{P}^+$ is $x^r y^s q^t$, then the contribution of $10u$ is $xyq\,(x^r y^s q^{t+s})$ as each 0 in u causes an inversion with the initial 1. Similarly the contribution of $100u$ is $xy^2q^2\,(x^r y^s q^{t+s})$. It follows that $\widetilde{g}$ satisfies

$$\widetilde{g}(x,y,q) = xy^2q^2 + xyq\,\widetilde{g}(x,yq,q) + xy^2q^2\,\widetilde{g}(x,yq,q) . \tag{4.12}$$

Therefore $c_1(y,q) = y^2q^2$ and for $n \geq 2$,

$$c_n(y,q) = yq\,c_{n-1}(yq,q) + y^2q^2\,c_{n-1}(yq,q) = yq(1+yq)\,c_{n-1}(yq,q) .$$

Now (4.10) follows by induction on n.

To prove (4.11), we use the q-binomial theorem with $z = yq$ to obtain the expansion

$$c_n(y,q) = \sum_{k=0}^{n-1} y^{n+1+k} q^{\frac{1}{2}n(n+3)+\frac{1}{2}k(k-1)+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q ,$$

and the coefficient of y^m here is

$$\mathcal{I}_{n,m}(q) = q^{\frac{1}{2}n(n+3)+\frac{1}{2}(m-n)(m-n-1)} \begin{bmatrix} n-1 \\ m-n-1 \end{bmatrix}_q . \tag{4.13}$$

This proves (4.11). □

TABLE 1. Restricted Fibonacci words with 3 or fewer occurrences of 1 with their major index and the number of inversions.

	Restricted Fibonacci words w	maj (w)	inv (w)
$\lvert w\rvert_1 = 1$	100	1	$1+1=2$
$\lvert w\rvert_1 = 2$	100100	$1+4=5$	$1+1+2+2=6$
	10100	$1+3=4$	$1+2+2=5$
$\lvert w\rvert_1 = 3$	100100100	$1+4+7=12$	$1+1+2+2+3+3=12$
	10010100	$1+4+6=11$	$1+1+2+3+3=10$
	10100100	$1+3+6=10$	$1+2+2+3+3=11$
	1010100	$1+3+5=9$	$1+2+3+3=9$

As a corollary of Theorem 4.2, we have the following.

Proposition 4.3. *Let $\mathcal{P}^*_{n,m}$ denote the number of restricted Fibonacci words w with $|w|_1 = n$ and $|w|_0 = m$ and let $\mathcal{M}_{n,m}(q)$, $\mathcal{I}_{n,m}(q)$ be the generating functions of* maj *and* inv *given in Definition 3.3. Then*

$$\mathcal{M}_{n,m}(q) = q^{\frac{1}{2}n(n-3)}\,\mathcal{I}_{n,m}(q) . \tag{4.14}$$

Proof. The expressions for the the generating function of restricted Fibonacci words in $\mathcal{P}^*_{n,m}$ by major index and by inversions are as given in (4.7) and (4.13), respectively. □

5. q-Fibonacci numbers

If we set $x = 1$ in $b_m(x, q)$ in (4.4), we obtain the polynomial $b_m(q)$, which is the enumerator of restricted Fibonacci words with m occurrences of 0 by major index. Similarly, if we set $x = 1$ in $d_m(x, q)$ in (4.11), we obtain the polynomial $d_m(q)$, which is the enumerator of restricted Fibonacci words with m occurrences of 0 by inversions. For $q = 1$, both $b_m(q)$ and $d_m(q)$ specialize to the Fibonacci number f_{m-1}. This is because the generating function (3.5) becomes the (shifted) generating function of the Fibonacci numbers. Therefore both $b_m(q)$ and $d_m(q)$ are q-analogues of the Fibonacci numbers. Consequently, from either of the expressions (4.4) or (4.11) with $x = q = 1$, we have for $m \geq 1$,

$$f_{m-1} = \sum_{k=\lceil \frac{m}{2} \rceil}^{m-1} \binom{k-1}{m-k-1} .$$

There are various q-analogues of the Fibonacci numbers in the literature [3, 4, 6, 11, 15] and it is natural to try and identify $b_m(q)$ and $d_m(q)$ in terms of one of these. Considering $d_m(q)$ first, from (4.12) and (4.9) we have the recursion

$$d_m(x,q) = xq^m \, d_{m-1}(x,q) + xq^m \, d_{m-2}(x,q) \tag{5.1}$$

and therefore

$$d_m(q) = q^m \, d_{m-1}(q) + q^m \, d_{m-2}(q) \tag{5.2}$$

for $m \geq 2$ with $d_0 = 1$ and $d_1 = 0$. From this recursion we can easily prove by induction that the degree of the polynomial $d_m(q)$ is $\frac{1}{2}(m-1)(m+2)$ for $m \geq 1$.

Corollary 5.1. *Define the polynomials $p_m(q)$ by the recursion*

$$p_m(q) = p_{m-1}(q) + q^{m-1} \, p_{m-2}(q) \tag{5.3}$$

for $m \geq 2$ with $p_0 = 1$, $p_1 = 0$. Then $p_m(1) = f_{m-1}$ and

$$d_m(q) = q^{\frac{1}{2}m(m+1)} \, p_m(\tfrac{1}{q}) . \tag{5.4}$$

Proof. We simply verify that the right hand side of (5.4) satisfies the recurrence relation (5.2). ☐

Note that the q-analogue $p_m(q)$ of the Fibonacci numbers in (5.3) is different from the standard one defined by

$$F_m(q) = F_{m-1}(q) + q^{m-2} F_{m-2}(q) \tag{5.5}$$

due to Schur, which was studied by Carlitz, Cigler and others in the literature [3, 4, 6], and also different from

$$F_m(q) = q \, F_{m-1}(q) + F_{m-2}(q)$$

with $F_1 = 1$ and $F_2 = q$, which are commonly known as *Fibonacci polynomials* [11].

In the case of the $b_m(q)$, the recursion analogous to (5.2) is

$$b_m(x,q) = xq \, b_{m-1}(xq^2, q) + xq \, b_{m-2}(xq^3, q) \tag{5.6}$$

for $m \geq 2$. This is a consequence of (4.5). However the recursion (5.6) is fundamentally different from (5.2). It is not clear what plays role of $p_m(q)$ in (5.3) for the major index polynomials $b_m(q)$.

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Using (4.4) with $x = 1$, after $b_0 = 1$ and $b_1 = 0$, the first few polynomials $b_m(q)$ are as follows:

$$\begin{aligned}
b_2(q) &= q \\
b_3(q) &= q^4 \\
b_4(q) &= q^5(1+q^4) \\
b_5(q) &= q^{10}(1+q+q^6) \\
b_6(q) &= q^{12}(1+q^5+q^6+q^7+q^{13}) \\
b_7(q) &= q^{19}(1+q+q^2+q^7+q^8+q^9+q^{10}+q^{17}) \\
b_8(q) &= q^{22}(1+q^6+q^7+2q^8+q^9+q^{10}+q^{15}+q^{16}+q^{17}+q^{18}+q^{19}+q^{27})
\end{aligned}$$

From (4.4) we calculate that the degree of $b_m(q)$ is $(m-1)^2$, and the lowest degree term which is shown as factored above, has degree $\frac{1}{8}m(3m-1)$ for m even and $\frac{1}{8}(3m^2+5)$ for m odd.

6. Consequences of the Second identity

Now we consider the consequences of the Second identity (3.3). It expresses the fact that a restricted Fibonacci word w is either null, or starts with s_1, or s_2, and so on. First we consider the major index statistic. If $w = s_k u$ with $u \in \mathcal{P}^*$, then the contribution of s_k to $\operatorname{maj}(w)$ is

$$q^{1+3+5+\cdots+(2k-1)} = q^{k^2} .$$

The contribution to the major index that comes from u is increased by $2k+1$, the length of s_k, for every occurrence of 1 in u. It follows that the generating function (4.1) satisfies the functional identity

$$f(x,y,q) = 1 + xy^2q^1\, f(xq^3,y,q) + x^2y^3q^4\, f(xq^5,y,q) + x^3y^4q^9\, f(xq^7,y,q) + \cdots \tag{6.1}$$

Writing it in the form $f(x)$ as in (4.2) we have

$$f(x) = 1 + \sum_{m\geq 1} x^m y^{m+1} q^{m^2} f(xq^{2m+1}) . \tag{6.2}$$

Noting that

$$f(xq^{2m+1}) = 1 + \sum_{k\geq 1} a_k(y,q) q^{(2m+1)k}\, x^k ,$$

substituting in (6.2) and equating the coefficient of x^n $(n \geq 1)$ on both sides find that the $a_n = a_n(y,q)$ of (4.3) is the solution to the recurrence relation

$$a_n = \sum_{k=0}^{n-1} y^{n+1-k} q^{n^2-k(k-1)} a_k .$$

Considering the generating function in (4.1) and writing it in the form $f(y)$ as in (4.2), and equating the coefficient of y^m $(m \geq 1)$ on both sides of (6.1) we find that the $b_m(x,q)$ as given in (4.4) is the solution to the recurrence relation

$$b_m(x,q) = \sum_{k=0}^{m-2} x^{m-k-1} q^{(m-k-1)^2} b_k(xq^{2m-2k-1}, q) . \tag{6.3}$$

For inversions, consider the generating function $g(x,y,q)$ of (4.8). If a nonnull restricted Fibonacci word w starts with s_k, then $w = s_k u$ with $u \in \mathcal{P}^*$ and the contribution of s_k itself to $\mathrm{inv}\,(w)$ is

$$q^{1+2+3+\cdots+k\,+\,k} = q^{\frac{1}{2}k(k+3)} \ .$$

The contribution to the number of inversions that comes from u is increased by k, the number of 1's in s_k, for every occurrence of 0 in u. It follows that $g(x,y,q)$ satisfies the functional equation

$$\begin{aligned} g(x,y,q) &= 1 + xy^2q^2 g(x,yq,q) + x^2y^3q^5 g(x,yq^2,q) + x^3y^4q^9 g(x,yq^3,q) + \cdots \\ &= 1 + \sum_{n\ge 1} x^n y^{n+1} q^{\frac{1}{2}n(n+3)} g(x,yq^n,q) \ . \end{aligned} \tag{6.4}$$

Comparing the coefficient of x^n on both sides of (6.4) we obtain that $c_n(y,q)$ of (4.10) are the solution to the recurrence relation

$$c_n(y,q) = \sum_{k=0}^{n-1} y^{n-k+1} q^{\frac{1}{2}(n-k)(n-k+3)} c_k(yq^{n-k},q) \tag{6.5}$$

with $c_0 = 1$. For the analogous result with the $d_m(x,q)$'s, we write with the notation of (4.9)

$$g(y) = 1 + \sum_{m\ge 1} y^{m+1} q^{\frac{1}{2}m(m+3)} g(yq^m) \ , \tag{6.6}$$

use

$$g(yq^m) = 1 + \sum_{k\ge 1} d_k(x,q) q^{mk} y^k$$

in (6.6) and compare coefficients to find that the polynomials $d_m = d_m(x,q)$ of (4.11) satisfy the recurrence relation

$$d_m = \sum_{k=0}^{m-2} x^{m-k-1} q^{\frac{1}{2}(m-k-1)(m+k+2)} d_k \tag{6.7}$$

for $m \ge 2$ with $d_0 = 1$, $d_1 = 0$.

We remark that the recurrence relations (6.5) and (6.7) can be proved independently by grouping terms and making use of the q-binomial identity (1.4).

Going back to the representation of restricted Fibonacci words as words over the alphabet of the s_k, we note that replacing a occurrence of $s_i s_j$ in a $w \in \mathcal{P}^+$ by $s_{i-1}s_{j+1}$ increases the major index by 1 and decreases the number of inversions by 1. Similarly replacing $s_i s_j$ by $s_{i+1}s_{j-1}$ decreases the major index by 1 and increases the number of inversions by 1. In the first case we need $i \ge 2$ and in the second, $j \ge 2$.

In $\mathcal{P}^*_{n,m}$ there are two special words which no such shift is possible: when $m = n+1$ and $w = s_n$, and when $m = 2n$ and $w = s_1^n$. Here n and m denote the number of 1's and the number of 0's in the word, respectively with $n+1 \le m \le 2n$ for restricted Fibonacci words in general.

We can easily calculate the two statistics for these two words as shown in Table 2:

TABLE 2. Two extreme cases.

Restricted Fibonacci word	Major index	Inversions
s_n	n^2	$\frac{1}{2}n(n+3)$
s_1^n	$\frac{1}{2}n(3n-1)$	$n(n+1)$

Aside from the two extreme cases of $m = n+1$ and $m = 2n$, writing $m = n+k+1$ with $1 \leq k \leq n-2$, there are two special elements in $\mathcal{P}^*_{n,m}$, which are $s_1^k s_{n-k}$ and $s_{n-k}s_1^k$.

If a word $u \in \mathcal{P}^*$ starts with letter $s_i s_j$ with $i > 1$, then using the shift $s_{i-1}s_{j+1}$ repeatedly, we end up with a word that starts with s_1, and the major index goes up by $i-1$. Summing the contributions over i,

$$\begin{aligned}
\mathcal{M}_{n,m}(q) &= \left(1 + \tfrac{1}{q} + \cdots + (\tfrac{1}{q})^{n-k-1}\right) \sum_{w \in \mathcal{P}^*_{n-1,m-2}} q^{\mathrm{maj}\,(s_1 w)} \\
&= \frac{1 - (\frac{1}{q})^{2n-m+1}}{1 - \frac{1}{q}} \cdot q^{3n-2}\, \mathcal{M}_{n-1,m-2}(q) \\
&= \frac{1 - q^{2n-m+1}}{1-q} \cdot q^{n+m-2}\, \mathcal{M}_{n-1,m-2}(q).
\end{aligned}$$

Similarly,

$$\mathcal{I}_{n,m}(q) = \frac{1 - q^{2n-m+1}}{1-q} \cdot q^m\, \mathcal{I}_{n-1,m-2}(q)\ .$$

These give us again the closed form expressions (4.7) and (4.13). Also we calculate directly

$$\frac{\mathcal{M}_{n,m}(q)}{\mathcal{I}_{n,m}(q)} = q^{n-2}\, \frac{\mathcal{M}_{n-1,m-2}(q)}{\mathcal{I}_{n-1,m-2}(q)}$$

and obtain (4.14) without actually computing each expression.

Remark 6.1. *The generating function of $\mathcal{P}^+$ by length in* (3.6) *is identical to the generating function of compositions $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ (i.e. ordered partitions) of n where $\pi_i = 2$ or 3 and $\pi_r = 3$. This last condition accounts for the t^3 in the numerator in* (3.6). *Inversion statistics on compositions themselves was studied in* [10]. *In our case the bijection simply maps a maximal length run of k 2's followed by a 3 in π to the letter s_{k+1}. For instance the composition $\pi = (2,3,2,2,2,3,3,2,2,3)$ of $n = 24$ corresponds to the restricted Fibonacci word $s_2s_4s_1s_3$ of length* 24.

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