Algorithms for the Character Theory of the Symmetric Group

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0. Introduction

The representation theory of the symmetric groups $\mathbf{S}_n$ - aside from being extremely elegant and interesting in its own right - can be used in a number of ways to obtain information about the representation theory of other classes of groups. This theory also turns out to have applications in such diverse areas of interest as atomic physics and quantum chemistry to graph isomorphism and PI algebras.

Prior to the advent of the electronic computing devices, the computational aspect of the character theory of $\mathbf{S}_n$ had to stay essentially at the level of examples. This is because of the formidable computational difficulties that arise in the resolution of various products defined on the irreducible characters of $\mathbf{S}_n$. Following the introduction of the first electronic computing machines, there emerged a number of approaches and significant computational results in this area, mainly focused on the construction of the character tables of $\mathbf{S}_n$. Theoretical results of Frobenius [Frob1-Frob2], Murnaghan [Murn2], Nakayama [Naka], Yamanouchi [Yama] enabled Bivins et al [BiMSW] to calculate the characters of symmetric groups of degrees up to 16 in 1950's. This was followed by the results of Comet [Come1-Come3], Gabriel [Gabri] among others which extended and improved these considerably. In recent years, the classification theory of finite simple groups spawned comprehensive packages such as the Character Algorithm System (CAS) [CAS1-CAS3] for handling characters of arbitrary finite groups.

The development of the special case of the operations on the characters of the symmetric groups was initiated by Young's fundamental series of monographs [Young] and the isomorphism between the multiplicative properties of group characters and the algebra of Schur functions introduced by Littlewood [Litt1-Litt4]. This approach lead to immense simplifications in the computational aspect of this theory. Furthermore, the combinatorial interpretation of Schur functions as the weight generating function of Weyl tableaux of the corresponding Young frame have opened up new avenues of combinatorial techniques for the understanding of the underlying structures. Recent developments in this area are too numerous to quote. Our purpose in undertaking this particular implementation of a compact interactive system for the resolution of various products of irreducible characters of $\mathbf{S}_n$, in particular the computation of *plethysms*, has been to exploit this recent understanding and to make use of the efficient combinatorial algorithms that it brought to the fore.

In section 1, the basic operations under consideration on the characters of $\mathbf{S}_n$ are presented. Section 2 covers the descriptions of the algorithms to be used and the extent of the experimental implementations realized so far. We briefly elaborate on each one of the operations, pointing out the nature and the limitations of some of the algorithms that were proposed to compile tables in the past. A summary of previous work concerning the computation of characters of $\mathbf{S}_n$ together with more recent and extensive packages such as the CAS system are also mentioned in this section.

In section 3 some applications of plethysms and Kronecker products in physics are presented. The nature of the package in some detail from the implementation point of view is discussed in section 4 . Finally, concluding remarks as a short summary are presented in section 5 .

1. Basic Operations

First we remark that the irreducible representations of $\mathbf{S}_n$ are in one-to-one correspondence with the *partitions* of n : a sequence of nonnegative integers $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ is a partition of n if $\lambda_1 + \lambda_2 + .. + \lambda_n = n$. The correspondence between the irreducible representations and partitions is more than just a count: it is well known that one can construct an irreducible $\mathbf{S}_n$ $-$module for each partition λ of n in a natural way. (Facts of this nature about the representation theory of the symmetric groups as well as the interplay between this theory and combinatorics that are not referenced explicitly here can be found in e.g. [JaKe], [MacD], [Isaac], [CuRe], [Feit], [Stan]).

Thus the basic high-level primitive operations that form the building blocks of the character theory of $\mathbf{S}_n$ take partitions of n as their arguments. The following operations to be implemented on the irreducible characters are of particular importance:

PLET $(\lambda;\mu)$:

Returns the expansion of the outer plethysm $\{\lambda\}\otimes\{\mu\}$ of the irreducible representations of $\mathbf{S}_n$ corresponding to the partitions λ and μ .

INNER $(\lambda;\mu)$:

Returns the expansion of the inner or the Kronecker product $\{\lambda\}\circ\{\mu\}$ of the corresponding irreducible representations of $\mathbf{S}_n$.

OUTER $(\lambda;\rho;...;\mu)$:

Returns the expansion of the outer product $\{\lambda\}\{\rho\}\cdots\{\mu\}$ of the irreducible representations of $\mathbf{S}_n$ corresponding to the partitions $\lambda, \rho, \ldots, \mu$.

$EXPAN(\lambda;\mu)$:

Returns the expansion of the *skew* representation of $\mathbf{S}_n$ that corresponds to the skew-shape λ/μ . This operation requires that the nodes of the Young frame of μ be contained in those of λ .

$COEF(\lambda; arg)$:

This primitive returns the coefficient or the multiplicity of the irreducible representation λ in the expansion of the operation *arg* . Here *arg* can be a file name which has the common I/O format of the package or any one of the operations *PLET*, *OUTER*, *INNER*, *EXPAN* .

$CHAR(\lambda;\mu)$:

Returns the value of the character λ at the conjugacy class of permutations with cycle type μ .

$CCOL(\lambda)$:

Generates the character values of all of the irreducible representations of $\mathbf{S}_n$ at the conjugacy class of cycle type λ .

Valid nested operations which involve computations on intermediate results are to be automatically interpreted, stacked and then executed. For example the compound operation

$$INNER\,(PLET\,(\lambda;\mu);OUTER\,(\rho;\pi))$$

would automatically generate the expansion of $(\{\lambda\}\otimes\{\mu\})\circ(\{\rho\}\{\pi\})$, and

$$COEFF\,(\nu;INNER\,(PLET\,(\lambda;\mu);OUTER\,(\rho;\pi)))$$

would return the multiplicity of the representation $\{\nu\}$ in the analysis of the compound representation $(\{\lambda\}\otimes\{\mu\})\circ(\{\rho\}\{\pi\})$.

2.1 Plethysms

The term *plethysm* refers to an operation on the algebra of symmetric polynomials introduced by Littlewood [Litt1-Litt4]. This operation can be defined loosely as follows:

Suppose we are given two symmetric polynomials p and q with integer coefficients the latter having nonnegative coefficients only. We can then write the polynomial q as a sum of monomials

$$q(x) = m_1 + m_2 + .. + m_r$$

where these monomials are ordered in some fashion. Now expanding p as a polynomial in r variables y_1 , y_2 , .. , y_r by means of the elementary symmetric functions, we set

$$p[q] = p(m_1\,,\,m_2\,,\,..,m_r)$$

and refer to the resulting symmetric polynomial as the plethysm of p and q .

For example if we have

$$p(x_1, x_2) = x_1 x_2 + x_1 + x_2 \qquad \text{and} \qquad q(x_1, x_2, x_3) = 2x_1^2 x_2^2 x_3^2 + x_1 x_2 x_3 \ ,$$

first we express p in terms of the elementary symmetric functions a_k in the form $p \equiv a_2 + a_1$. Given this, we augment the number of variables to three and write

$$p(y_1, y_2, y_3) = a_2(y_1, y_2, y_3) + a_1(y_1, y_2, y_3) \ .$$

Now setting $y_1 = y_2 = x_1^2 x_2^2 x_3^2$ and $y_3 = x_1 x_2 x_3$ gives

$$p[q] = (x_1^4 x_2^4 x_3^4 + 2x_1^3 x_2^3 x_3^3) + (2x_1^2 x_2^2 x_3^2 + x_1 x_2 x_3) \ .$$

We will use the notation $p[q]$ for the plethysm of p and q as well as the more commonly used notation $q \otimes p$ introduced by Littlewood.

One of the fundamental problems in the theory of symmetric functions is to find an efficient way to calculate the coefficients $\pi_{\lambda\mu}^{\nu}$ that arise in the plethysm $\{\mu\} \otimes \{\lambda\}$, or equivalently the plethysm of the corresponding Schur symmetric functions:

$$S_\lambda[S_\mu] = \sum_\nu \pi_{\lambda\mu}^{\nu} S_\nu \ ,$$

which corresponds to the resolution of the outer plethysm of the irreducible representations of $\mathbf{S}_n$ carried by the partitions λ and μ.

In other words, it is desirable to have a construction analoguous to the Littlewood-Richardson (LR) rule [LiRic] by which the coefficients $g_{\lambda\mu}^{\nu}$ in the expansion of the ordinary (outer) product

$$S_\lambda S_\mu = \sum_\mu g_{\lambda\mu}^{\nu} S_\nu$$

can be found.

A number of such algorithms have been proposed in the literature. Among these are several algorithms due to Littlewood [Litt2], and three further ones found in [Todd], [Robi] and [Foul1-Foul2]. Murnaghan also proposed inductivé algorithms for plethysms [Murn1].

These algorithms however, have not been very suitable for computer implementations for a number of reasons. Some of these algorithms, especially those of Littlewood, require human ingenuity and a certain amount of experimentation to carry through. On the other hand, Todd's algorithm requires an extravagant use of random access storage. These undesirable features render them quite inefficient to adapt to the limitations of an automatic process. Furthermore, other less sophisticated but equally important limitations must also be taken into account. For example, it turns out that efficiency has to be achieved by a deeper understanding of the nature of the calculations than at the expense of storage. For instance, in the expansion of the totally symmetric plethysm $S_5[S_7]$ the number of terms that can arise is potentially $p(35) = 14883$ terms, but actually only 901 Schur functions occur in the final expansion. Since the number of partitions $p(n)$ of n grows quite rapidly with n, it is of utmost importance that the algorithm produces as few as possible *dead* partitions during the intermediate computations. By this we mean the partitions that do not occur in the final expansion. In particular, a reject-accept type of approach which requires the generation of a substantial fraction, if not all of the partitions in question is unacceptable. In other words, even a characterization of the 901 *live* partitions in the above expansion is not necessarily helpful if each one of the 14883 partitions of 35 has to be individually tested before being rejected or accepted. Thus an efficient algorithm for plethysms needs to construct rather than recognize the live partitions to be practical.

One such algorithm was developed in University of California, San Diego recently under the auspices of Prof. A.M. Garsia [ChGaRe] which will be referred as the SD algorithm here. Its basic ingredients are as follows:

(i) Fast multiplication of Schur functions implementing an algorithmic modification of the LR rule, due to Remmel and Whitney [ReWhi].

(ii) The expansion

$$S_m(x^p) = S_m(x_1^p, x_2^p, \dots) = \sum_{\lambda} \pm S_\lambda(x)$$

where the summation is over all partitions λ of mp with void p-core. A result of Chen [Chen] is used to *generate* those partitions that appear in this expansion. This algorithm will be referred to as the SXP algorithm.

(iii) The identity

$$S_n = \frac{1}{n} \sum_{p=1}^{n} \psi_p \, S_{n-p}$$

where ψ_p is the p-th power symmetric function, which yields the expansion

$$S_n[S_\mu] = \frac{1}{n} \sum_{p=1}^{n} S_\mu(x^p) \, S_{n-p}[S_\mu] \quad .$$

(iv) The Jacobi-Trudi [Jacob], [Trudi] identity $S_\lambda = \det[h_{\lambda_i + j - i}]$ which can be used to express a Schur function as a linear combination of homogeneous symmetric functions h_p. This identity appears in a number of contexts in the literature: [JaKe], [MacD], [Ege3]. The algorithm to calculate $S_\lambda(x_1^p, x_2^p, \dots)$ by making use of the SXP algorithm and the Jacobi-Trudi identity will be referred to as the SλXP algorithm.

(v) The fact that the Polya enumerator

$$S_n = P_{\mathbf{S}_n}(\psi_1, \psi_2, \dots, \psi_n)$$

can be used to express the plethysm $S_n[S_\mu]$ as sums of products of expansions of the form $S_\mu(x^p)$:

$$S_n[S_\mu] = P_{\mathbf{S}_n}(S_\mu(x), S_\mu(x^2), \dots, S_\mu(x^n))$$

A limited version of the SD algorithm was implemented in 1984 to generate tables of the symmetric and antisymmetric plethysms $S_n[S_m]$ and $S_{1^n}[S_m]$ by Egecioglu and Remmel [EgeRe].

We remark that we were able to carry out the calculations for these cases on a minicomputer under UNIX for up to $mn = 40$. This compares very favorably with the tables of plethysms due to Butler and Wybourne [BuWybl] in which the general cases are tabulated for $mn \leq 16$, the results of Ibrahim [Ibra1-Ibra2] where the degrees of the underlying representations are bounded by 15, and Makar and Missiha's particular results [MaMis].

2.2 Kronecker products

Suppose α and β are two irreducible representations of a finite group G . Then the reduction of the Kronecker or inner product representation $\alpha \circ \beta$ is of general interest, with particular importance for $G = \mathbf{S}_n$.

For a subgroup $H \leq G$, denote by $\gamma_H \uparrow G$ the representation of G induced by the representation γ_H of H . Similarly, given a representation β_G of G , denote by $\beta_G \downarrow H$ the representation of H obtained by restricting β_G to H .

It is well known and easy to prove by Frobenius reciprocity that

$$(1_H \uparrow G) \circ \beta_G = (\beta_G \downarrow H) \uparrow G \quad . \qquad [2.2.1]$$

Robinson and Taulbee observed that this identity in effect reduces the expansion of the Kronecker product $\{\lambda\} \circ \{\mu\}$ of $\mathbf{S}_n$ to multiplication of Schur functions [RobTa].

In view of the Jacobi-Trudi identity and the distributivity of $\circ$ over addition, it suffices to note that for any partition $\rho = (\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n)$ of n , the inner product $h_\rho \circ S_\mu$ can be so decomposed where $h_\rho = h_{\rho_1} h_{\rho_2} \cdots h_{\rho_n}$ is the homogeneous symmetric function corresponding to ρ .

In view of [2.2.1], $h_\rho \circ S_\mu$ is a sum of products of Schur functions, each obtained by restricting the representation $\{\mu\}$ to the Young subgroup of $\mathbf{S}_n$ corresponding to the partition ρ .

For instance we have

$$h_{2^2} \circ S_{13} = S_2 S_1 S_1 + S_{1^2} S_2 \quad .$$

Thus to compute the Kronecker product $S_{1^2 2} \circ S_{13}$ we proceed as follows: by the Jacobi-Trudi identity $S_{1^2 2}$ can be expressed as a linear combination of homogeneous symmetric functions

$$S_{1^2 2} = \det \begin{bmatrix} h_1 & h_2 & h_4 \\ 1 & h_1 & h_3 \\ 0 & 1 & h_2 \end{bmatrix} = h_{1^2 2} + h_4 - h_{13} - h_{2^2} \ .$$

Next, for each h_ρ that appears in the expansion of the above determinant, we compute $h_\rho \circ S_{13}$ making use of the identity (2.2.1) :

$$\begin{aligned} h_{1^2 2} \circ S_{13} &= 2 S_2 S_1 S_1 + S_{1^2} S_1 S_1 \\ h_4 \circ S_{13} &= S_{13} \\ h_{13} \circ S_{13} &= S_3 S_1 + S_{12} S_1 \\ h_{2^2} \circ S_{13} &= S_2 S_1 S_1 + S_{1^2} S_2 \end{aligned}$$

Thus

$$\begin{aligned} S_{1^2 2} \circ S_{13} &= 2 S_2 S_1 S_1 + S_{1^2} S_1 S_1 + S_{13} - S_3 S_1 - S_{12} S_1 - S_2 S_1 S_1 - S_{1^2} S_2 \\ &= S_{13} + S_{2^2} + S_{1^2 2} + S_{1^4} \end{aligned}$$

after the multiplication of the Schur functions and the arithmetic manipulations are carried out.

We should remark at this point that in certain instances the commutativity of the Kronecker product can be used to shorten the calculations involved considerably. For example, using the above algorithm to calculate $S_{1^2 2} \circ S_{13}$ in the form $S_{13} \circ S_{1^2 2}$, the determinant that has to be evaluated is reduced to 2×2 giving

$$S_{1^2 2} \circ S_{13} = S_{13} \circ S_{1^2 2} = \det \begin{bmatrix} h_1 & h_4 \\ 1 & h_3 \end{bmatrix} \circ S_{1^2 2}$$

$$= h_{13} \circ S_{1^2 2} - h_4 \circ S_{1^2 2} \quad .$$

This is one of the reduction rules to be used in the computation of *INNER* $(\lambda;\mu)$. Other reduction rules and short cuts such as conjugation relations that simplify the construction of *PLET* $(\lambda;\mu)$ also turn out to be useful for Kronecker products.

We also remark that Frame [Frame] recently introduced a recursive method to determine the Kronecker powers of a fixed irreducible character χ that is less time consuming than Murnaghan's approach [Murn3]. It could be possible to incorporate Frame's algorithm as a part of *INNER* to be invoked in the case of the equality of the input arguments to increase efficiency.

2.3 Outer Products

It is well known that the product of two Schur functions $\{\lambda\}$ and $\{\mu\}$ of degrees n and m respectively, can be expressed as a nonnegative integral linear combination of Schur functions $\{\rho\}$ of degree $n+m$:

$$\{\lambda\}\{\mu\} = \sum_{\rho} g^{\rho}_{\lambda,\mu} \{\rho\} \quad . \qquad [2.3.1]$$

The correspondence [LiRic] between the construction of the outer product $\lambda \times \mu$ of two irreducible representations λ and μ of the symmetric group and the (outer) product of the Schur functions determined by these partitions gives the multiplicity of the irreducible constituents of the representation $\lambda \times \mu$ once the expansion [2.3.1] is known.

Similarly, the coefficient $g^{\rho}_{\lambda^1,\lambda^2,..,\lambda^r}$ in the expansion

$$\{\lambda^1\}\{\lambda^2\} \cdots \{\lambda^r\} = \sum_{\rho} g^{\rho}_{\lambda^1,\lambda^2,..,\lambda^r} \{\rho\} \qquad [2.3.2]$$

of the Schur functions $\{\lambda^1\},\{\lambda^2\},..,\{\lambda^r\}$ gives the multiplicity of the corresponding irreducible representation ρ in the analysis of $\lambda^1 \times \lambda^2 \times \cdots \times \lambda^r$ induced from the corresponding Young subgroup.

The LR rule is a combinatorial algorithm to compute the coefficients $g^{\rho}_{\lambda,\mu}$ that occur in [2.3.1] [LiRic]. [Litt3] is a more complete reference on this result.

We note that the backbone of the algorithms for *PLET* $(\lambda;\mu)$ and *INNER* $(\lambda;\mu)$ consists of the resolution of products of the form [2.3.2]. Therefore it is of paramount importance to have an efficient procedure for this expansion.

In [Egel], an algorithmic modification of the Littlewood-Richardson rule developed by Remmel-Whitney [ReWhi] was implemented to this end. The Remmel-Whitney result is extremely suitable for computer implementation and it makes it possible to multiply an arbitrary number of Schur functions directly without excessive computational effort. A version of this particular implementation was also used as a subprocedure in the calculation of symmetric and antisymmetric outer plethysms of Schur functions that we have already mentioned [EgeRe].

We also remark that with minor modifications, skew representations of $\mathbf{S}_n$ can also be expanded with this algorithm. Thus *OUTER* $(\lambda;\rho;...;\mu)$ can actually be constructed by the primitive *EXPAN* using suitable input parameters.

Algorithms to expand the outer products and skew representations of $\mathbf{S}_n$ via Schur functions and the LR rule can be traced back to [HuWil] . Manipulations with symmetric functions and suitable representation schemes appear in [McKa1], [BrMcK] .

2.4 Character Values

One of the well known ways of calculating the value of an irreducible character of the symmetric group $\mathbf{S}_n$ at a given conjugacy class is the recursive formulation due to Murnaghan [Murn2]. This formula appears in a variety of forms in the literature [JaKe], [Litt3], [Ege2]. Calculation of all the entries in an arbitrary column of the character table of $\mathbf{S}_n$ (i.e. the values of all the irreducible characters at a fixed conjugacy class) can be realized by making use of the Frobenius formula [Frob1] :

$$\psi_\mu = \sum_\lambda \chi_\lambda(\mu)\, S_\lambda \qquad [2.4.1]$$

in conjunction with Murnaghan's rule where μ and λ are partitions of n , $\chi_\lambda(\mu)$ is the (λ,μ)th entry in the character table of $\mathbf{S}_n$, ψ_μ is the power symmetric function corresponding to the partition μ and S_λ is the Schur function corresponding to λ . A careful implementation of Murnaghan's rule provides a fast and compact algorithm for calculating the values of the irreducible characters of the symmetric groups in an efficient manner. This was done in [EgeCo] .

For example, a Pascal routine for the computation of the character values

$$\chi_{(1^2 3 7 10)}(2\,4^2\,6^2) = -2 \qquad \chi_{(5^2 7 8 15)}(3\,4^3\,7\,9^2) = -24$$

implementing this rule took 17 and 266 milliseconds respectively, on a VAX-11/780 minicomputer. Considering the fact that the underlying symmetric groups are of rather large degrees ($\mathbf{S}_{22}$ and $\mathbf{S}_{40}$ respectively), the performance of the algorithm and its implementation are seen to be quite satisfactory.

Of course the degree of the irreducible representation λ of $\mathbf{S}_n$ can be calculated by calling *CHAR* with the arguments λ and 1^n , though our algorithm here would essentially generate all Young tableaux of shape λ . These degrees on the other hand, can be readily calculated by the celebrated hook formula of Frame, Robinson and Thrall [FrRoTh] .

We should also remark that the repeated application of a simple combinatorial rule - the *slinky* rule due to Rodriguez [Rodri] for the expansion of $S_\lambda \psi_p$ in terms of Schur functions, yields the expansion in [2.4.1] without generating too many dead partitions. Thus the slinky rule can be used to generate the stream of character values of all irreducible representations at a single conjugacy class. This forms the basis of the $CCOL(\lambda)$ operation.

Machine computation of the character values and the construction of the character table of $\mathbf{S}_n$ goes back to the beginning of 1950's. Bivins et al [BiMSW] computed the characters of $\mathbf{S}_{15}$ and $\mathbf{S}_{16}$ on MANIAC at Los Alamos in 1954, using a direct implementation of Murnaghan's recursion by removing cycles of various lengths. In this particular approach, preparation of all partitions beforhand was necessary.

One of the major problems of the time seems to be the unavailability of high level languages which are necessary for the symbolic methods of Young [Young] and Yamanouchi [Yama] . Nevertheless, improved results were obtained through that decade by Comet [Come1-Come3] and by Gabriel [Gabri] . For example, Comet was able to compute the character values for up to $n = 20$ by using Nakayama's recursion [Naka] . Later works include [McKa2] .

In very recent years, the classification problem of finite simple groups and general developments in the theory of finite groups produced a wealth of computational algorithms and results for the calculation of characters: for example [Dixon], [McKa3-McKa4], [Neub], [SiFra], among many others. A number of further references can be found in [CAS3] . Furthermore, extensive packages for operations with the characters of large classes of finite groups were developed such as the CAS [CAS1-CAS3]. The CAS allows for extensive analysis of characters of an arbitrary finite group, given a partial knowledge of its structure. Operations such as induction and reduction among many other operations useful in a general setting are made possible. In particular the character tables for the symmetric groups can be constructed by decomposing the tensor powers of a faithful representation. The power and the general nature of the CAS necessarily requires some overhead in terms of the size of the package. For the particular case of the symmetric group, especially in physical applications that require the calculation of plethysms of representations, the symbolic-combinatorial nature of the SD algorithm has advantages in terms of speed and compactness.

3. Some Applications

The algebra of plethysms for Schur functions has proved to be an extremely useful tool in calculations of branching rule coefficients for various subgroups of the full linear group $GL(n)$ and of resolutions of Kronecker products of irreducible representations.

Knowledge of such branching rule coefficients and resolutions of Kronecker products has a number of applications in atomic, nuclear and particle physics and quantum chemistry. For example, in the 1950s Elliot exploited the basic identification between plethysms $\{\mu\}\otimes\{\lambda\}\ (=S_\lambda[S_\mu])$ and branching rule coefficients for the decomposition of $U(N) \supset U(M)$ together with Ibrahim's tables of plethysms [Ibra3] to establish the branching rules for the decomposition $U(N) \supset U(3)$ which were then used in the study of the $SU(3)$ shell model of nuclei [Elli].

In the late 1960s and early 1970s, several authors used plethysms to attack a number of problems in complex spectra. For example, Smith and Wybourne [SmWyb] gave applications of plethysms to the classification of the atomic states of n−electron configurations, the analysis and classification of the N−particle operators that arise in the application of perturbation theory to atomic problems, and the derivation of selection rules for matrix elements of operators. Related work in this area during this period includes [Judd], [Gram], [Wybo1], [BuWyb2], [BuKin].

Work within the last ten years includes Men at al. [MeChMe], [MeMe], [MeVaMe], [MeLeMe], who have used and extended the applications of plethysms in nuclear theory and the study of the electron configuration of atoms; Sullivan [Sull1], [Sull2], who has used plethysms for applications to shell theory; and Dehuai and Wybourne [DeWyb], who have worked on plethysms for spin representations. More recent work includes [KiWyb], [Newm], [PaSha1], [PaSha2], [SaJuBe].

Wybourne's book [Wybo2] furnishes further examples of the uses of the outer plethysm of Schur functions for calculating branching rule coefficients for the unitary $U(N)$, orthogonal (O_n), and symplectic (Sp_n) groups and the resolution of Kronecker squares into symmetric and antisymmetric terms. A number of applications of such calculations to various problems in atomic spectroscopy along with tables of expansions [Butle] are also included.

4.1 Software Considerations

In the existing implementations of $OUTER(\lambda,\rho,\ldots,\mu)$ and in the various utility procedures required in the calculations of plethysms such as merging a large number of files, the use of AVL-trees turned out to be a sufficient data structure. For more general computations proposed here, the extent of the calculations and the number of intermediate partitions generated demands a more careful consideration in this respect. To carry this out more efficiently on the computer, we can proceed as follows: each linear combination of Schur functions

$$\sum_{\lambda} c_\lambda S_\lambda$$

is represented as a dictionary. We recall that these are binary-tree-like structures with four fields: the first field gives the key, the second stores the information and the last two fields are pointers to left and right sons of the node. In our case each node represents a term of the form $c_\lambda S_\lambda$, the key being the underlying partition λ and the information being the coefficient c_λ. As new partitions are generated by one of the algorithms these records are constantly kept arranged so that when we read the dictionary in symmetric order the partitions generated come out in lexicographic order.

This given, as new partitions are generated, they are recorded in the dictionary by locating the node with the proper key and updating the coefficient field. If the key is nonexistent, a new node is added in the appropriate location and the dictionary is updated to keep a suitable balance. Of course the frequencies c_λ with which the various shapes appear are not known in advance, as these are the coefficients we are trying to determine. Efficiency of the updating procedure can be increased by making use of a recent updating procedure for dictionaries discovered by Sleator and Tarjan [SleTa].

The input-output files have a common format as follows. Each Schur function is represented by three fields: multiplicity, the number of parts and the parts themselves in *ascending* order of magnitude:

$c_\lambda \quad k \quad \lambda_1 \quad \lambda_2 \cdots \lambda_k$

For instance the file S2X3 which is generated by the SXP algorithm would look like

1	1	6		
1	2	3	3	
−1	2	1	5	
−1	3	1	2	3
1	3	2	2	2
1	3	1	1	4

coding the expansion

$$S_2(x^3) = S_6 + S_{3^2} - S_{15} - S_{123} + S_{2^3} + S_{1^24} \quad .$$

The internal representation of partitions can be most economically realized by coding them as binary strings as in [Come4] : we simply scan the boundary of the Young frame (= the Ferrers' diagram) of λ from top to bottom, recording each horizontal step as a 1 and each vertical step as a 0 . For instance the partition (124^26) would correspond to the binary word 10101100110 .

(Figure 1)

For all practical purposes, two 32-bit words would suffice to represent a partition this way. We use the first two bits as special purpose flags, and the content of the next 6 bits point to the index of the last 0 in the representation of λ , the offset being the ninth bit. This way we are left with 56 bits for the binary word of the partition.

(Figure 2)

Clearly, partitions of up to $n = 56$ can be represented with this scheme.

The high-level dependencies between the primitives and internal utility routines are schematically described in (Figure 3) for the SD algorithm and the *INNER* operation.

(Figure 3)

The core of the package has the following basic functions:

a) Parsing the input expressions and controlling interprocess I/O,

b) Generating symbolic expressions to be evaluated by the various primitives, such as the Jacobi-Trudi expansion and cycle-index polynomial generation,

c) Generating sequential source code for the primitives to execute,

d) Optimising algorithms by using the conjugacy relations, etc., before the computations are carried out.

Also, for efficiency in speed, keeping short and frequently used data files on disk should be helpful. For instance

a) SXP output for small n and p ,

b) Symbolic expansions for the cycle-index polynomials $P_{\mathbf{S}_n}$ for small integers n ,

c) Symbolic expansions of determinants of small sizes,

d) Totally symmetric and antisymmetric plethysms that have to be frequently recalculated.

The extent of the speedup from such a lookup procedure remains to be experimentally determined.

Another factor which effects speed is the choice of a language for implementation. Even though some of the existing applications were coded in Pascal, we feel that for the general case a better choice is the C-language in a UNIX environment. Various facilities of UNIX such as pipes and I/O redirection are time-savers for the generation of such a package.

We feel that modularity is an essential aspect of this type of software development. Even though a certain number of shared procedures are necessary, the modification of the total package for particular applications and/or accommodation of new algorithms should be facilitated with this approach.

5. Summary

Description of a comprehensive package of routines for the character theory of the symmetric groups $\mathbf{S}_n$ have been presented. The efficiency of these algorithms derives from the reduction of the expansions of plethysms and Kronecker products to multiplication of Schur functions. This in turn is done with minimal computational effort by making use of an algorithmic modification of the LR rule due to Remmel and Whitney.

Existing implementations of a number of these algorithms turned out to be very efficient. The calculation of totally symmetric and antisymmetric plethysms for instance, could be carried out further than the existing tables in the literature.

The package is centered around a core which coordinates the various primitives and sequentially invokes the necessary routines to carry out the required calculations. The common features of the various algorithms implemented make it possible to construct this package in a highly modular and compact form in a UNIX environment.

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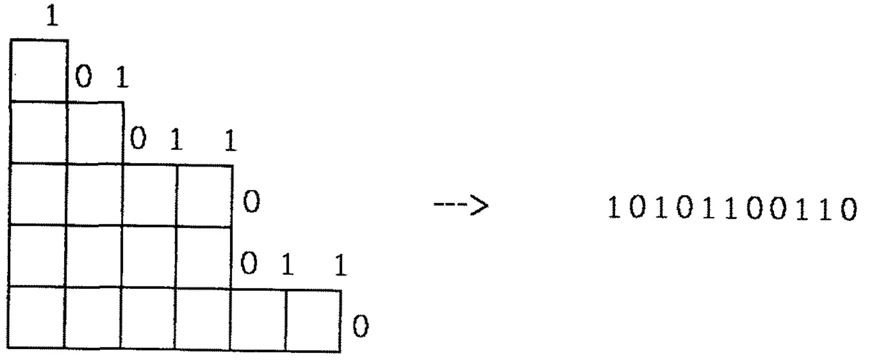


Figure 1

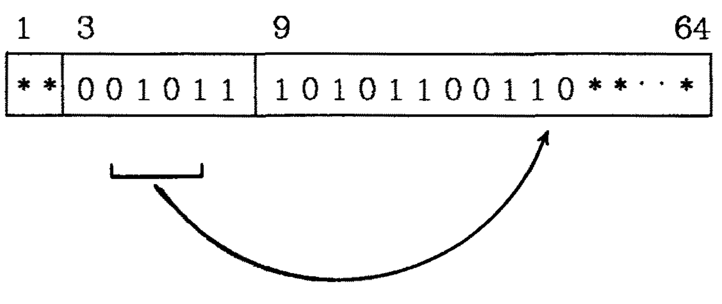
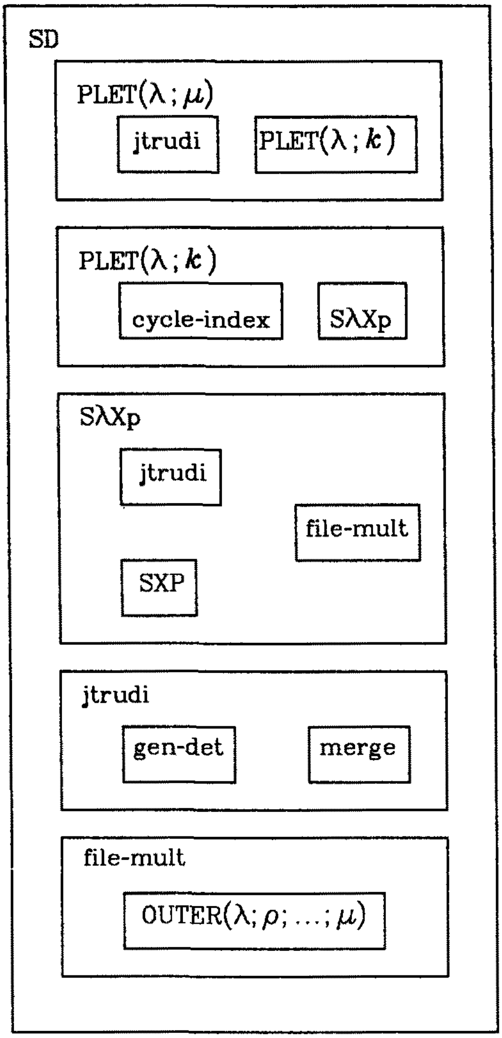


Figure 2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jtrudi
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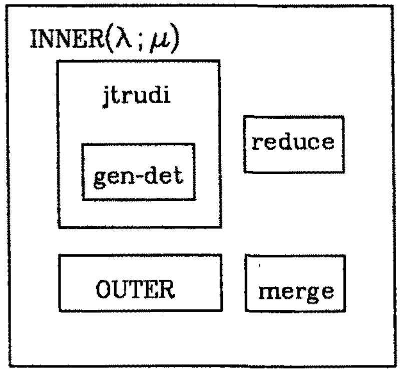
INNER(λ ; μ)
jtrudi
gen-det
reduce
OUTER
merge

Figure 3