Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 14 No. 4 (2025), pp. 261-270.

https://toc.ui.ac.ir
www.ui.ac.ir

ON TWIN EP NUMBERS

ÖMER EĞECIOĞLU* AND BÜNYAMIN ŞAHIN

Abstract. EP numbers were introduced by Estrada and Pogliani in 2008. These are positive integers $E(n)$ defined as the product of n and the sum of the digits of n. Estrada and Pogliani suspected that there may be infinitely many twin EP numbers; i.e. those pairs in this sequence that differ by one. It is relatively easy to show that three consecutive EP numbers do not exist, and that no pair $E(n)$ and $E(m)$ can be twins for infinitely many bases b. The main contribution of our work is the result that indeed there are infinitely many twin EP numbers over any base. The proof is constructive and makes use of elementary properties of natural numbers. The forms of the twin EP numbers presented are derived from continued fractions. The behavior of the series of the reciprocals of twin EP numbers is also considered.

1. Introduction

As building blocks of number systems, digits of a number in relation to the number itself has been broadly studied in number theory and elsewhere. For example Estrada [8, 7] considered artistic creations based on integer digit functions as well as their relation to visual arts.

Perhaps the most widely known function of the digits of a positive integer is the "sum-of-digits" function $s_b(n)$, which is the sum of the digits of n expressed in base b. Bush [3] computed the average

MSC(2020): Primary: 11A63; Secondary: 11A67, 40A05.

Keywords: EP number, twin EP number, continued fractions, reciprocal series.

Article Type: Research Paper.

Communicated by Alireza Abdollahi.

*Corresponding author.

Received: 16 July 2023, Accepted: 04 October 2024, Published Online: 17 November 2024.

Cite this article: Ö. Eğecioğlu and Bünyamin Şahin, On twin EP numbers, *Trans. Comb.*, **14** no. 4 (2025) 261-270.
http://dx.doi.org/10.22108/toc.2024.138412.2092 .

order of this function as

$$\frac{1}{N}\sum_{n=1}^{N} s_b(n) = \frac{b-1}{2\log b}\,\log N\,. \tag{1.1}$$

Properties, including generating functions and infinite products for the sum-of-digits function were studied in the papers [1, 2, 13, 14, 17]. It was proved by Allouche and Shallit [2] that

$$\sum_{n=1}^{\infty} \frac{s_b(n)}{n(n+1)} = \frac{b}{b-1}\log b\,. \tag{1.2}$$

There are a number of interesting integer sequences which are constructed by consideration of the digits of a number, such as the Kaprekar and the Harshad numbers. These also have attracted attention in the literature. We can only give some examples and name a few references.

An r-digit number n is a *Kaprekar number* if it is equal to the sum of the right r digits of n^2 and the left r (or $r-1$) digits [11]. For instance $9^2 = 81$ with $8+1 = 9$, and $297^2 = 88209$ with $88+209 = 297$, so that 9 and 297 are Kaprekar numbers. This is the sequence A006886/M4625 in the OEIS, The On-Line Encyclopedia of Integer Sequences [15].

Another related family is *Harshad numbers*. Also referred to as Niven numbers or multidigital numbers, a Harshad number is a positive integer that is divisible by the sum of its digits. For example, 2023 and 2024 are Harshad numbers since $2023 = 289(2+2+3)$ and $2024 = 253(2+2+4)$. Many interesting properties of this sequence (Reference number A005349 in OEIS) were studied [9, 4, 5].

In 2008 Estrada and Pogliani [6] noticed that in addition to being a sacred number in Hindu tradition (alongside 108 and 1008), 2008 can be expressed as the product of a number and the sum of the digits of that number, i.e., 2008=251(2+5+1). Inspired by this observation, they defined a sequence of integers by applying the same rule to each integer and noted a number of properties of these numbers. This integer sequence, formed by the product of a number and the sum of its digits, is called the *EP number sequence*. It appears in the OEIS with reference number A117570. Estrada and Pogliani also gave examples of *twin EP numbers* which are consecutive EP numbers, i.e., those pairs that differ exactly by one.

In this paper, we show that there are infinitely many twin EP numbers for every base. The proof is constructive. The forms of the twin EP numbers were established by considering data from the study of continued fractions. We also show that three consecutive EP numbers do not exist, and that no pair $E(n)$ and $E(m)$ can be twins for infinitely many bases b. Finally we provide computational evidence that indicates that series of the reciprocals of twin EP numbers converges.

2. **Preliminaries**

Let n be a integer with base $b \geq 2$ expansion

$$n = \sum_{i=0}^{r} d_i b^i$$

http://dx.doi.org/10.22108/toc.2024.138412.2092

where $0 \leq d_i < b$ are the base b digits of n with $d_r > 0$. The EP number $E_b(n)$ in base b is defined as

$$E_b(n) = n\, s_b(n)$$

where

$$s_b(n) = \sum_{i=0}^{r} d_i \tag{2.1}$$

is the sum of digits function in base b [6].

We omit the subscript b in $s_b(n)$ and $E_b(n)$ for $b = 10$ and write $E(n)$ and $s(n)$, respectively.

Example 1. *We have* $s(15) = 6$ *with* $E(15) = 15 \times 6 = 90$*, and* $s_2(15) = 4$ *with* $E_2(15) = 15 \times 4 = 60$*.*

EP numbers $E(n)$ and $E(m)$ are called *twin EP numbers* if they are consecutive integers, that is $E(m) = N$ and $E(n) = N + 1$ for some integer N. This means that $E(n) = E(m) + 1$.

We will make frequent use of Bézout's theorem, which says that the greatest common divisor $\gcd(n, m)$ of two positive integers n and m can be expressed in the form $an + bm$ for some integers a and b. The coefficients a, b are called *Bézout coefficients* (See, for example [12]).

3. Main Results

We start with some elementary properties of EP and twin EP numbers. Since $n \pmod 9 = s(n) \pmod 9$, $E(n)$ is a square modulo 9. This is the special case of the more general observation that for $b > 2$, $E_b(n)$ is a square modulo $b - 1$. Therefore

Lemma 1. $E(n) \equiv 0, 1, 4, 7 \pmod 9$.

Using Lemma 1 we obtain

Proposition 1. *No three consecutive integers can be EP numbers.*

Proof. Assume $N, N+1, N+2$ are three EP numbers. Then by Lemma 1,

$$\begin{aligned} N &\equiv 0, 1, 4, 7 \pmod 9, \\ N+1 &\equiv 0, 1, 4, 7 \pmod 9, \\ N+2 &\equiv 0, 1, 4, 7 \pmod 9. \end{aligned}$$

However if $N \equiv 0, 1, 4, 7 \pmod 9$, then $N+1 \equiv 1, 2, 5, 8 \pmod 9$. Since we also have $N+1 \equiv 0, 1, 4, 7 \pmod 9$, it follows that $N + 1 \equiv 1 \pmod 9$. Therefore $N + 2 \equiv 2 \pmod 9$, but $2 \notin \{0, 1, 4, 7\}$, a contradiction. □

Next, we note that by definition twin EP numbers $E(n)$ and $E(m)$ satisfy

$$n\, s(n) - m\, s(m) = 1\,. \tag{3.1}$$

This implies a number of properties of such pairs.

http://dx.doi.org/10.22108/toc.2024.138412.2092

Proposition 2. *Suppose $E(m) = N$ and $E(n) = N + 1$ are twin EP numbers. Then*

(1) $\gcd(n, m) = 1$ *with Bézout coefficients $s(n)$ and $-s(m)$,*
(2) $\gcd(s(n), s(m)) = 1$ *with Bézout coefficients n and $-m$,*
(3) $s(n) \equiv \pm 1 \pmod 9$ *and $s(m) \equiv 0 \pmod 3$.*

Proof. The first two statements are direct consequences of (3.1). For the last one, we use (3.1) together with Lemma 1. □

3.1. **Decimal twin EP numbers.** Consider the identity

$$(54 \times 10^k + 251) \times 17 - (51 \times 10^k + 237) \times 18 = 1 \tag{3.2}$$

for $k \geq 1$. This can be proved by direct calculation. For $k \geq 1$, let

$$\begin{aligned} n_k &= 54 \times 10^k + 251\,, \\ m_k &= 51 \times 10^k + 237\,. \end{aligned} \tag{3.3}$$

Then $s(n_k) = 17$ and $s(m_k) = 18$ independently of the value of k. In other words the identity (3.2) is equivalent to

$$E(n_k) - E(m_k) = 1\,,$$

or $E(n_k) = E(m_k) + 1$. Therefore we have a given a constructive proof of the following proposition.

Proposition 3. *There are infinitely many twin EP numbers in base $b = 10$.*

3.2. **The general case.** The result of Proposition 3 can be extended to any base $b \geq 2$ as the following proposition shows.

Proposition 4. *There are infinitely many twin EP numbers for any base $b \geq 2$.*

Proof. The proof is again constructive. It is handled in two cases, depending on whether $b > 2$ or $b = 2$.

Case 1: $b > 2$

Let

$$\begin{aligned} n_k &= (b-1)b^k + b^{b-3}\,, \\ m_k &= b^{k+1} + b^{b-3} + b^{b-4} + \cdots + b + 1 \end{aligned} \tag{3.4}$$

for $k > b - 3$. Then the sum of the base b digits of n_k is $s_b(n_k) = b - 1 + 1 = b$, and those of m_k is $s_b(m_k) = 1 + b - 2 = b - 1$. Note that

$$b^{b-3} + b^{b-4} + \cdots + b + 1 = \frac{b^{b-2} - 1}{b - 1}\,,$$

and therefore

$$\begin{aligned} E(n_k) - E(m_k) &= \Big((b-1)b^k + b^{b-3}\Big) \times b - \left(b^{k+1} + \frac{b^{b-2}-1}{b-1}\right) \times (b-1) \\ &= b^{k+2} - b^{k+1} + b^{b-2} - \Big(b^{k+1}(b-1) + b^{b-2} - 1\Big) \\ &= 1\,. \end{aligned}$$

Therefore $E(n_k) = E(m_k) + 1$ for every $k > b-3$ and these two numbers are twin EP numbers in base b.

Case 2: $b = 2$

The above approach fails for $b = 2$ because of the appearance of b^{b-3} in the definition of n_k in (3.4). Therefore we need a special construction in the binary case. Let

$$\begin{aligned} n_k &= 2^k + 2^{k-1} + 2^5 + 2 + 1\,, \\ m_k &= 2^k + 2^{k-2} + 2^4 + 2^3 + 2^2 + 1 \end{aligned} \tag{3.5}$$

for $k \geq 7$. Then $s_2(n_k) = 5$, and $s_2(m_k) = 6$ for every $k \geq 7$. Since $2^5+2+1 = 35$ and $2^4+2^3+2^2+1 = 29$, we quickly calculate

$$\begin{aligned} E(n_k) - E(m_k) &= (2^k + 2^{k-1}) \times 5 - (2^k + 2^{k-2}) \times 6 + 35 \times 5 - 29 \times 6 \\ &= 2^{k-2} \times 6 \times 5 - 2^{k-2} \times 5 \times 6 + 1 \\ &= 1\,. \end{aligned}$$

□

It is interesting that the general result in Proposition 4 for $b = 10$, gives a different sequence of twin EP numbers than the construction (3.3) used in the proof of Proposition 3. Proposition 4 with $b = 10$ gives the pair

$$\begin{aligned} n_k &= 9 \times 10^k + 10^7\,, \\ m_k &= 10^{k+1} + 10^7 + 10^6 + \cdots + 10 + 1 \end{aligned}$$

for $k > 7$. In this case $s(n_k) = 10$, $s(m_k) = 9$ and

$$\begin{aligned} (9 \times 10^k + 10^7) \times 10 - \left(10^{k+1} + \frac{10^8 - 1}{9}\right) \times 9 &= 9 \times 10^{k+1} + 10^8 - \big(9 \times 10^{k+1} + 10^8 - 1\big) \\ &= 1\,. \end{aligned}$$

We also have the following negative result.

Proposition 5. *There exists no pair n, m for which $E(n)$ and $E(m)$ are twin EP numbers for infinitely many bases $b \geq 2$.*

Proof. Pick a base b for which we have the base b expansions

$$n = \sum_{i=0}^{r} d_i b^i \,, \; m = \sum_{i=0}^{s} e_i b^i$$

with $E_b(n) - E_b(m) = 1$, i.e.,

$$\left(\sum_{i=0}^{r} d_i b^i \right) \left(\sum_{i=0}^{r} d_i \right) - \left(\sum_{i=0}^{s} e_i b^i \right) \left(\sum_{i=0}^{s} e_i \right) = 1 \,.$$

By assumption this identity holds for infinitely many $b' > b$, since the digits of n and m in base $b' > b$ are identical to their digits in base b. It follows that setting

$$p(x) = \sum_{i=0}^{r} d_i x^i \,, \; q(x) = \sum_{i=0}^{s} e_i x^i$$

we have the polynomial identity

$$p(x)p(1) - q(x)q(1) = 1 \,.$$

Putting $x = 1$, this implies $(p(1) - q(1))(p(1) + q(1)) = 1$. Since $p(1)$ and $q(1)$ are positive integers, this is a contradiction. □

3.3. **Continued fractions.** Once the forms of the twin EP numbers are established, such as (3.3), (3.4) and (3.5), then it is straightforward to prove that they are indeed twin EP numbers as claimed. Here it may be of interest to mention the means used to arrive at some of these expressions. These consist of continued fractions and experimentation using Mathematica.

There are numerous sources on continued fractions. For a concise introduction, see [16]. The nth convergent of a continued fraction $[c_0, c_1, \ldots]$ is

$$\frac{p_n}{q_n} = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}} = [c_0, c_1, \ldots, c_n] \,.$$

It can be shown that with the initial conditions $p_{-1} = 1$, $p_0 = c_0$, $q_{-1} = 0$, $q_0 = 1$, the convergents satisfy

$$\begin{aligned} p_{n+1} &= c_{n+1} p_n + p_{n-1}, \\ q_{n+1} &= c_{n+1} q_n + q_{n-1} \end{aligned}$$

with

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \tag{3.6}$$

http://dx.doi.org/10.22108/toc.2024.138412.2092

for $n \geq 0$. Starting with a positive rational number A/B in lowest terms with continued fraction expansion

$$\frac{A}{B} = [c_0, c_1, \ldots, c_n]$$

with n odd, (3.6) implies that

$$Aq_{n-1} - Bp_{n-1} = 1\,, \tag{3.7}$$

so that the $(n-1)$st convergent $[c_0, c_1, \ldots, c_{n-1}]$ gives the Bézout coefficients q_{n-1} and $-p_{n-1}$ in (3.7). The easiest computational approach to explore twin EP numbers seemed to be to take $n = 3$ and experiment with continued fractions

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3}}} \quad \text{and} \quad c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2}}$$

where c_0, c_1, c_2, c_3 range over an interval of positive integers. If $A/B = [c_0, c_1, c_2, c_3]$ and $b/a = [c_0, c_1, c_2]$ are such that $a = s(A)$ and $b = s(B)$, then in view of (3.7) $A\,a - B\,b = 1$ and A and B are twin EP numbers. Some numerical experimentation produced the following continued fractions with this property:

$$[1, 16, 1, 313],\ [1, 16, 1, 3013],\ [1, 16, 1, 30013],\ [1, 16, 1, 300013],\ [1, 16, 1, 3000013]$$

from this data, it is a quick generalization that $A = 54 \times 10^k + 251$ and $B = 51 \times 10^k + 237$ have these properties. These are the twin EP pairs we gave in (3.3). Twin EP numbers in other bases were developed from similar experimental computations with Mathematica and consequent generalizations using continued fractions.

4. Series of reciprocals

We next consider the series of reciprocals of EP and twin EP numbers.

4.1. Reciprocals of EP numbers.

Proposition 6. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{E_b(n)}$$

diverges for any base $b \geq 2$.

Proof. The number of digits of n in base b is $\lfloor \log_b n \rfloor + 1$. Since a base b digit is at most $b-1$, we have

$$E_b(n) \leq n\,(b-1)\,(\lfloor \log_b n \rfloor + 1) \tag{4.1}$$

and

$$\frac{1}{E_b(n)} \geq \frac{1}{b-1} \frac{1}{n(\lfloor \log_b n \rfloor + 1)} \tag{4.2}$$

However the series with the terms that are the right hand side of (4.2) diverges by the integral test. The proposition follows by the inequality (4.2) and the comparison test. □

Among r-digit numbers n, i.e., those in the range $b^{r-1} \leq n < b^r$, the expected value of $E_b(n)$ is $c_b\, n \log n$, where c_b is a constant that depends only on b. When we consider the prime numbers, we have $p_n \sim n \log n$ by the prime number theorem. We also know by Brun's theorem that the series of reciprocals of twin primes converges [10]. Brun's theorem leaves open the possibility that the number of twin primes is finite; however if there are infinitely many of them, then they are spread out thinly enough to make the series of reciprocals converge.

4.2. **Reciprocals of twin EP numbers.** We have proved that the number of twin EP numbers is infinite over any base b. Thus in view of the expected behavior of $E_b(n)$, it is reasonable to conjecture that the series of reciprocals of twin EP numbers converges. In this section we provide computational evidence for this conjecture.

Here we are considering the behavior of the series

$$\sum \left(\frac{1}{E(n)} + \frac{1}{E(n)+1} \right) \tag{4.3}$$

where the summation is over all $n > 0$ such that $E(n)+1$ is also an EP number, i.e., $E(n)+1 = E(m)$ for some m.

Using Mathematica, we have generated the first 250 million EP numbers $E(n)$, and then sorted this list and eliminated duplicate entries. There were 238,256,274 elements in the final collection of distinct EP numbers in this range. Then we have computed the partial sums of the reciprocals of the pairs of consecutive elements in this list. We considered twin EP numbers among the first five million entries and calculated the sum of their reciprocals, then those among the first ten million entries, etc., and continued computing the partial sums of (4.3) with increments of five million elements. The values of the partial sums of the series of twin EP numbers in these intervals are plotted in Fig. 1.

The final value, which is the sum of the reciprocals of 2,455,354 twin EP numbers is about 0.28167. It seems highly likely that the series converges. In case of twin primes, it is well-known that the series converges to Brun's constant, which is about 1.90216.

4.3. **Behavior over bases.** It is also possible to consider this problem over different bases. To this end, we generated the first twenty five million EP numbers in base b, for b ranging from 2 to 20, and then calculated the sum of the reciprocals of the twin EP numbers in this range. Compared to the calculations we did above for $b = 10$, this is a very limited range. The resulting values are given in Fig. 2.

http://dx.doi.org/10.22108/toc.2024.138412.2092

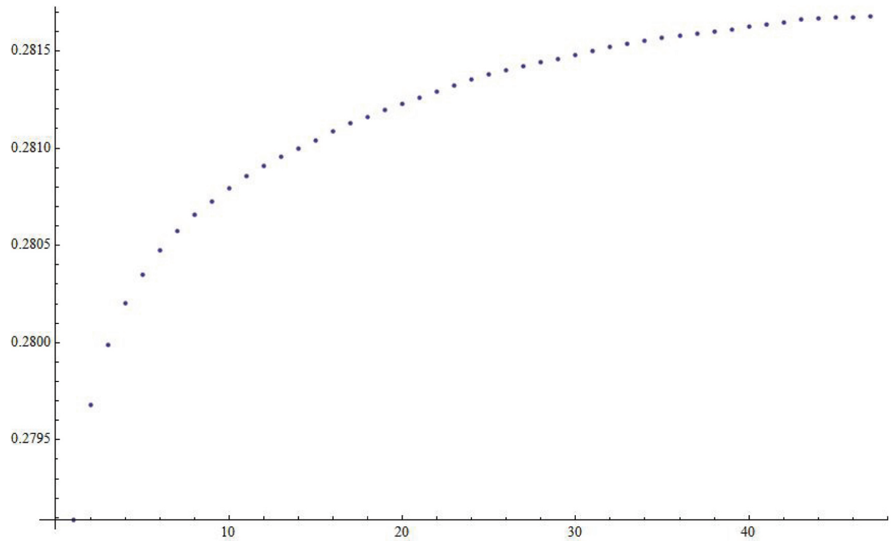

FIGURE 1. Partial sums of the reciprocals of the first 2,455,354 twin EP numbers in base $b = 10$.

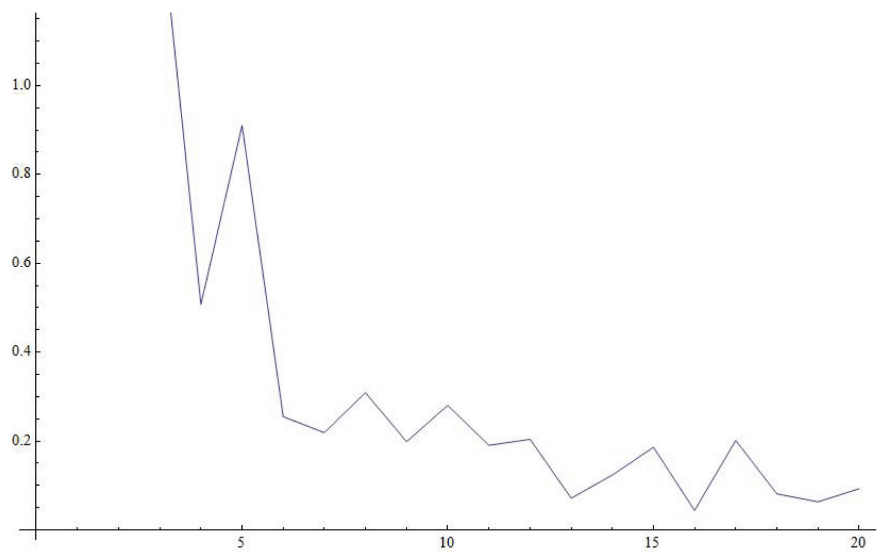

FIGURE 2. Partial sums of the reciprocals of twin EP numbers in base $b = 2, 3, \ldots, 20$.

There are also some trivial corollaries of known results to series involving the sequence of EP numbers. Since $E_b(n) = ns_b(n)$, Bush's average order result (1.1) immediately implies

$$\frac{1}{N}\sum_{n=1}^{N}\frac{E_b(n)}{n} = \frac{b-1}{2\log b}\,\log N\,,$$

and from the identity (1.2) of Allouche and Shallit we obtain

$$\sum_{n=1}^{\infty} \frac{E_b(n)}{n^2(n+1)} = \frac{b}{b-1} \log b\,. \tag{4.4}$$

References

[1] F. T. Adams-Watters and F. Ruskey, Generating functions for the digital sum and other digit counting sequences, *J. Integer Seq.*, **12** (2009).

[2] J,-P. Allouche and J. Shallit, Sums of digits and the Hurwitz zeta function, *Analytic number theory (Tokyo, 1988)*, Lecture Notes in Math., Springer, Berlin, 1990 19–30.

[3] L. E. Bush, An asymptotic formula for the average sum of the digits of integers, *Amer. Math. Monthly*, **47** (1940) 154–156.

[4] T. Cai, On 2-Niven numbers and 3-Niven numbers, *Fibonacci Quart.*, **34** no. 2 (1996) 118–120.

[5] C. Cooper and R. E. Kennedy, On consecutive Niven numbers, *Fibonacci Quart.*, **31** no. 2 (1993) 146–151.

[6] E. Estrada and L. Pogliani, A new integer sequence based on the sum of the digits of integers, *Kragujevac J. Science*, **30** (2008) 45–50.

[7] E. Estrada and P. Pereira-Ramos, Spatial "Artistic" Networks: From Deconstructing Integer-Functions to Visual Arts. *Complexity,* Vol. 2018, Article ID 9893867.

[8] E. Estrada, Integer-digit functions: an example of math-art integration, *Math. Intelligencer*, **40** no. 1 (2018) 73–78.

[9] H. G. Grundman, Sequences of consecutive n-Niven numbers, *Fibonacci Quart.*, **32** no. 2 (1994) 174–175.

[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 5th ed., 1980.

[11] D. R. Kaprekar, Problems involving reversal of digits, *Scripta Math.*, **19** (1953) 81–82.

[12] K. H. Rosen, *Discrete Mathematics and its Applications*, 7th Edition, McGraw-Hill, New York, 2012, 269–270.

[13] M. Schneider and R. Schneider, Digit sums and generating functions, *Ramanujan J.*, **52** no. 2 (2020) 291–302.

[14] J. O. Shallit, On infinite products associated with sums of digits, *J. Number Theory*, **21** no. 2 (1985) 128–134.

[15] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences (OEIS), `http://oeis.org`.

[16] A. J. Van der Poorten, An introduction to continued fractions, *Diophantine analysis (Kensington, 1985)*, London Math. Soc. Lecture Note Ser., 109, Cambridge Univ. Press, Cambridge, 1986 99–138.

[17] C. Vignat and T. Wakhare, Finite generating functions for the sum-of-digits sequence, *Ramanujan J.*, **50** no. 3 (2019) 639–684.

Ömer Eğecioğlu

Department of Computer Science, University of California Santa Barbara, CA 93106, USA

`Email: omer@cs.ucsb.edu`

BünyaminŞahin

Department of Mathematics, Faculty of Science, Selçuk University, Konya 42130, Turkey

`Email: bunyamin.sahin@selcuk.edu.tr`

http://dx.doi.org/10.22108/toc.2024.138412.2092