Artificial Intelligence

CS 165A

Oct 15, 2020

Instructor: Prof. Yu-Xiang Wang

Today

→ Probability notations

→ Counting number of parameters

→ Probabilistic modeling

→ Factorization and conditional independence

Recap: Last lecture

- Logistic loss and its gradient
- Stochastic gradient descent
- From linear logistic regression to neural networks
- Discriminative vs. Generative modelling

Plan for today

- Basics
  - Probability notations
  - Joint distributions, marginal, conditional
  - Representing these quantities as arrays / matrices
- Modeling:
  - Case study of “author classification”
- Conditional independences and factorization
- Introduction to BayesNet

- Tuesday next week:
  - BayesNet examples
  - d-separation, reasoning and inference, probabilistic modelling

Probability notation and notes

- Probabilities of *propositions / events*
  - P(A), P(the sun is shining)
- Probabilities of *random variables (r.v.)*
  - $P(X = x_1)$, $P(Y = y_1)$, $P(x_1 < X < x_2)$

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- P(A) usually means P(A = True) (A is a proposition, not a variable)
  - This is a probability **value**
  - Technically, P(A) is a probability *function*

P(¬A)

- $P(X = x_1)$
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- P(X)
  - This is a **probability distribution** function, a.k.a probability mass function (**p.m.f.**) for discrete r.v. or a probability density function (**p.d.f.**) for continuous r.v.

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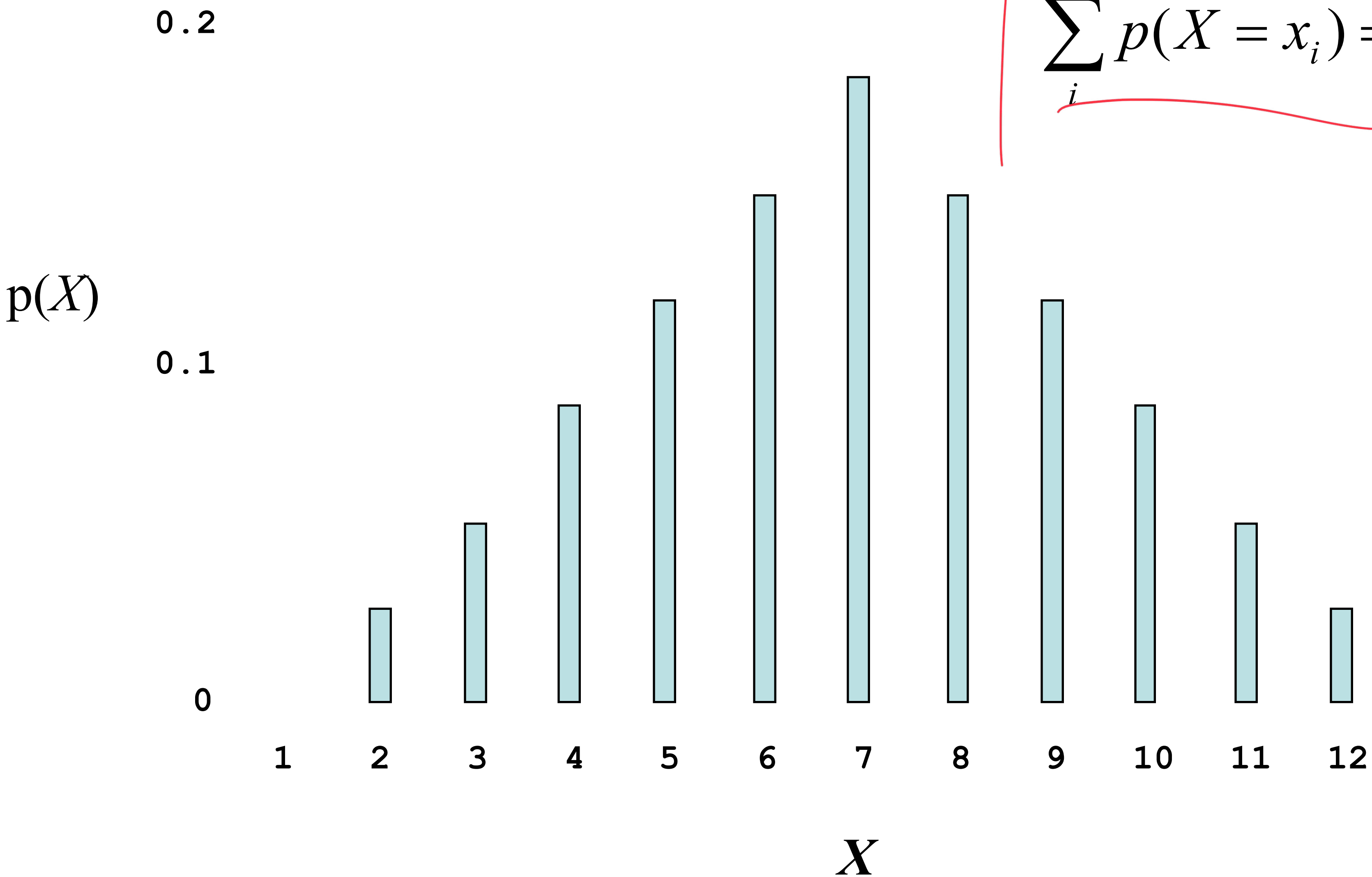
  - This is a **probability distribution** function, a.k.a probability mass function (**p.m.f.**) for discrete r.v. or a probability density function (**p.d.f.**) for continuous r.v.
- Technically, if X is an r.v., we should not write $P(X) = 0.5$
  - But rather $P(X = x_1) = 0.5$

$P(X) = [P(X=0), P(X=1)]$

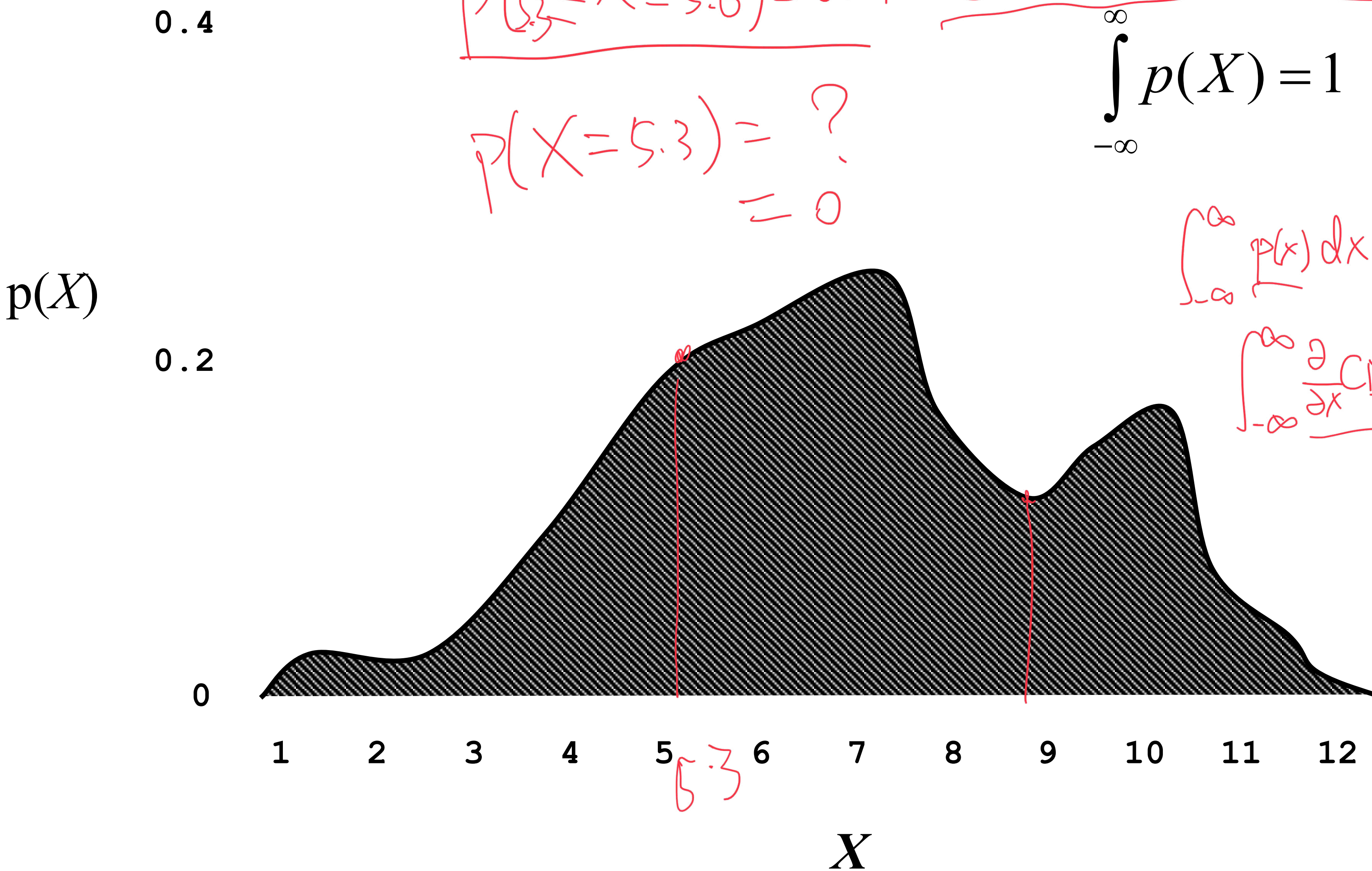
Discrete probability distribution

$$\sum_i p(X = x_i) = 1$$



Continuous probability distribution

$$\int_{-\infty}^{\infty} p(X) = 1$$



Joint Probabilities

- A **complete probability model** is a single joint probability distribution over all propositions/variables in the domain
  - $P(X_1, X_2, \ldots, X_i, \ldots)$
- A particular instance of the world has the probability
  - $P(X_1{=}x_1 \wedge X_2{=}x_2 \wedge \ \ldots \wedge X_i{=}x_i \wedge \ \ldots) = p$

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  - P(Raining, WetGrass) = 0.15
  - P(Raining, ¬WetGrass) = 0.01
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	¬WetGrass	WetGrass
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Raining	0.01	0.15

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Marginal and Conditional Probability

- Marginal Probability
  - Marginal probability (distribution) of X: $P(X) = \sum_Y P(X, Y)$
  - **Bayesian interpretation:** Probabilities associated with one proposition or variable, **prior** to any evidence
  - E.g., P(WetGrass), P(¬Raining)
- Conditional Probability
  - P(A | B) – "The probability of A given that we know B"
  - **Bayesian interpretation:** After (**posterior** to) procuring evidence
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$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)} \quad \text{or} \quad P(X \mid Y)\, P(Y) = P(X, Y)$$

Assumes P(Y) nonzero

The chain rule: factorizing a joint distribution into marginal and conditionals

$$P(X,Y) = P(X \mid Y)\, P(Y)$$

By the Chain Rule

$$P(X,Y,Z) = P(X \mid Y,Z)\, P(Y,Z)$$

= P(Y|Z) P(Z)

= P(Z|Y) P(Y)

$$= P(X \mid Y,Z)\, P(Y \mid Z)\, P(Z)$$

or, equivalently

$$= P(X) P(Y \mid X) P(Z \mid X,Y)$$

Notes:

- Precedence: '|' is lowest
- E.g., P(X | Y, Z) means which?

  P((X | Y), Z)

  P(X | (Y, Z))

Chain Rule implies Bayes' Rule

Thomas Bayes: 1701 - 1761

- Since $P(X,Y) = P(X \mid Y)\, P(Y)$

  and $P(X,Y) = P(Y \mid X)\, P(X)$

- Then $P(X \mid Y)\, P(Y) = P(Y \mid X)\, P(X)$

Posterior of X|Y; Conditional of Evidence; Prior

$$P(X \mid Y) = \frac{P(Y \mid X)\, P(X)}{P(Y)}$$

$$P(Y) = \sum_{X} P(Y \mid X) \cdot P(X)$$

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  and $P(X,Y) = P(Y \mid X)\, P(X)$

- Then $P(X \mid Y)\, P(Y) = P(Y \mid X)\, P(X)$

$$P(X \mid Y) = \frac{P(Y \mid X)\ P(X)}{P(Y)}$$ Bayes' Rule

Funny fact: Thomas Bayes is arguably a frequentist.

Stephen Fienberg. "When did Bayesian inference become 'Bayesian'?." *Bayesian analysis* 1.1 (2006): 1-40.
https://projecteuclid.org/euclid.ba/1340371071

Representing Probability Distributions using linear algebraic data structures (in python)

	Continuous vars	Discrete vars
P(X)	Function (of one variable)	m vector
P(X=x)	Scalar*	Scalar
P(X,Y)	Function of two variables	m×n matrix
P(X|Y)	Function of two variables	m×n matrix
P(X|Y=y)	Function of one variable	m vector
P(X=x|Y)	Function of one variable	n vector
P(X=x|Y=y)	Scalar*	Scalar

* - actually zero. Should be $P(x_1 < X < x_2)$

Example: Joint probability distribution

From $P(X,Y)$, we can always calculate:

$P(X)$	$P(X=x_1)$
$P(Y)$	$P(Y=y_2)$
$P(X\|Y)$	$P(X\|Y=y_1)$
$P(Y\|X)$	$P(Y\|X=x_1)$
	$P(X=x_1\|Y)$
	etc.

Example: Joint probability distribution

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$P(Y)$ $P(Y=y_2)$

$P(X|Y)$ $P(X|Y=y_1)$

$P(Y|X)$ $P(Y|X=x_1)$

$P(X=x_1|Y)$

etc.

X

Y		x_1	x_2	x_3
	y_1	0.2	0.1	0.1
	y_2	0.1	0.2	0.3

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X,Y)

	x_1	x_2	x_3
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P(X)

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

x_1	x_2	x_3
0.3	0.3	0.4

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

x_1	x_2	x_3
0.3	0.3	0.4

P(Y)

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

x_1	x_2	x_3
0.3	0.3	0.4

P(Y)

y_1	0.4
y_2	0.6

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

x_1	x_2	x_3
0.3	0.3	0.4

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P(X|Y)

P(X,Y)

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y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

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0.3	0.3	0.4

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y_1	0.4
y_2	0.6

P(X|Y)

$$\frac{0.2}{0.2+0.1+0.1} =$$

	x_1	x_2	x_3
y_1	0.5	0.25	0.25
y_2	0.167	0.333	0.5

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

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y_2	0.167	0.333	0.5

P(Y|X)

	x_1	x_2	x_3
y_1	0.667	0.333	0.25
y_2	0.333	0.667	0.75

P(X,Y)

	x_1	x_2	x_3
y_1	0.2	0.1	0.1
y_2	0.1	0.2	0.3

P(X)

x_1	x_2	x_3
0.3	0.3	0.4

P(Y)

y_1	0.4
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$P(X=x_1, Y=y_2) = ?$

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y_2	0.333	0.667	0.75

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Quick checkpoint

- Probability notations
  - P(A) is a number when A is an event / predicate.
  - P(X) is a vector/function when X is a random variable.

- Joint probability distribution
  - Enumerating all combinations of events.
  - All values the random variables can take.
  - Assign a non-negative value to each.

- Marginals, conditionals
  - How they are related: Chain rule, Bayes rule

You should know HOW TO do the following:

- For discrete probability distributions for multiple random variables
  - Know **the number of possible values** these RVs can take
  - Know the **shape of the numpy arrays** that you need to represent Joint-distribution,   conditional distribution
  - Know the **number of independent parameters** you need to specify these distributions. (we often need to -1 here or there. Why is that?)
- More generally: Know the distinctions between
  - p.m.f -- probability mass function (for discrete distribution)
  - p.d.f. -- probability density function (for continuous distribution)
  - CDF -- cumulative distribution function (for both)

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  - Handles missing data/variables
  - Easy to incorporate prior knowledge

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  - Design classifiers

$$\hat{y} = \text{arg}max_y P(Y = y \,|X)$$

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    - Sherlock Holmes: P(Murderer | Observed Evidence)

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  - We can make Bayesian inference (probabilistic reasoning)
    - Sherlock Holmes: P(Murderer | Observed Evidence)
    - Doctor: P(Disease | Symptoms) , P(Effect | Treatment)
    - Parenting:
      - P(Dirty Diaper, Hungry, Lonely | 5 a.m., Baby crying)
      - P(Baby crying at 5 a.m. | feeding at 2 a.m.)
      - P(Baby crying at 5 a.m. | feeding at 1 a.m.)

(3 min discussion) Modeling the world with probability distribution

- Example: Author attribution task as in HW1
  - Variables: *Word 1, Word 2, Word 3, …, Word N, Author*
  - 15 authors in total: {Dickens, Shakespeare, Kafka, Jane Austen, Tolkien, George RR. Martin, … , Xueqin Cao, Douglas Adams}
  - A vocabulary of size 3000

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$P(W_1, \dots W_N, Author)$

3000×15 ?

$W_1 \in \{w_1, \dots w_{3000}\}$

- Questions:
  - What is the dimension(s) of the joint distribution?
  - How many free parameters are needed to represent this distribution?

$3000^N \cdot 15$

$3000^N \cdot 15 - 1$

Statistical Independences

(Marginal / absolute) Independence

$P(X,Y) = P(X)\,P(Y|X)$

- X and Y are independent iff
  - $P(X,Y) = P(X)\,P(Y)$ [by definition]
  - $P(X \mid Y) = P(X)$ Since $P(X \mid Y) = P(X, Y)/P(Y) = P(X)\,P(Y)/P(Y)$

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Conditional Independence

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  - $P(X \mid Y, Z) = P(X \mid Z)$

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Conditional Independence

- If X and Y are (conditionally) independent given Z, then
  - $P(X \mid Y, Z) = P(X \mid Z)$
  - Example:
    - P(WetGrass | Season, Rain) = P(WetGrass | Rain)

Example of Conditional Independence

- In practice, conditional independence is more common than marginal independence.
  - P(Final exam grade | Weather) ≠ P(Final exam grade)
    - i.e., they are not independent
  - P(Final exam grade | Weather, Effort) = P(Final exam grade | Effort)
    - But they are conditionally independent given Effort

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  - 15 authors in total: {Dickens, Shakespeare, Tolkien, George RR. Martin, … ,Douglas Adams}
  - A vocabulary of size 3000

- In addition, assume that: *Word 1, …, Word N* are **mutually independent** given *Author*
  - $P(Word\ 1, \ldots, Word\ N \mid Author) = P(Word\ 1 \mid Author) \times \ldots \times P(Word\ N \mid Author)$

(Example continued) Modeling the world with probability distribution

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  - 15 authors in total: {Dickens, Shakespeare, Tolkien, George RR. Martin, … ,Douglas Adams}
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- In addition, assume that: *Word 1, …, Word N* are **mutually independent** given *Author*
  - $P(\text{Word } 1, \ldots, \text{Word } N \mid \text{Author}) = P(\text{Word } 1 \mid \text{Author}) \times \ldots \times P(\text{Word } N \mid \text{Author})$
- Question:
  - What are the dimensions of each factor?
  - How many “free parameters” are needed in total?

(Example continued) Modeling the world with probability distribution

- Example: Author attribution as in HW1
  - Variables: *Word 1, Word 2, Word 3, …, Word N, Author*
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Fully general

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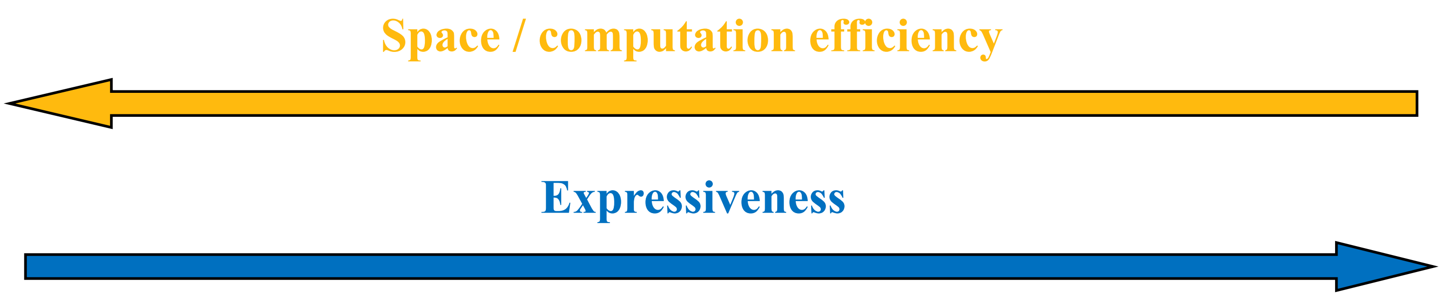

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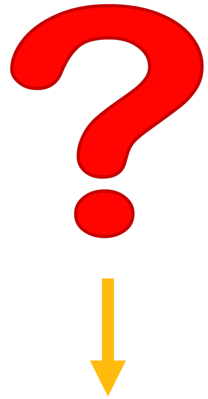

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Space / computation efficiency ←

Expressiveness →

Idea:

1. Independent groups of variables?
2. **Conditional independences?**

Benefit of conditional independence

$(X \perp Y \mid Z) \Leftrightarrow P(X \mid Y, Z) = P(X \mid Z)$

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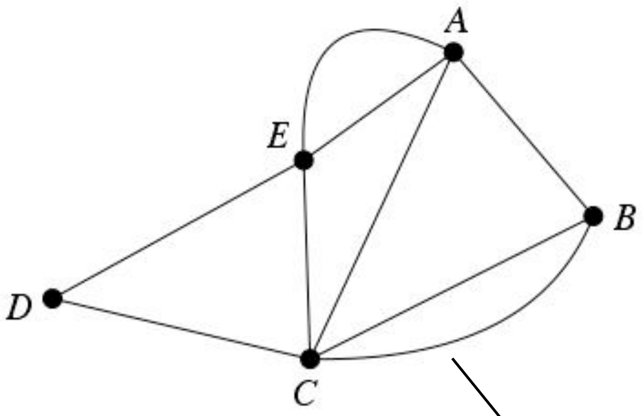
- This is often the case in real problems.

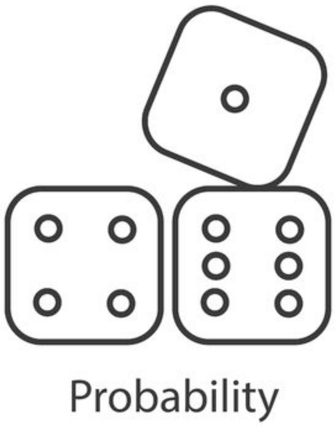
When given a problem with many variables.

[CS165A Lecture attendance,
HW1,HW2, HW3, HW4,
Readings, Piazza, Final Grade,
Weather, Election Result, Job Offer]

How do we know **which conditional independence**(s) to include in the joint distribution?

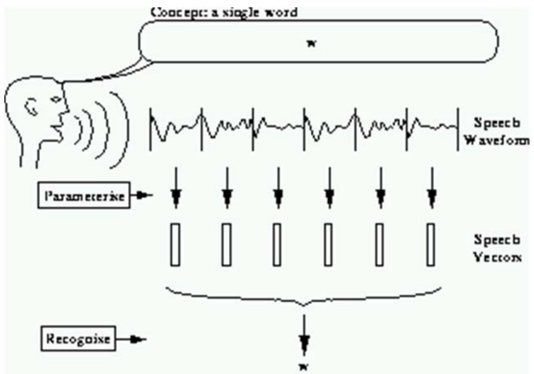
Graphical models come out of the marriage of graph theory and probability theory



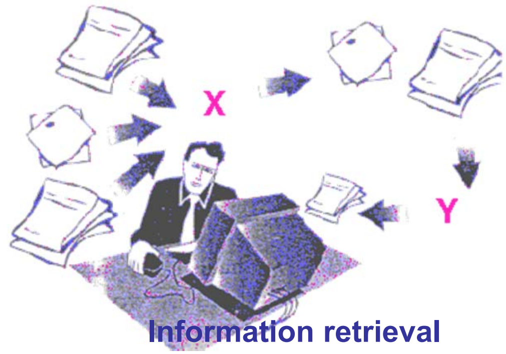


Directed Graph => Bayesian Networks / Belief Networks
Undirected Graph => Markov Random Fields

Used as a modeling tool. Many applications!

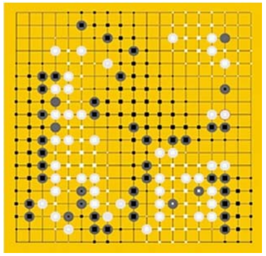


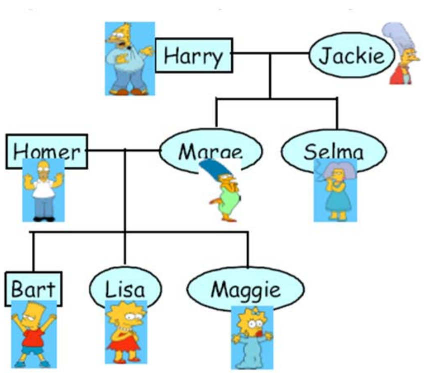
Speech recognition



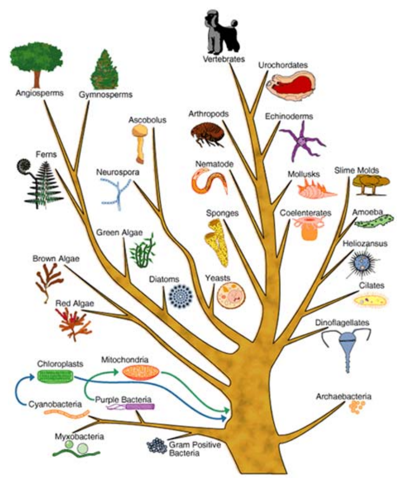
Information retrieval


Computer vision


Games

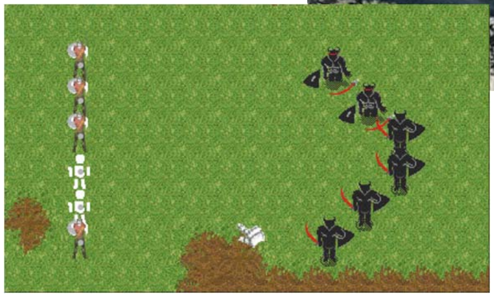


Pedigree



Evolution


Robotic control


Planning

(Slides from Prof. Eric Xing)

Two ways to think about Graphical Models

- A particular factorization of a joint distribution
  - $P(X,Y,Z) = P(X)\ P(Y|X)\ P(Z|Y)$
- A collection of conditional independences
  - $\{ X \perp Z \mid Y, \dots \}$

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Represented using a graph!

Belief Networks

a.k.a. Probabilistic networks, Belief nets, Bayes nets, etc.

- Belief network
  - A data structure (depicted as a graph) that represents the dependence among variables and allows us to concisely specify the joint probability distribution
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- A belief network is a **directed acyclic graph** where:
  - The nodes represent the set of random variables (one node per random variable)
  - Arcs between nodes represent *influence*, or *causality*
    - A link from node X to node Y means that X "directly influences" Y
  - Each node has a ***conditional probability table*** (CPT) that defines **P(node | parents)**

Example

- Random variables X and Y
  - X – It is raining
  - Y – The grass is wet
- X has an *effect* on Y

  Or, Y is a *symptom* of X
- Draw two nodes and link them

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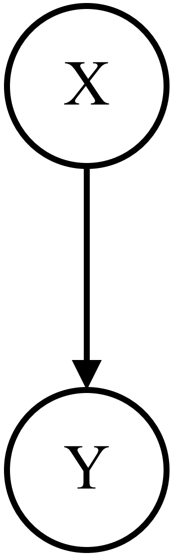

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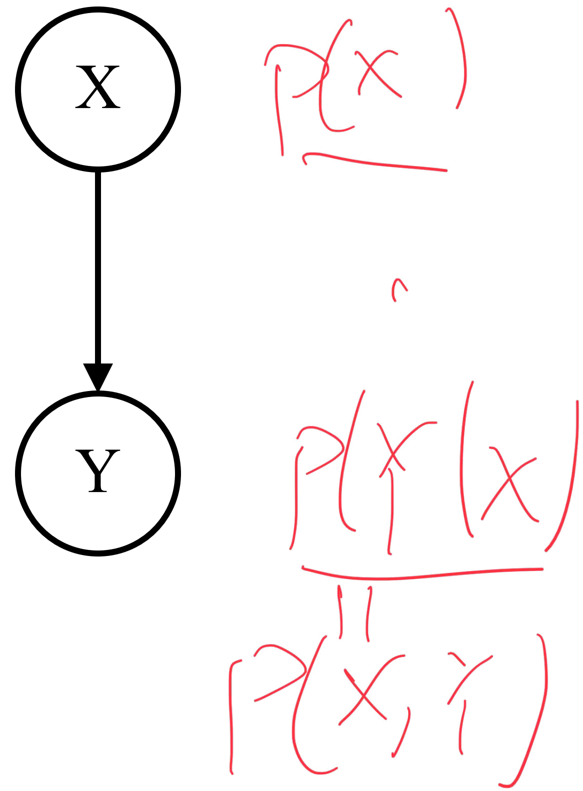

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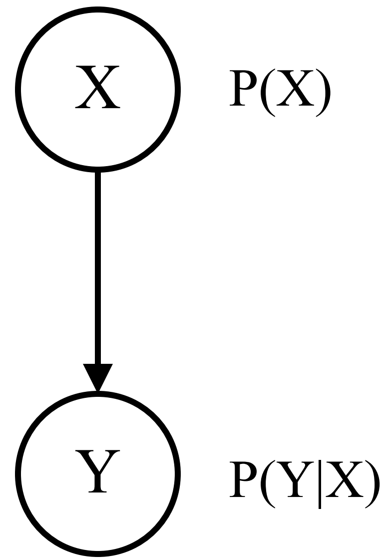

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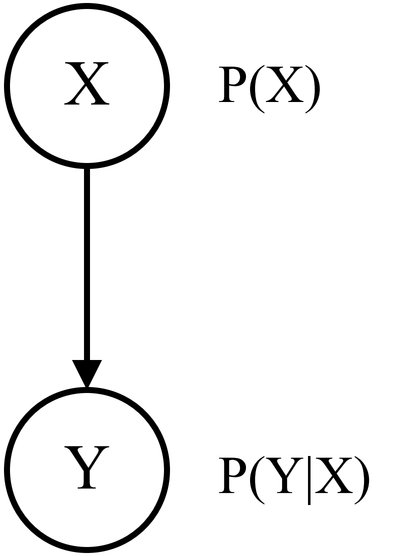
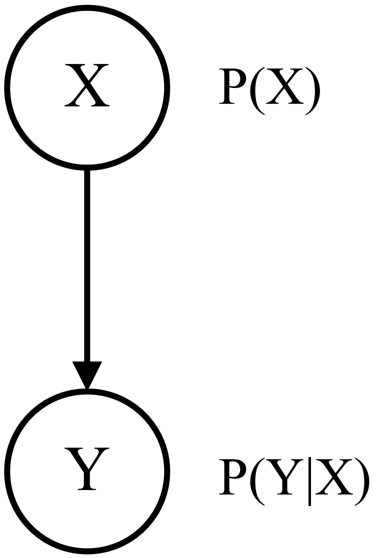


- Define the CPT for each node
  - P(X) and P(Y | X)
- Typical use: we observe Y and we want to query P(X | Y)
  - Y is an *evidence variable*
  - X is a *query variable*

We can write everything we want as a function of the CPTs. Try it!

- What is P(X | Y)?
  - Given that we know the CPTs of each node in the graph



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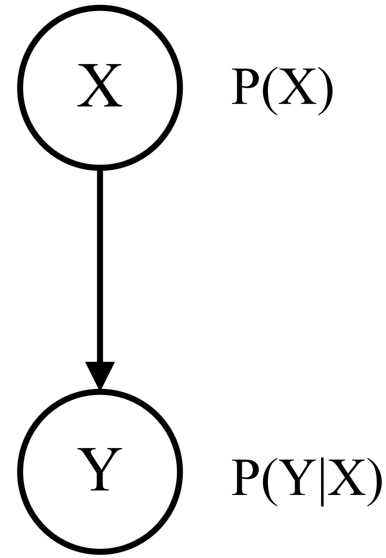

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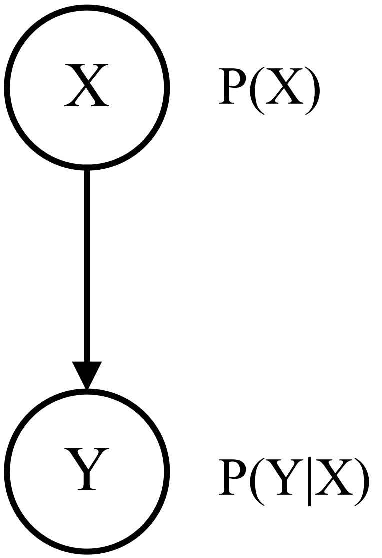

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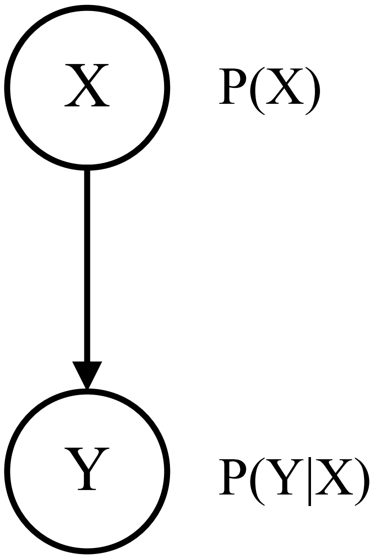

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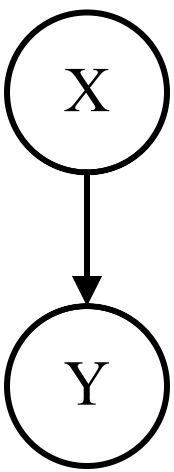

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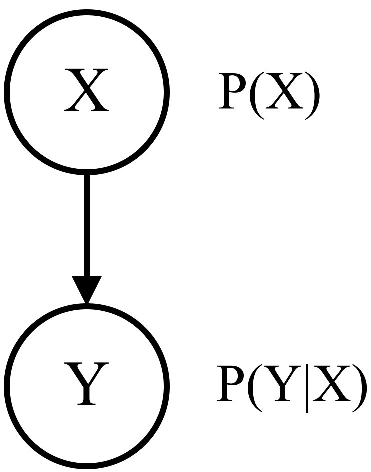

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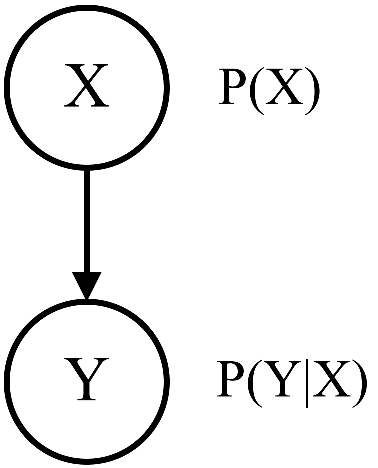

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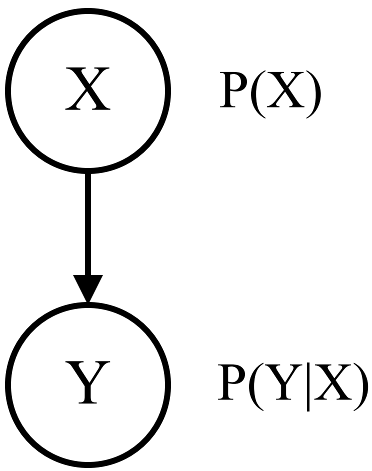

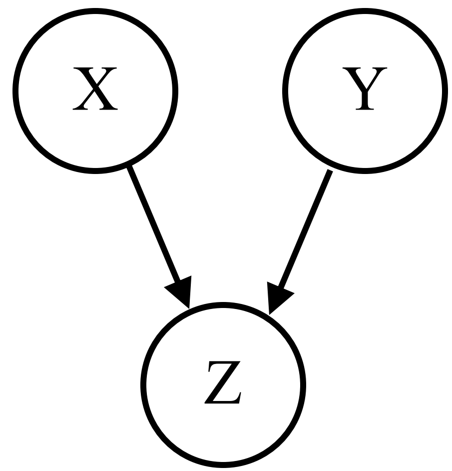

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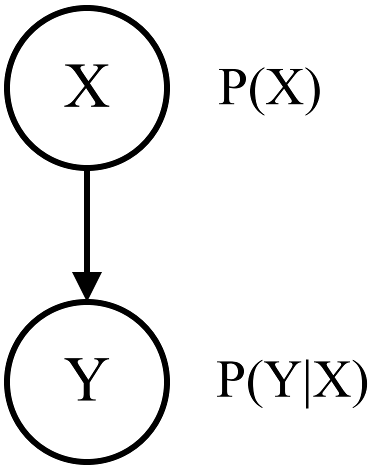

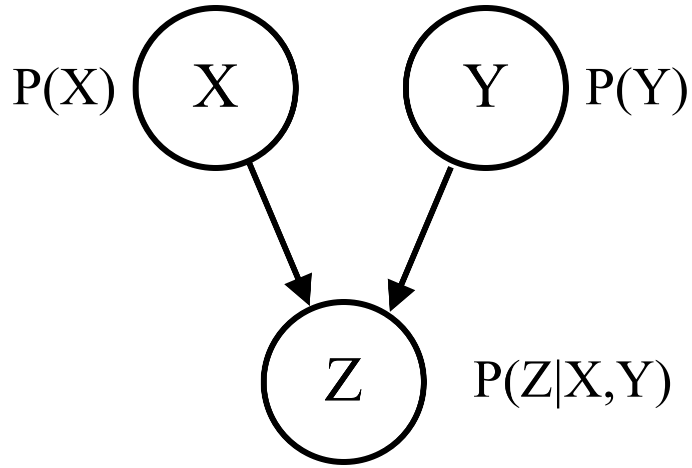

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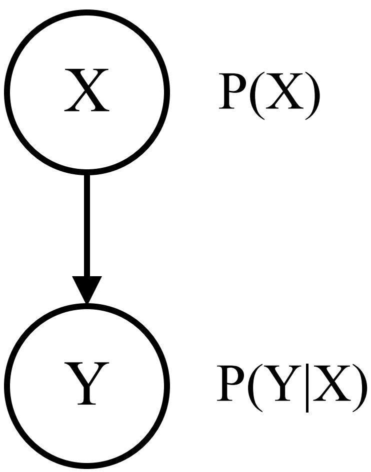

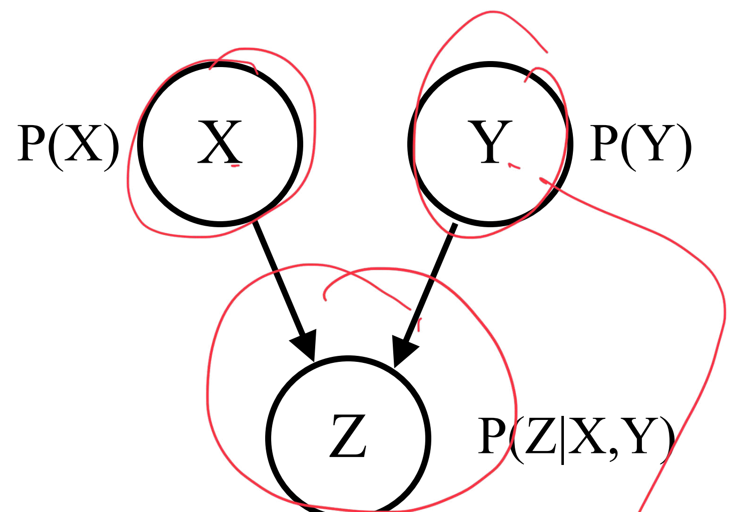

$P(X,Y) = P(X)\, P(Y|X)$

$P(X,Y,Z) = P(X)\, P(Y)\, P(Z|X,Y)$

Three steps in modelling with Belief Networks

1. Choose variables in the environments, represent them as nodes.

2. Connect the variables by inspecting the "direct influence": cause-effect

3. Fill in the probabilities in the CPTs.

Example: Modelling with Belief Net

I'm at work and my neighbor John called to say my home alarm is ringing, but my neighbor Mary didn't call. The alarm is sometimes triggered by minor earthquakes. Was there a burglar at my house?

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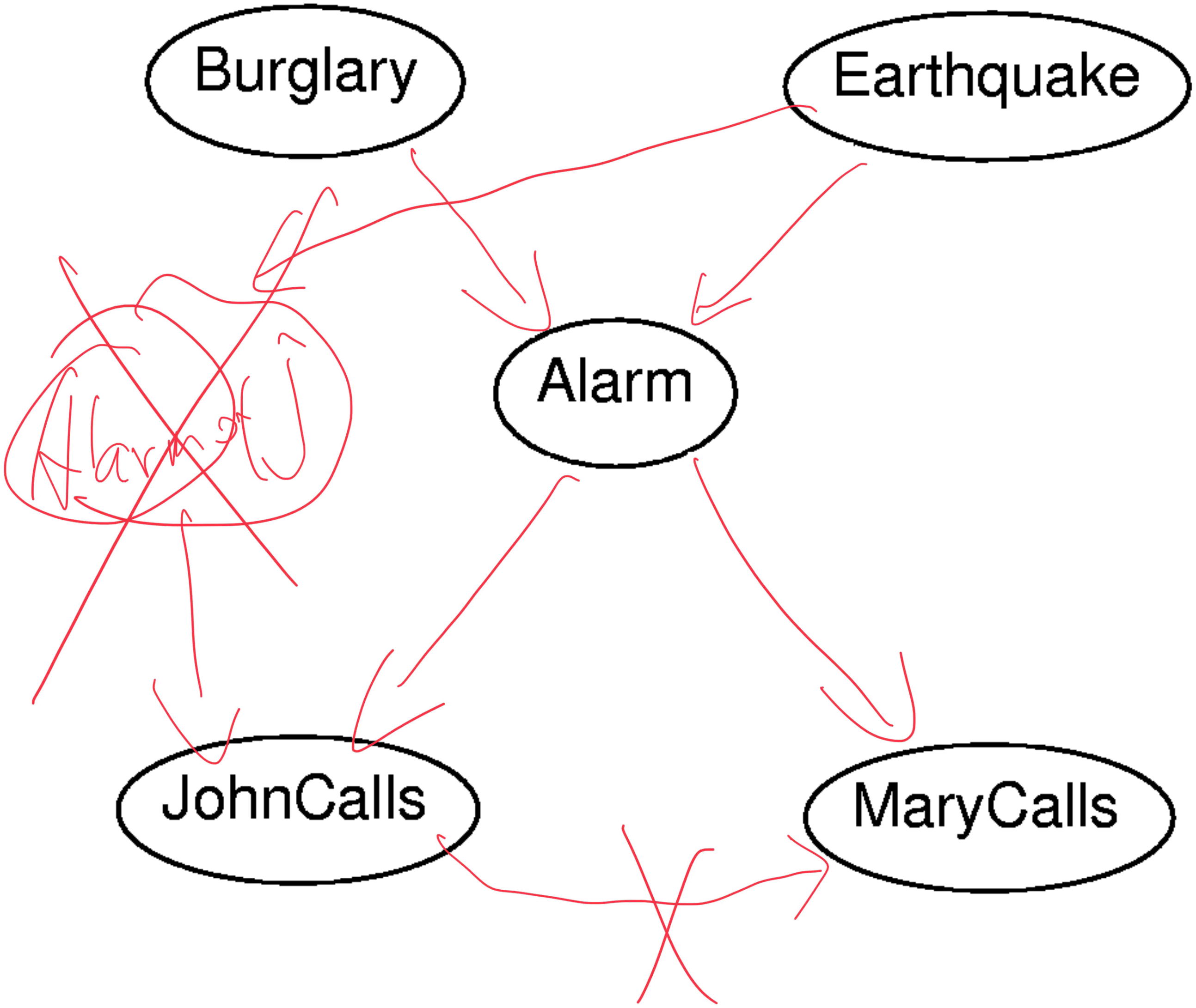
$P(B \mid J, \neg M)$

How should we connect the nodes? (3 min discussion)



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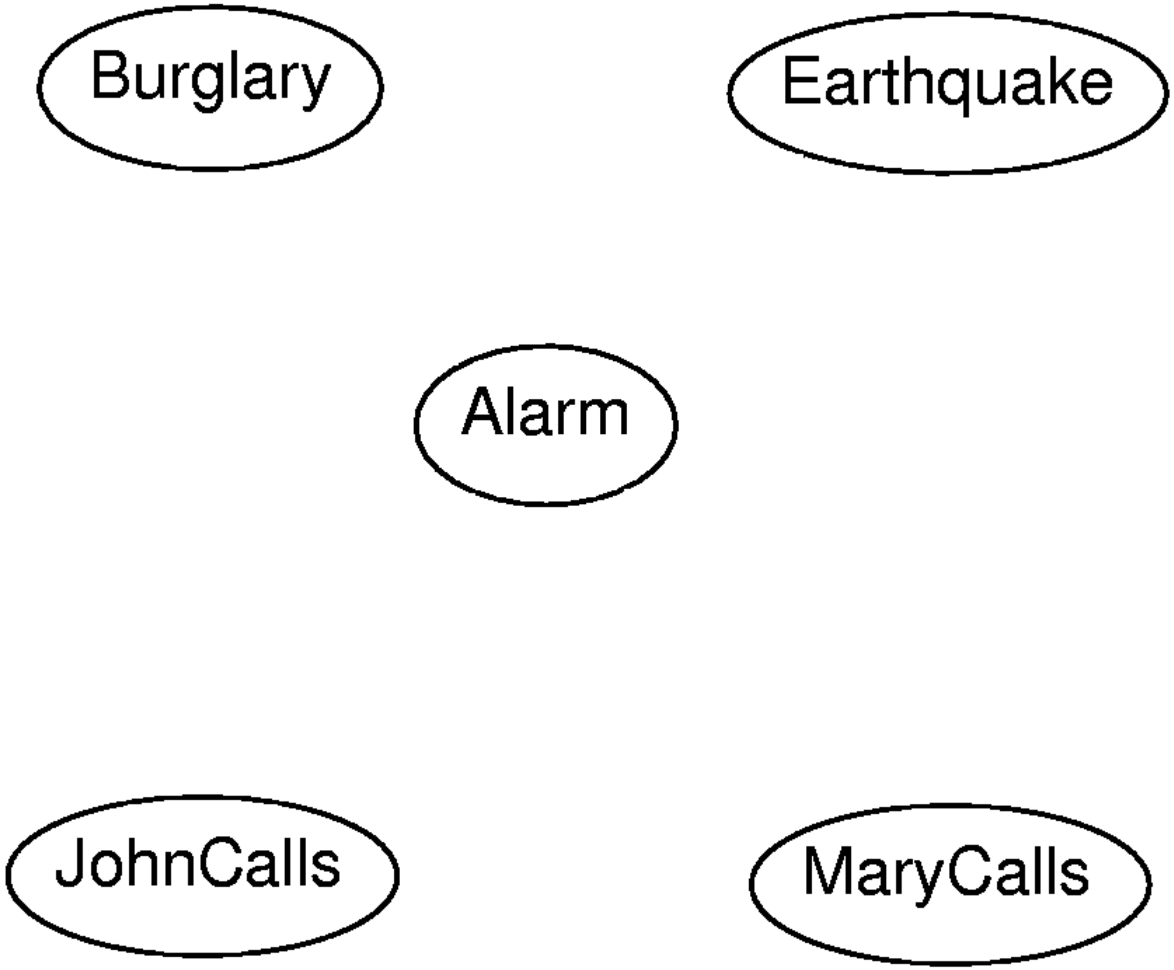
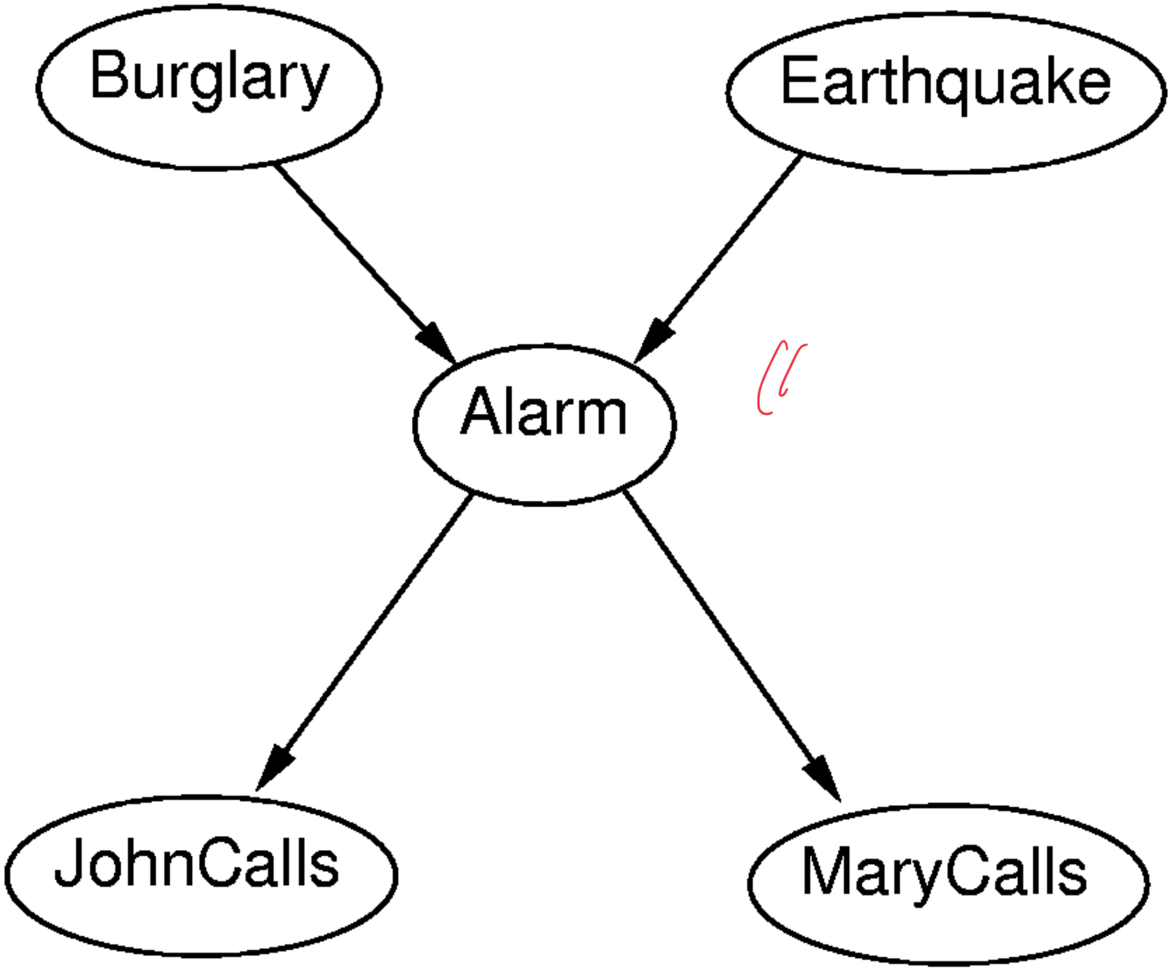


Links and CPTs?

What are the CPTs? What are their dimensions?



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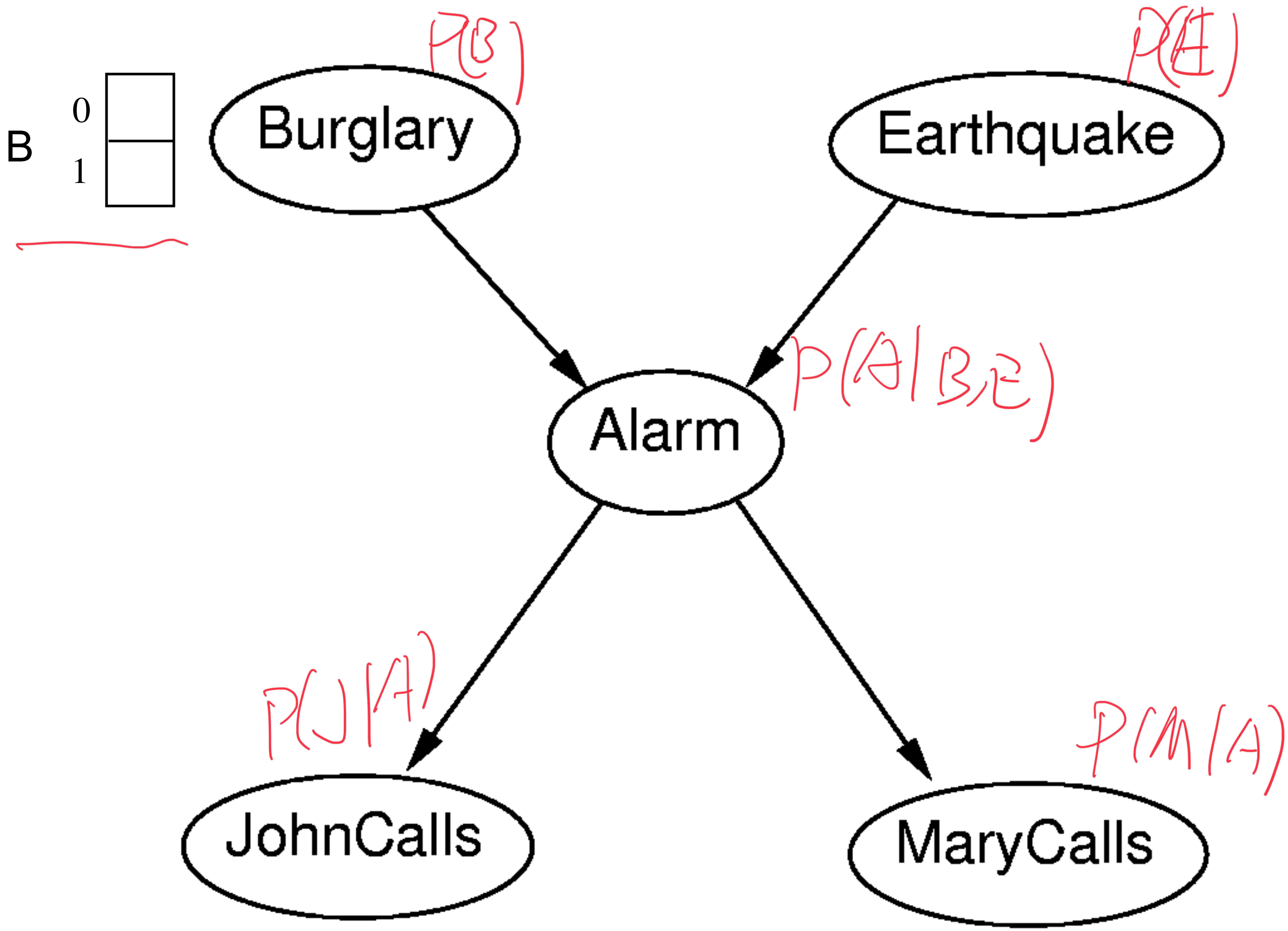

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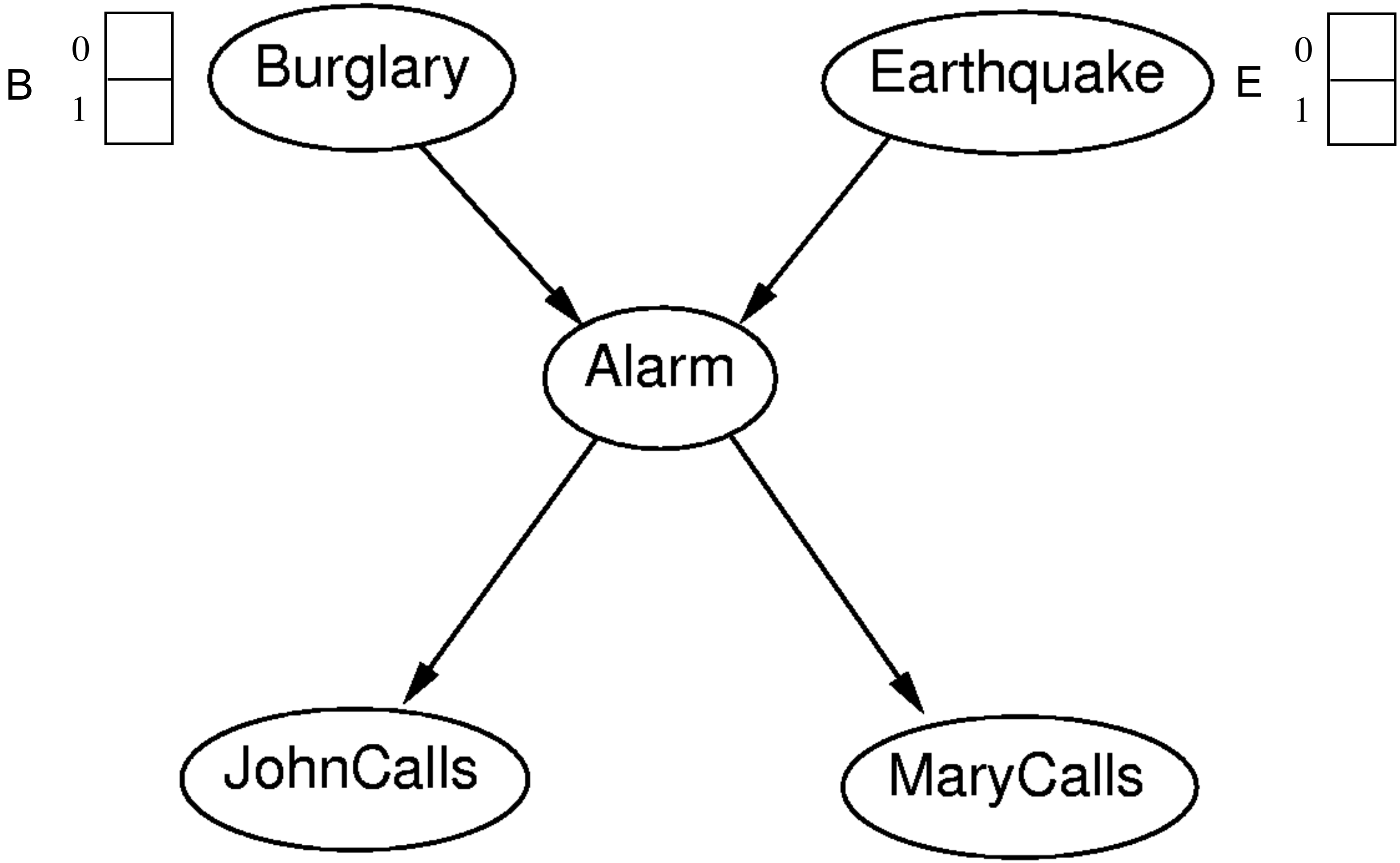

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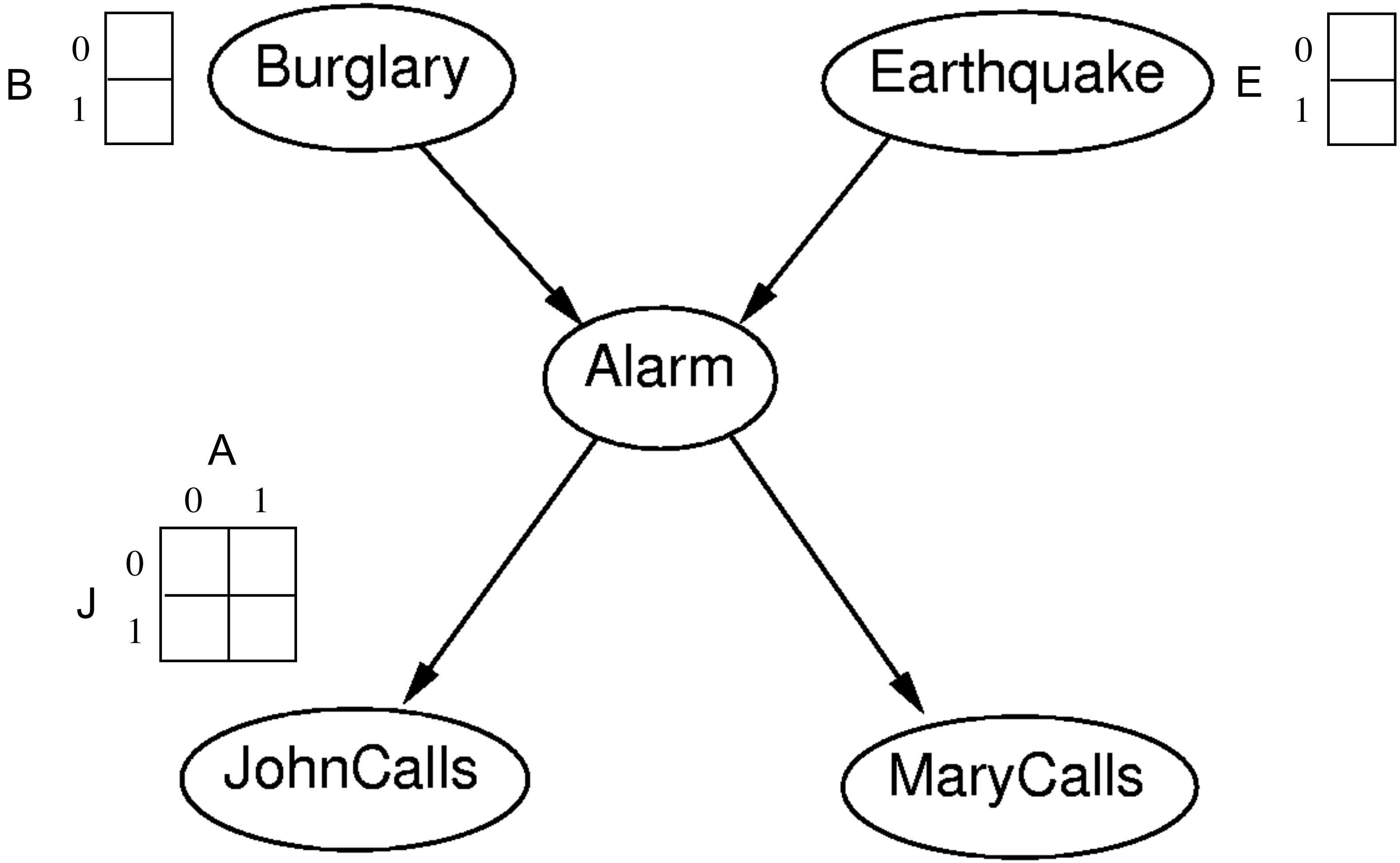

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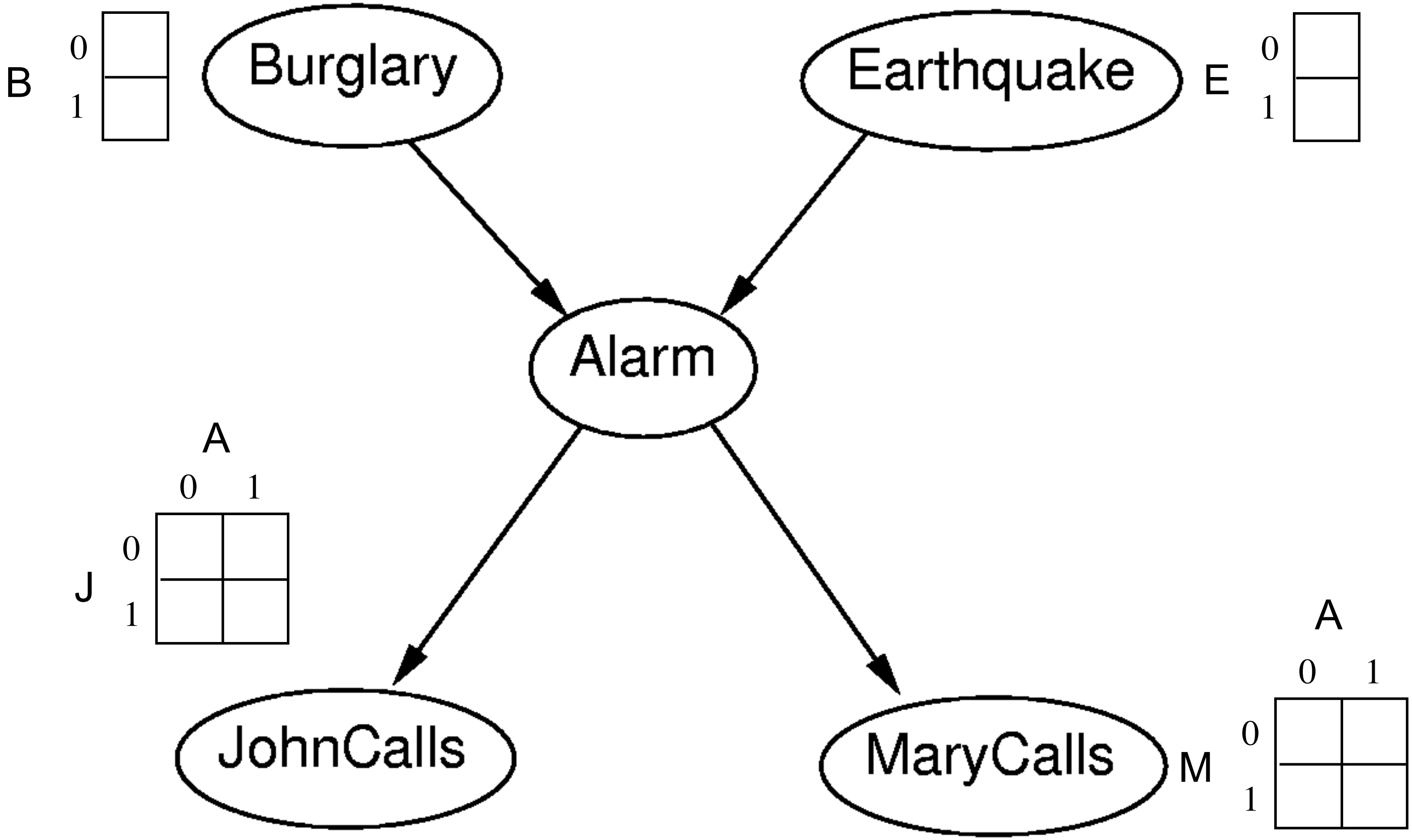

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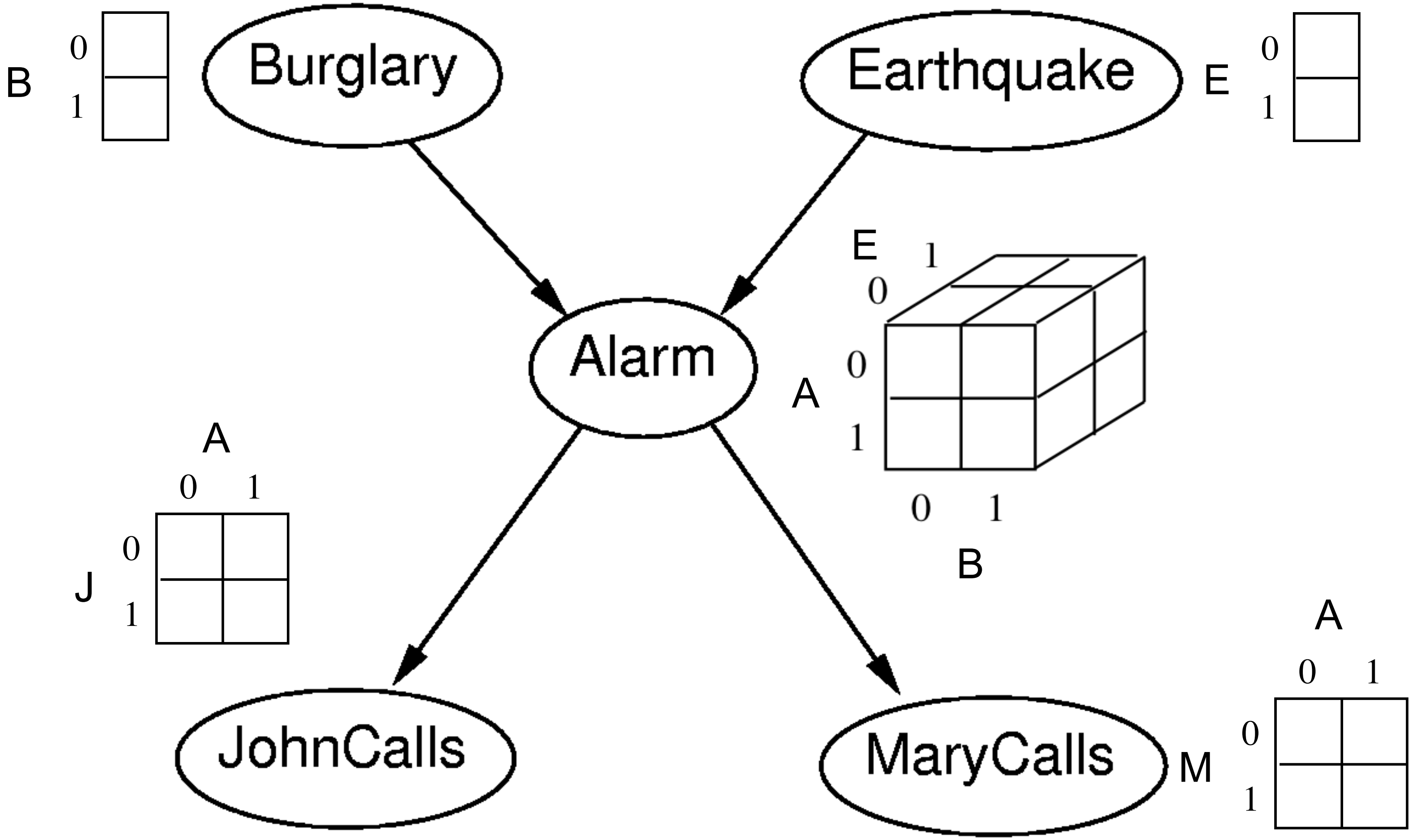

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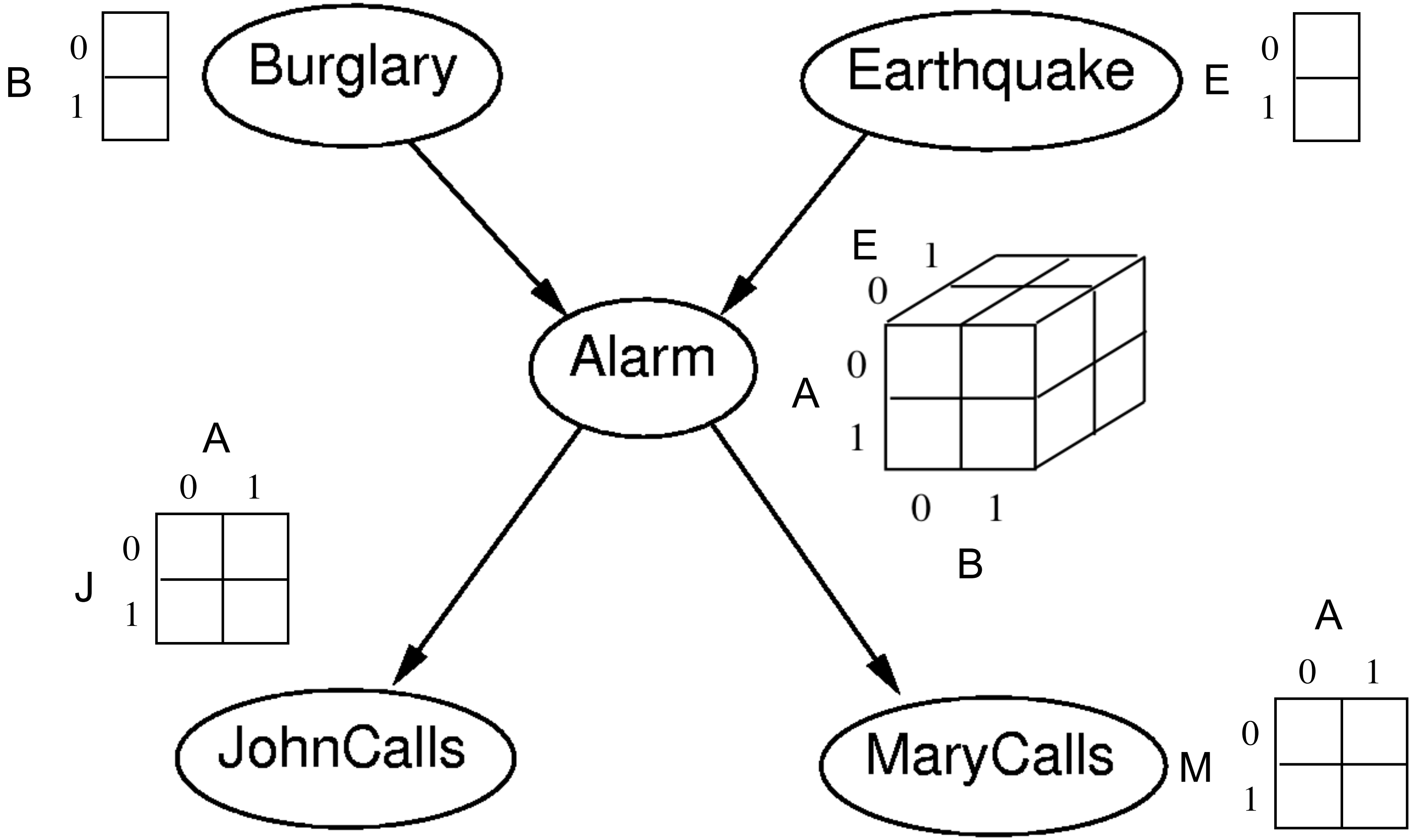

Question: How to fill values into these CPTs?

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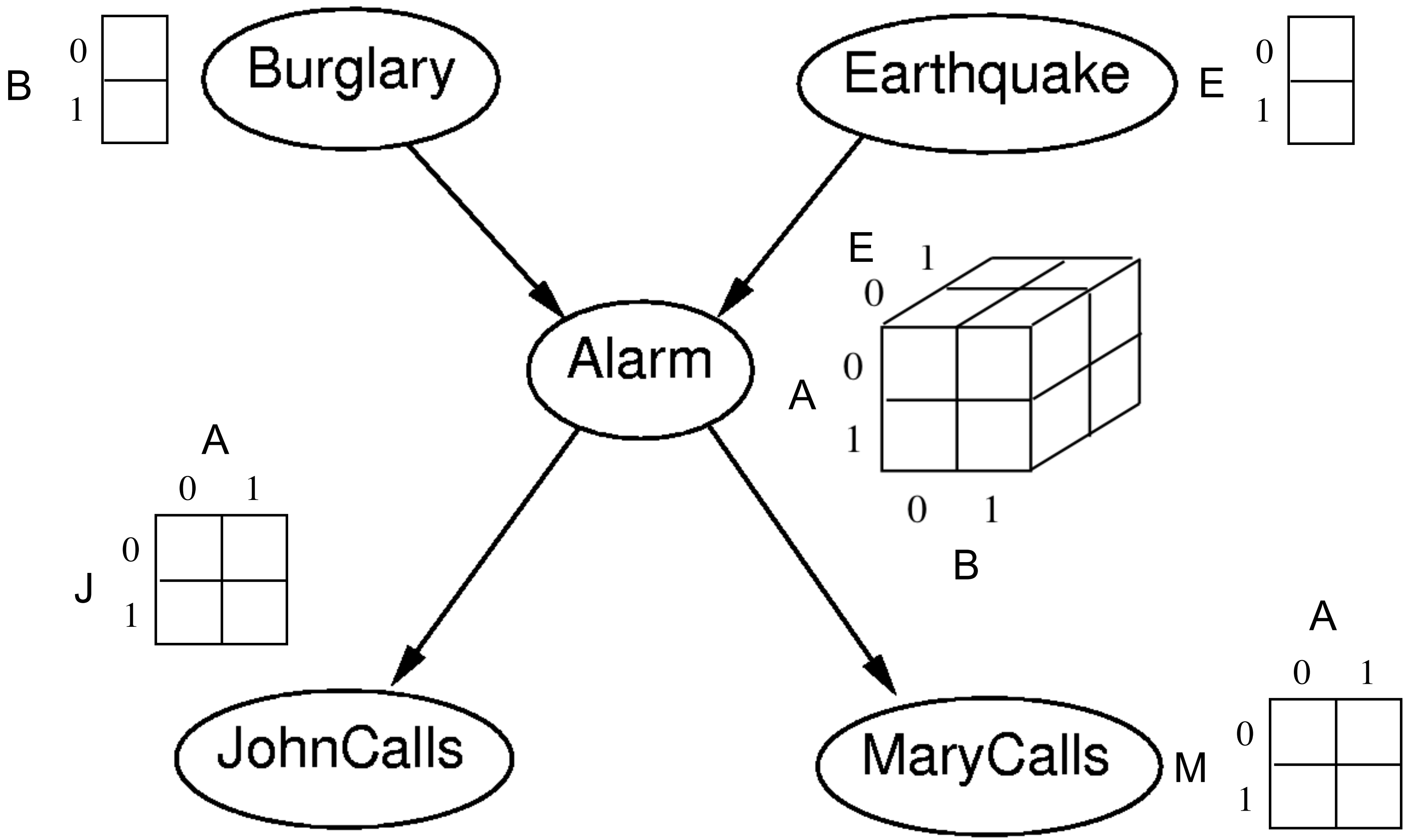
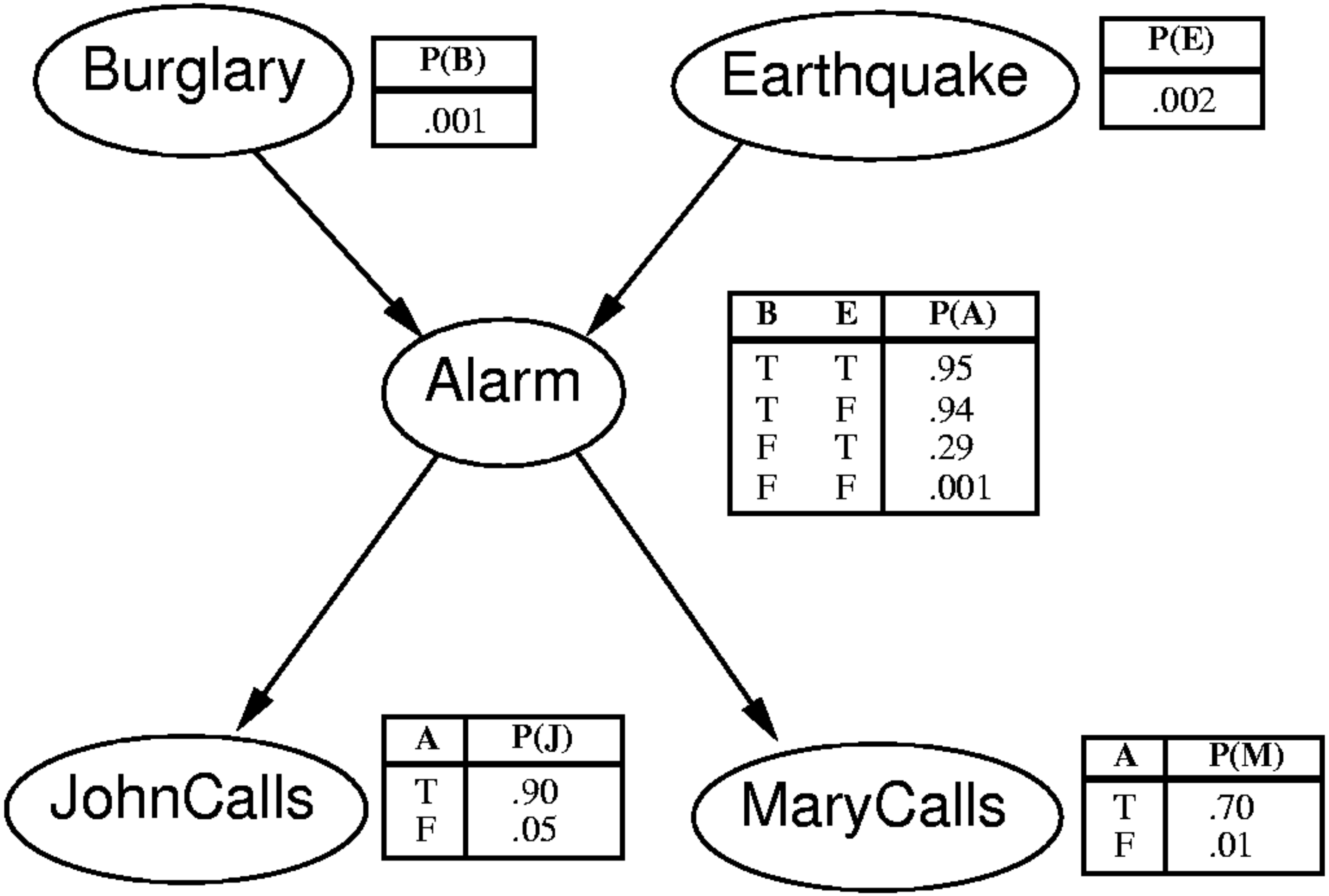


Question: How to fill values into these CPTs?

Ans: Specify by hands. Learn from data (e.g., MLE).

Example



P(B)
.001

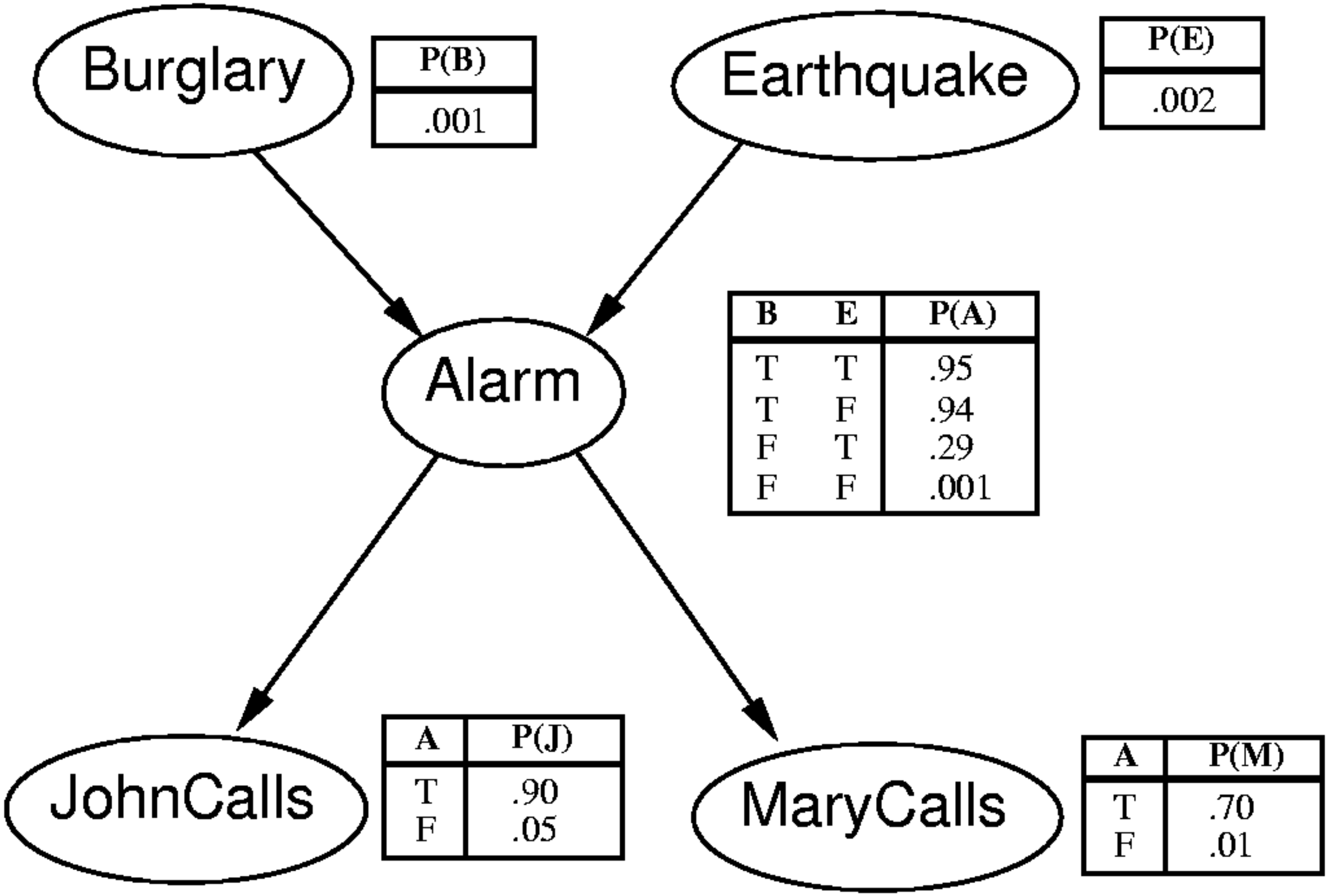
P(E)
.002

B	E	P(A)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J)
T	.90
F	.05

A	P(M)
T	.70
F	.01

Example



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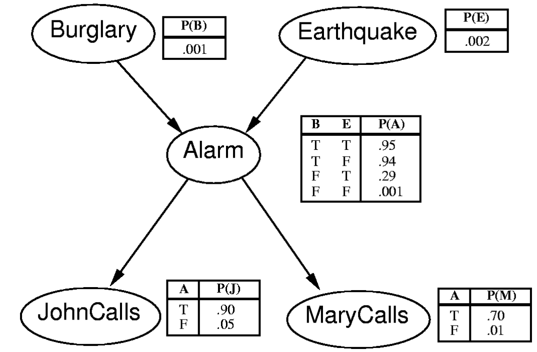
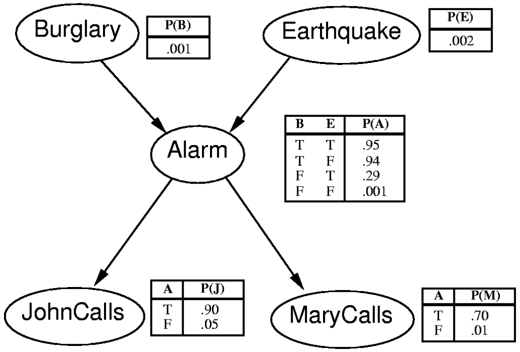
Joint probability? P(J, ¬M, A, B, ¬E)?



Calculate $P(J, \neg M, A, B, \neg E)$

Read the joint pf from the graph:

$$P(J, M, A, B, E) = P(B)\, P(E)\, P(A|B,E)\, P(J|A)\, P(M|A)$$



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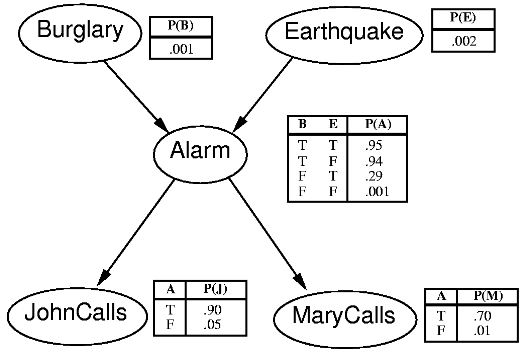

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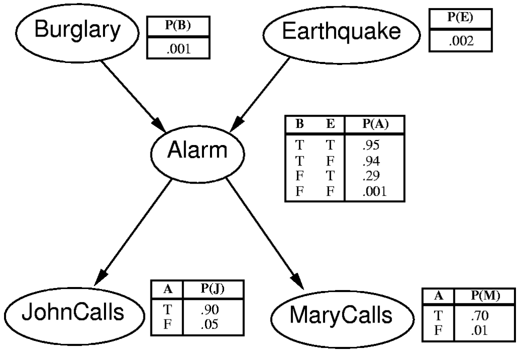
Calculate P(J, ¬M, A, B, ¬E)



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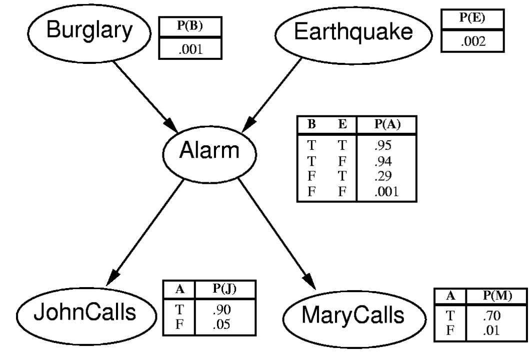
P(J, ¬M, A, B, ¬E) = P(B) P(¬E) P(A|B,¬E) P(J|A) P(¬M|A)

= 0.001 * 0.998 * 0.94 * 0.9 * 0.3



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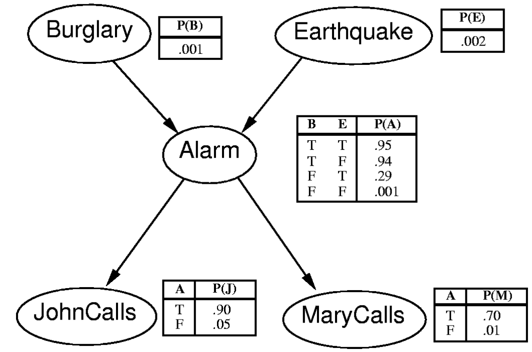

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How about P(B | J, ¬M) ?

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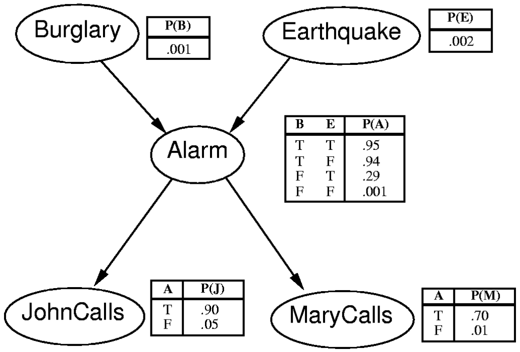

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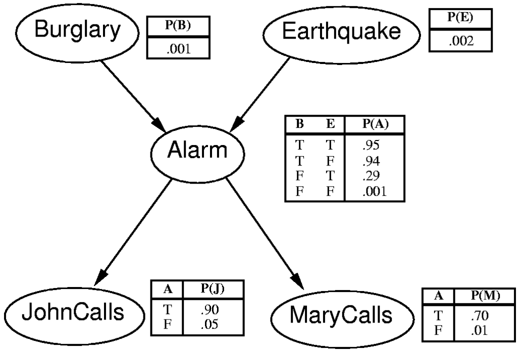
How about P(B | J, ¬M) ?

Remember, this means P(B=true | J=true, M=false)

Calculate P(B | J, ¬M)

$$P(B \mid J, \neg M) = \frac{P(B, J, \neg M)}{P(J, \neg M)}$$



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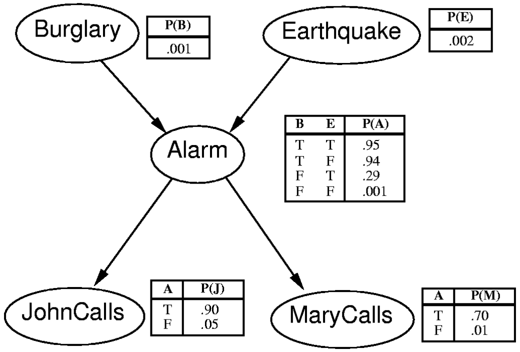

$$P(B \mid J, \neg M) = \frac{P(B, J, \neg M)}{P(J, \neg M)}$$

By marginalization:

$$= \frac{\sum_i \sum_j P(J, \neg M, A_i, B, E_j)}{\sum_i \sum_j \sum_k P(J, \neg M, A_i, B_j, E_k)}$$

$$= \frac{\sum_i \sum_j P(B)P(E_j)P(A_i \mid B, E_j)P(J \mid A_i)P(\neg M \mid A_i)}{\sum_i \sum_j \sum_k P(B_j)P(E_k)P(A_i \mid B_j, E_k)P(J \mid A_i)P(\neg M \mid A_i)}$$

Quick checkpoint

- Belief Net as a modelling tool
- By inspecting the cause-effect relationships, we can draw directed edges based on our domain knowledge
- The product of the CPTs give the joint distribution
  - We can calculate P(A | B) for any A and B
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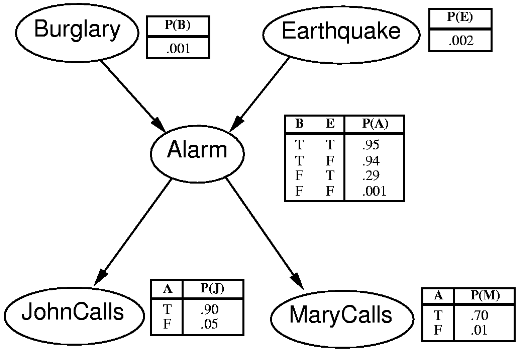
What else can we get?

Example: Conditional Independence



- Conditional independence is seen here
  - P(JohnCalls | MaryCalls, Alarm, Earthquake, Burglary) = P(JohnCalls | Alarm)
  - So JohnCalls is independent of MaryCalls, Earthquake, and Burglary, given Alarm

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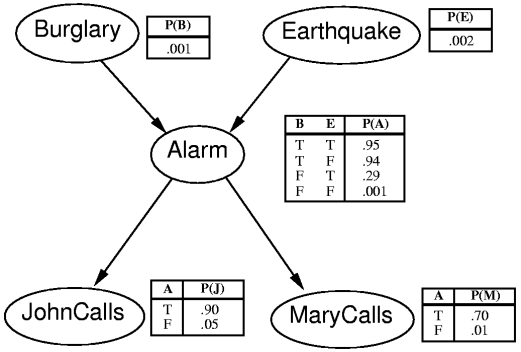

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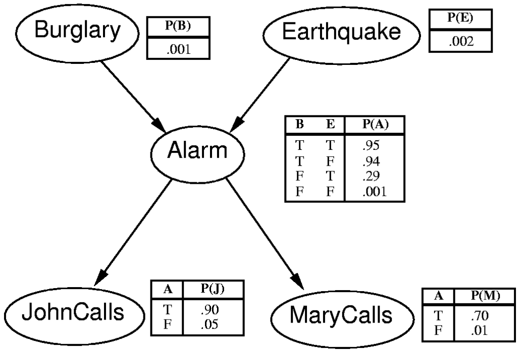

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  - JohnCalls is <u>conditionally</u> independent of Earthquake, but not <u>absolutely</u> independent of it

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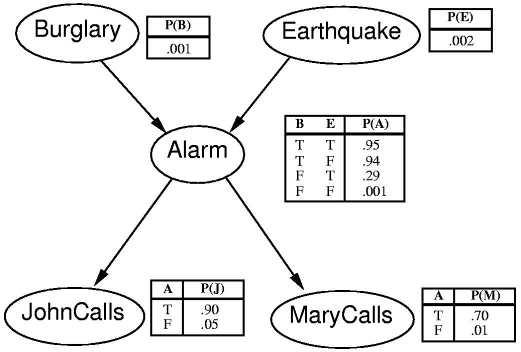

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***This conclusion is independent to values of CPTs!**

Question

If X and Y are independent, are they therefore independent given any variable(s)?

I.e., if $P(X, Y) = P(X)\, P(Y)$ [i.e., if $P(X|Y) = P(X)$], can we conclude that
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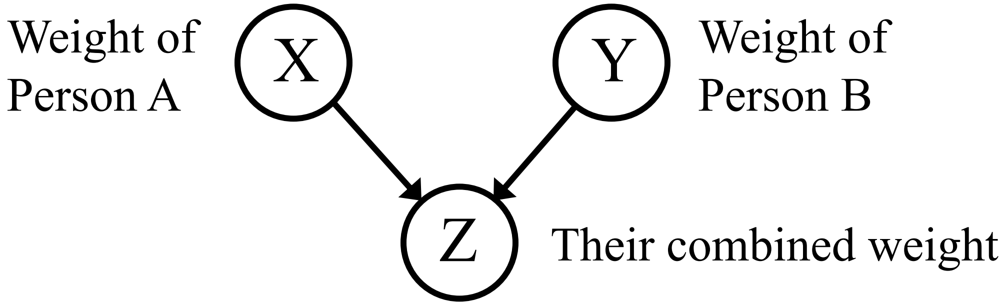
The answer is **no**, and here's a counter example:



$$P(X \mid Y) = P(X)$$
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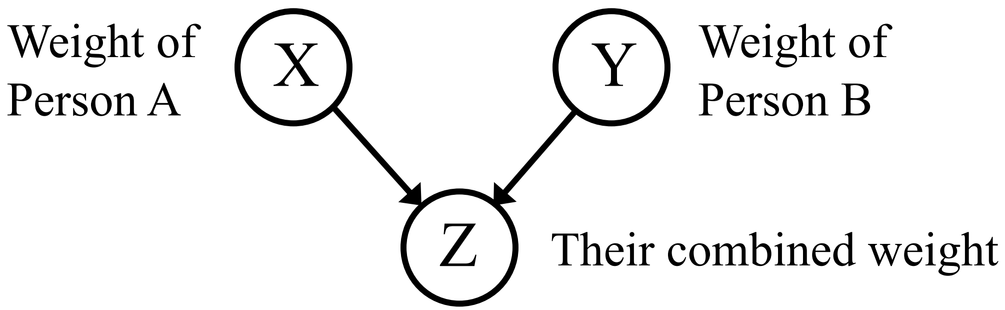

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***Again: This conclusion is independent to values of CPTs!**

Key points of today's lecture

- Probability notations
  - Distinguish between events and random variables, apply rules of probabilities
- Representing a joint-distribution
  - number of parameters exponential in the number of variables
  - Calculating marginals and conditionals from the joint-distribution.
- Conditional independences and factorization of joint-distributions
  - Saves parameters, often exponential improvements
- Intro to Bayesian networks / directed graphical models.

Next lectures

- More Bayesian networks, directed graphical models.
- Read off conditional independences from the graph!
- More examples