Duality

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(Based on Ryan Tibshirani's 10-725)

Last time: stochastic gradient descent

Consider

$$\min_x \; \frac{1}{m} \sum_{i=1}^m f_i(x)$$

Stochastic gradient descent or SGD: let $x^{(0)} \in \mathbb{R}^n$, repeat:

$$x^{(k)} = x^{(k-1)} - t_k \cdot \nabla f_{i_k}(x^{(k-1)}), \quad k = 1, 2, 3, \ldots$$

where $i_k \in \{1, \ldots m\}$ is chosen uniformly at random. Step sizes t_k chosen to be fixed and small, or diminishing

Compare to full gradient, which would use $\frac{1}{m} \sum_{i=1}^m \nabla f_i(x)$. Upside of SGD: much (potentially much, much) cheaper iterations, optimal for stochastic optimization.

Downside: can be slow to converge, suboptimal for finite sum problems (one of our advanced topics...)

Lower bounds in linear programs

Suppose we want to find lower bound on the optimal value in our convex problem, $B \le \min_x f(x)$

E.g., consider the following simple LP

$$\begin{aligned} \min_{x,y} \quad & x + y \\ \text{subject to} \quad & x + y \ge 2 \\ & x, y \ge 0 \end{aligned}$$

What's a lower bound? Easy, take $B = 2$

But didn't we get "lucky"?

Try again:

$$\begin{aligned}\min_{x,y} \quad & x + 3y \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0\end{aligned}$$

$$\begin{array}{rr} & x + y \geq 2 \\ + & 2y \geq 0 \\ = & x + 3y \geq 2 \end{array}$$

Lower bound $B = 2$

More generally:

$$\begin{aligned}\min_{x,y} \quad & px + qy \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0\end{aligned}$$

$$\begin{aligned} a + b &= p \\ a + c &= q \\ a, b, c &\geq 0 \end{aligned}$$

Lower bound $B = 2a$, for any a, b, c satisfying above

What's the best we can do? Maximize our lower bound over all possible a, b, c:

$$\begin{aligned} \min_{x,y} \quad & px + qy \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

Called primal LP

$$\begin{aligned} \max_{a,b,c} \quad & 2a \\ \text{subject to} \quad & a + b = p \\ & a + c = q \\ & a, b, c \geq 0 \end{aligned}$$

Called dual LP

Note: number of dual variables is number of primal constraints

Try another one:

$$\begin{aligned} \min_{x,y} \quad & px + qy \\ \text{subject to} \quad & x \geq 0 \\ & y \leq 1 \\ & 3x + y = 2 \end{aligned}$$

Primal LP

$$\begin{aligned} \max_{a,b,c} \quad & 2c - b \\ \text{subject to} \quad & a + 3c = p \\ & -b + c = q \\ & a, b \geq 0 \end{aligned}$$

Dual LP

Note: in the dual problem, c is unconstrained

Outline

Today:

- Duality in LP
- Examples (Max-Flow Min-Cut, Minimax Theorem)
- Lagrange Duality in General Convex Programs
- Examples (QP, SVM)

Duality for general form LP

Given $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $G \in \mathbb{R}^{r \times n}$, $h \in \mathbb{R}^r$:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & Gx \leq h \end{aligned}$$

Primal LP

$$\begin{aligned} \max_{u,v} \quad & -b^T u - h^T v \\ \text{subject to} \quad & -A^T u - G^T v = c \\ & v \geq 0 \end{aligned}$$

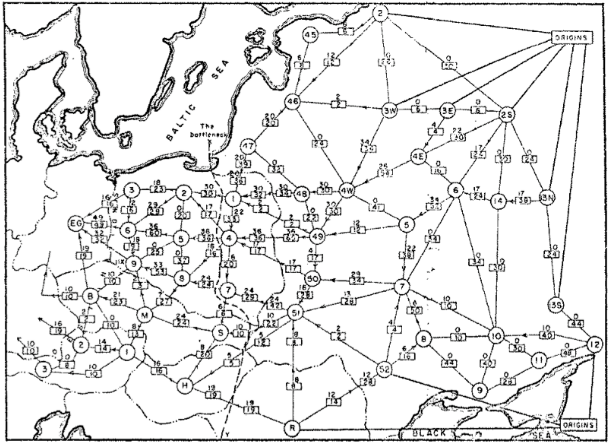
Dual LP

Explanation: for any u and $v \geq 0$, and x primal feasible,

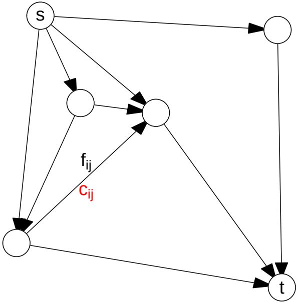
$$u^T(Ax - b) + v^T(Gx - h) \leq 0, \quad \text{i.e.,}$$
$$(-A^T u - G^T v)^T x \geq -b^T u - h^T v$$

So if $c = -A^T u - G^T v$, we get a bound on primal optimal value

Example: max flow and min cut



Soviet railway network (from Schrijver (2002), "On the history of transportation and maximum flow problems")



Given graph $G = (V, E)$, define flow f_{ij}, $(i, j) \in E$ to satisfy:

- $f_{ij} \geq 0$, $(i, j) \in E$
- $f_{ij} \leq c_{ij}$, $(i, j) \in E$
- $\displaystyle\sum_{(i,k)\in E} f_{ik} = \sum_{(k,j)\in E} f_{kj}$, $k \in V \backslash \{s, t\}$

Max flow problem: find flow that maximizes total value of the flow from s to t. I.e., as an LP:

$$\begin{array}{lll} \displaystyle\max_{f\in\mathbb{R}^{|E|}} & \displaystyle\sum_{(s,j)\in E} f_{sj} & \\ \text{subject to} & 0 \leq f_{ij} \leq c_{ij} & \text{for all } (i, j) \in E \\ & \displaystyle\sum_{(i,k)\in E} f_{ik} = \sum_{(k,j)\in E} f_{kj} & \text{for all } k \in V \setminus \{s, t\} \end{array}$$

Follow the steps before, just flip the logic:
Find the tightest upper bound of the objective by taking linear combinations of the constraints, subject to the constraints from the primal objective's coefficients.

Dual LP of max flow: The dual problem is (minimize over b, x to get best upper bound):

$$\begin{aligned}
\min_{b \in \mathbb{R}^{|E|},\, x \in \mathbb{R}^{|V|}} \quad & \sum_{(i,j) \in E} b_{ij} c_{ij} \\
\text{subject to} \quad & b_{ij} + x_j - x_i \geq 0 \quad \text{for all } (i,j) \in E \\
& b \geq 0,\ x_s = 1,\ x_t = 0
\end{aligned}$$

Suppose that at the solution, it just so happened that

$$x_i \in \{0, 1\} \quad \text{for all } i \in V$$

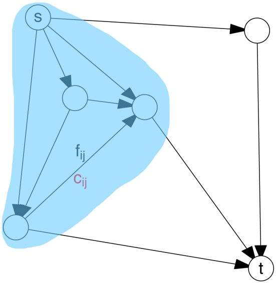
Let $A = \{i : x_i = 1\}$, $B = \{i : x_i = 0\}$; note $s \in A$, $t \in B$. Then

$$b_{ij} \geq x_i - x_j \quad \text{for } (i,j) \in E, \quad b \geq 0$$

imply that $b_{ij} = 1$ if $i \in A$ and $j \in B$, and 0 otherwise. Moreover, the objective $\sum_{(i,j)\in E} b_{ij} c_{ij}$ is the capacity of cut defined by A, B

I.e., we've argued that the dual is the LP relaxation of the min cut problem:

$$\begin{aligned} \min_{b \in \mathbb{R}^{|E|},\, x \in \mathbb{R}^{|V|}} \quad & \sum_{(i,j)\in E} b_{ij} c_{ij} \\ \text{subject to} \quad & b_{ij} \geq x_i - x_j \\ & b_{ij}, x_i, x_j \in \{0, 1\} \\ & \text{for all } i, j \end{aligned}$$



Therefore, from what we know so far:

value of max flow $\leq$

optimal value for LP relaxed min cut $\leq$

capacity of min cut

Famous result, called max flow min cut theorem: value of max flow through a network is exactly the capacity of the min cut

Hence in the above, we get all equalities. In particular, we get that the primal LP and dual LP have exactly the same optimal values, a phenomenon called strong duality

How often does this happen? More on this soon

Another perspective on LP duality

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & Gx \le h \end{aligned}$$

Primal LP

$$\begin{aligned} \max_{u,b} \quad & -b^T u - h^T v \\ \text{subject to} \quad & -A^T u - G^T v = c \\ & v \ge 0 \end{aligned}$$

Dual LP

Explanation # 2: for any u and $v \ge 0$, and x primal feasible

$$c^T x \ge c^T x + u^T(Ax - b) + v^T(Gx - h) := L(x, u, v)$$

So if C denotes primal feasible set, $f^\star$ primal optimal value, then for any u and $v \ge 0$,

$$f^\star \ \ge \ \min_{x \in C} \ L(x, u, v) \ \ge \ \min_x \ L(x, u, v) \ := \ g(u, v)$$

In other words, $g(u,v)$ is a lower bound on $f^\star$ for any u and $v \geq 0$

Note that

$$g(u,v) = \begin{cases} -b^T u - h^T v & \text{if } c = -A^T u - G^T v \\ -\infty & \text{otherwise} \end{cases}$$

Now we can maximize $g(u,v)$ over u and $v \geq 0$ to get the tightest bound, and this gives exactly the dual LP as before

This last perspective is actually completely general and applies to arbitrary optimization problems (even nonconvex ones)

Example: mixed strategies for matrix games

Setup: two players,  vs.  , and a payout matrix P

			R		
		1	2	$\dots$	n
	1	P_{11}	P_{12}	$\dots$	P_{1n}
J	2	P_{21}	P_{22}	$\dots$	P_{2n}
	$\dots$				
	m	P_{m1}	P_{m2}	$\dots$	P_{mn}

Game: if J chooses i and R chooses j, then J must pay R amount P_{ij} (don't feel bad for J—this can be positive or negative)

They use mixed strategies, i.e., each will first specify a probability distribution, and then

$$x: \quad \mathbb{P}(\text{J chooses } i) = x_i, \; i = 1, \dots m$$
$$y: \quad \mathbb{P}(\text{R chooses } j) = y_j, \; j = 1, \dots n$$

The expected payout then, from J to R, is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j P_{ij} = x^T P y$$

Now suppose that, because J is wiser, he will allow R to know his strategy x ahead of time. In this case, R will choose y to maximize $x^T P y$, which results in J paying off

$$\max \{x^T P y : \ y \geq 0, \ 1^T y = 1\} = \max_{i=1,\dots n} (P^T x)_i$$

J's best strategy is then to choose his distribution x according to

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{i=1,\dots n} (P^T x)_i \\ \text{subject to} & x \geq 0, \ 1^T x = 1 \end{array}$$

In an alternate universe, if R were somehow wiser than J, then he might allow J to know his strategy y beforehand

By the same logic, R's best strategy is to choose his distribution y according to

$$\begin{array}{ll} \max\limits_{y} & \min\limits_{j=1,\dots m} (Py)_j \\ \text{subject to} & y \geq 0,\ 1^T y = 1 \end{array}$$

Call R's expected payout in first scenario $f_1^\star$, and expected payout in second scenario $f_2^\star$. Because it is clearly advantageous to know the other player's strategy, $f_1^\star \geq f_2^\star$

But by Von Neumman's minimax theorem: we know that $f_1^\star = f_2^\star$ ... which may come as a surprise!

Recast first problem as an LP:

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{subject to} \quad & x \geq 0,\ 1^T x = 1 \\ & P^T x \leq t \end{aligned}$$

Now form what we call the Lagrangian:

$$L(x,t,u,v,y) = t - u^T x + v(1 - 1^T x) + y^T (P^T x - t1)$$

and what we call the Lagrange dual function:

$$\begin{aligned} g(u,v,y) &= \min_{x,t}\ L(x,t,u,v,y) \\ &= \begin{cases} v & \text{if } 1 - 1^T y = 0,\ Py - u - v1 = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Hence dual problem, after eliminating slack variable u, is

$$\begin{aligned} \max_{y,v} \quad & v \\ \text{subject to} \quad & y \geq 0,\ 1^T y = 1 \\ & Py \geq v \end{aligned}$$

This is exactly the second problem, and therefore again we see that strong duality holds

So how often does strong duality hold? In LPs, as we'll see, strong duality holds unless both the primal and dual are infeasible

Quick summary

We introduced duality in LP and offered two explanations.
Explanation #1: Use constraints set to construct lower bounds, make sure that the LHS of the lower bound matches the objective.
Explanation # 2: for any u and $v \geq 0$, and x primal feasible

$$c^T x \geq c^T x + u^T(Ax - b) + v^T(Gx - h) := L(x, u, v)$$

So if C denotes primal feasible set, $f^\star$ primal optimal value, then for any u and $v \geq 0$,

$$f^\star \;\geq\; \min_{x \in C} \; L(x, u, v) \;\geq\; \min_{x} \; L(x, u, v) \;:=\; g(u, v)$$

Finally, maximize the lower bound. This second explanation reproduces the same dual, but is actually completely general and applies to arbitrary optimization problems (even nonconvex ones)

Lagrangian

Consider general minimization problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & h_i(x) \le 0, \ i = 1, \dots m \\ & \ell_j(x) = 0, \ j = 1, \dots r \end{aligned}$$

Need not be convex, but of course we will pay special attention to convex case

We define the Lagrangian as

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x)$$

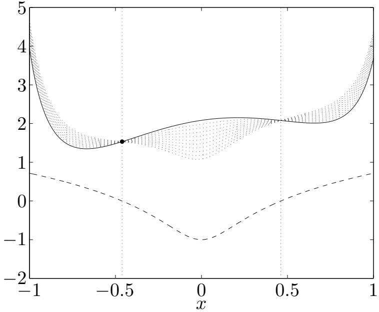
New variables $u \in \mathbb{R}^m, v \in \mathbb{R}^r$, with $u \ge 0$ (implicitly, we define $L(x, u, v) = -\infty$ for $u < 0$)

Important property: for any $u \geq 0$ and v,

$$f(x) \geq L(x, u, v) \quad \text{at each feasible } x$$

Why? For feasible x,

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i \underbrace{h_i(x)}_{\leq 0} + \sum_{j=1}^{r} v_j \underbrace{\ell_j(x)}_{=0} \leq f(x)$$



- Solid line is f
- Dashed line is h, hence feasible set $\approx [-0.46, 0.46]$
- Each dotted line shows $L(x, u, v)$ for different choices of $u \geq 0$

(From B & V page 217)

Lagrange dual function

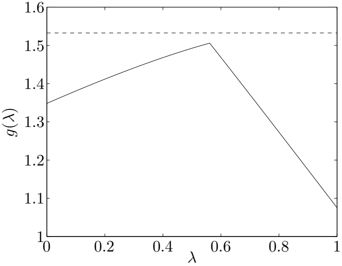
Let C denote primal feasible set, $f^\star$ denote primal optimal value. Minimizing $L(x,u,v)$ over all x gives a lower bound:

$$f^\star \;\geq\; \min_{x\in C}\; L(x,u,v) \;\geq\; \min_{x}\; L(x,u,v) \;:=\; g(u,v)$$

We call $g(u,v)$ the Lagrange dual function, and it gives a lower bound on $f^\star$ for any $u \geq 0$ and v, called dual feasible u,v

- Dashed horizontal line is $f^\star$
- Dual variable λ is (our u)
- Solid line shows $g(\lambda)$

(From B & V page 217)



Example: quadratic program

Consider quadratic program:

$$\begin{array}{ll} \min_x & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} & Ax = b, \; x \geq 0 \end{array}$$

where $Q \succ 0$. Lagrangian:

$$L(x,u,v) = \frac{1}{2}x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

Lagrange dual function:

$$g(u,v) = \min_x \; L(x,u,v) = -\frac{1}{2}(c-u+A^T v)^T Q^{-1}(c-u+A^T v) - b^T v$$

For any $u \geq 0$ and any v, this is lower a bound on primal optimal value $f^\star$

Same problem

$$\begin{array}{ll} \min\limits_{x} & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} & Ax = b,\ x \geq 0 \end{array}$$

but now $Q \succeq 0$. Lagrangian:

$$L(x, u, v) = \frac{1}{2}x^T Q x + c^T x - u^T x + v^T (Ax - b)$$

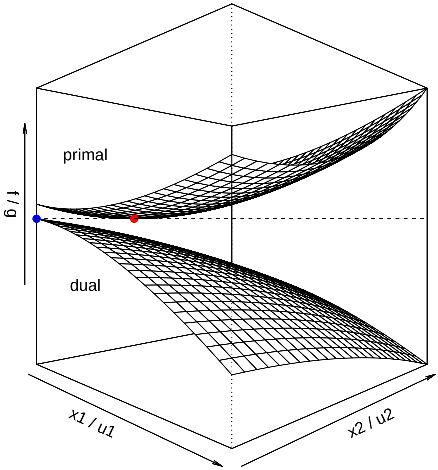
Lagrange dual function:

$$g(u, v) = \begin{cases} -\frac{1}{2}(c - u + A^T v)^T Q^+ (c - u + A^T v) - b^T v & \\ & \text{if } c - u + A^T v \perp \text{null}(Q) \\ -\infty & \text{otherwise} \end{cases}$$

where Q^+ denotes generalized inverse of Q. For any $u \geq 0$, v, and $c - u + A^T v \perp \text{null}(Q)$, $g(u, v)$ is a nontrivial lower bound on $f^\star$

Example: quadratic program in 2D

We choose $f(x)$ to be quadratic in 2 variables, subject to $x \geq 0$. Dual function $g(u)$ is also quadratic in 2 variables, also subject to $u \geq 0$



Dual function $g(u)$ provides a bound on $f^\star$ for every $u \geq 0$

Largest bound this gives us: turns out to be exactly $f^\star$ ... coincidence?

More on this later, via KKT conditions

Lagrange dual problem

Given primal problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & h_i(x) \leq 0, \ i = 1, \ldots m \\ & \ell_j(x) = 0, \ j = 1, \ldots r \end{aligned}$$

Our constructed dual function $g(u, v)$ satisfies $f^\star \geq g(u, v)$ for all $u \geq 0$ and v. Hence best lower bound is given by maximizing $g(u, v)$ over all dual feasible u, v, yielding Lagrange dual problem:

$$\begin{aligned} \max_{u,v} \quad & g(u, v) \\ \text{subject to} \quad & u \geq 0 \end{aligned}$$

Key property, called weak duality: if dual optimal value is $g^\star$, then

$$f^\star \geq g^\star$$

Note that this always holds (even if primal problem is nonconvex)

Another key property: the dual problem is a convex optimization problem (as written, it is a concave maximization problem)

Again, this is always true (even when primal problem is not convex)

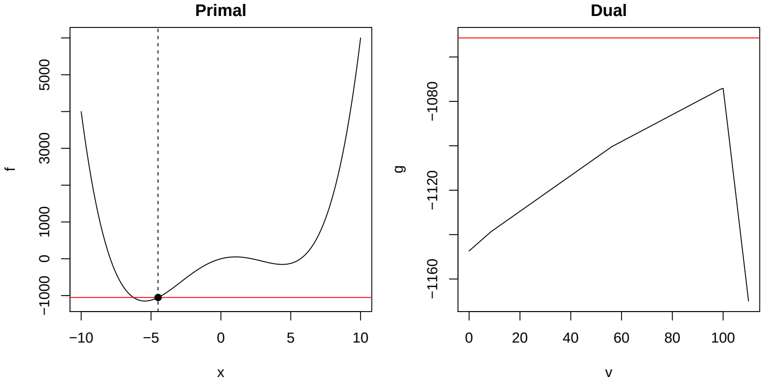
By definition:

$$\begin{aligned} g(u,v) &= \min_x \left\{ f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j \ell_j(x) \right\} \\ &= -\underbrace{\max_x \left\{ -f(x) - \sum_{i=1}^m u_i h_i(x) - \sum_{j=1}^r v_j \ell_j(x) \right\}}_{\text{pointwise maximum of convex functions in } (u,v)} \end{aligned}$$

I.e., g is concave in (u,v), and $u \geq 0$ is a convex constraint, hence dual problem is a concave maximization problem

Example: nonconvex quartic minimization

Define $f(x) = x^4 - 50x^2 + 100x$ (nonconvex), minimize subject to constraint $x \geq -4.5$



Dual function g can be derived explicitly, via closed-form equation for roots of a cubic equation

Form of g is rather complicated:

$$g(u) = \min_{i=1,2,3} \left\{F_i^4(u) - 50F_i^2(u) + 100F_i(u)\right\},$$

where for $i = 1, 2, 3$,

$$F_i(u) = \frac{-a_i}{12 \cdot 2^{1/3}}\Big(432(100-u)-\big(432^2(100-u)^2-4\cdot 1200^3\big)^{1/2}\Big)^{1/3}$$
$$-100\cdot 2^{1/3}\frac{1}{\Big(432(100-u) - \big(432^2(100-u)^2 - 4 \cdot 1200^3\big)^{1/2}\Big)^{1/3}},$$

and $a_1 = 1$, $a_2 = (-1 + i\sqrt{3})/2$, $a_3 = (-1 - i\sqrt{3})/2$

Without the context of duality it would be difficult to tell whether or not g is concave ... but we know it must be!

Strong duality

Recall that we always have $f^\star \geq g^\star$ (weak duality). On the other hand, in some problems we have observed that actually

$$f^\star = g^\star$$

which is called strong duality

Slater's condition: if the primal is a convex problem (i.e., f and $h_1, \ldots h_m$ are convex, $\ell_1, \ldots \ell_r$ are affine), and there exists at least one strictly feasible $x \in \mathbb{R}^n$, meaning

$$h_1(x) < 0, \ldots h_m(x) < 0 \quad \text{and} \quad \ell_1(x) = 0, \ldots \ell_r(x) = 0$$

then strong duality holds

This is a pretty weak condition. An important refinement: strict inequalities only need to hold over functions h_i that are not affine

LPs: back to where we started

For linear programs:

- Easy to check that the dual of the dual LP is the primal LP
- Refined version of Slater's condition: strong duality holds for an LP if it is feasible
- Apply same logic to its dual LP: strong duality holds if it is feasible
- Hence strong duality holds for LPs, except when both primal and dual are infeasible

(In other words, we nearly always have strong duality for LPs)

Example: support vector machine dual

Given $y \in \{-1, 1\}^n$, $X \in \mathbb{R}^{n\times p}$, rows $x_1, \ldots x_n$, recall the support vector machine problem:

$$
\begin{aligned}
\min_{\beta, \beta_0, \xi} \quad & \frac{1}{2}\|\beta\|_2^2 + C\sum_{i=1}^n \xi_i \\
\text{subject to} \quad & \xi_i \geq 0, \ i = 1, \ldots n \\
& y_i(x_i^T \beta + \beta_0) \geq 1 - \xi_i, \ i = 1, \ldots n
\end{aligned}
$$

Introducing dual variables $v, w \geq 0$, we form the Lagrangian:

$$
\begin{aligned}
L(\beta, \beta_0, \xi, v, w) = \frac{1}{2}\|\beta\|_2^2 + C\sum_{i=1}^n \xi_i - \sum_{i=1}^n v_i \xi_i + \\
\sum_{i=1}^n w_i \big(1 - \xi_i - y_i(x_i^T \beta + \beta_0)\big)
\end{aligned}
$$

Minimizing over β, β_0, ξ gives Lagrange dual function:

$$g(v, w) = \begin{cases} -\frac{1}{2} w^T \tilde{X} \tilde{X}^T w + 1^T w & \text{if } w = C1 - v, \ w^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

where $\tilde{X} = \text{diag}(y) X$. Thus SVM dual problem, eliminating slack variable v, becomes

$$\begin{array}{ll} \max\limits_{w} & -\dfrac{1}{2} w^T \tilde{X} \tilde{X}^T w + 1^T w \\ \text{subject to} & 0 \le w \le C1, \ w^T y = 0 \end{array}$$

Check: Slater's condition is satisfied, and we have strong duality. Further, from study of SVMs, might recall that at optimality

$$\beta = \tilde{X}^T w$$

This is not a coincidence, as we'll later via the KKT conditions

Duality gap

Given primal feasible x and dual feasible u, v, the quantity

$$f(x) - g(u, v)$$

is called the duality gap between x and u, v. Note that

$$f(x) - f^\star \le f(x) - g(u, v)$$

so if the duality gap is zero, then x is primal optimal (and similarly, u, v are dual optimal)

From an algorithmic viewpoint, provides a stopping criterion: if $f(x) - g(u, v) \le \epsilon$, then we are guaranteed that $f(x) - f^\star \le \epsilon$

Very useful, especially in conjunction with iterative methods ... more dual uses in coming lectures

References

- S. Boyd and L. Vandenberghe (2004), "Convex optimization", Chapter 5
- R. T. Rockafellar (1970), "Convex analysis", Chapters 28–30