Lecture 12 Interior Point Method.

$f(x^{(k+1)}) \le f(x^{(k)}) - \gamma$ for some $\gamma > 0$ $\qquad mI \preceq \nabla^2 f(x) \preceq LI$

non-self-concordant analysis $k_0 = \mathrm{poly}(\frac{L}{m})$

self concordant analysis $k_0 = O(1)$

$f(x^+) \le f(x) + \langle \nabla f(x), tv \rangle + \frac{Lt^2}{2}\|v\|^2$ by L-smoothness

$x^+ = x + tv$

$v = -\nabla^2 f(x)^{-1} \nabla f(x)$ $\qquad \nabla^T f(x) (\nabla^2 f(x))^{-2} \nabla f(x)$

$\le f(x) - t \underbrace{\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)}_{\lambda^2(x)} + \frac{Lt^2}{2} \|\nabla^2 f(x)^{-1} \nabla f(x)\|^2$

$\|\underbrace{\nabla^2 f(x)^{-\frac{1}{2}}}_{\uparrow} \nabla^2 f(x)^{-\frac{1}{2}} \nabla f(x)\|^2$

$\le f(x) - t\lambda^2(x) + \frac{Lt^2}{2m} \underbrace{\|\nabla^2 f(x)^{-\frac{1}{2}} \nabla f(x)\|_2^2}_{\lambda^2(x)}$ $\qquad \|\nabla^2 f(x)^{-\frac{1}{2}}\|_{op} \le \sqrt{\frac{1}{m}}$

$= f(x) - t(1 - \frac{Lt}{2m})\lambda^2(x)$ $\qquad$ Chose $t \le \frac{m}{L}$ $\qquad \gamma = \frac{m}{2L^2}\eta^2$

$k \le k_0$

$\le f(x) - \frac{t}{2} \|\nabla f(x)\|^2_{\nabla^2 f(x)^{-1}}$ $\qquad t = \frac{m}{L}$

$\|\nabla f(x)\|_2 \ge \eta$.

$\|\nabla f(x)\|^2_{\nabla^2 f(x)^{-1}} \ge \frac{1}{L}\|\nabla f(x)\|^2 \ge \frac{\eta^2}{L}$

Self concordant analysis

Define $f(x)$ is self-concordant if $|f'''(x)| \le 2 f''(x)^{\frac{3}{2}}$

R-self-concordant if $\quad\text{———} \le 2R \text{———}$

when $x \in \mathbb{R}^d$ this needs to apply to every direction..

Property of self-concordance.

$$(1-R\|v\|_{(f)})^2 \nabla^2 f(x) \preceq \nabla^2 f(x+v) \preceq \frac{1}{(1-R\|v\|_{\nabla^2 f(x)})^2} \nabla^2 f(x)$$

$$f(x+tv) = f(x) + \langle \nabla f(x), tv \rangle + t^2 v^T \nabla^2 f(\xi) v \quad \exists \xi \in [x, x+tv]$$

$$\{x + tv \cdot \tilde{t} \mid \tilde{t} \in [0,1]\}$$

$$= f(x) - t\lambda^2(x) + t^2 \nabla f(x)^T (\nabla^2 f(x))^{-1} \nabla^2 f(\xi) (\nabla^2 f(x))^{-1} \nabla f(x)$$

$$\le f(x) - t\lambda^2(x) + t^2 \nabla f(x)^T (\nabla^2 f(x))^{-1} \frac{\cancel{\nabla^2 f(x)} \, \cancel{(\nabla^2 f(x))^{-1}}}{(1-\tilde{t}\|v\|_{\nabla^2 f(x)})^2} \nabla f(x)$$

$$\le f(x) - t\lambda^2(x) + \frac{t^2}{(1-t\lambda(x))^2} \lambda^2(x) \le f(x) - t\lambda^2(x)\left(1 - \frac{t}{(\frac{1}{2})^2}\right)$$

$$t \le \min\left\{\frac{1}{2\lambda(x)}, \frac{1}{8}\right\}$$

$$\le f(x) - \frac{t}{2}\lambda^2(x) \le f(x) - \max\left\{\frac{1}{16}\lambda^2(x), \frac{\lambda(x)}{4}\right\}$$