CS292F StatRL Lecture 4 Finite-Horizon MDPs / Temporal Difference Learning

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UC Santa Barbara

Homework 1 released; project ideas shared

- You will learn the various elements of MDPs by solving problems. Also you will practice using Hoeffding's inequality and Bernstein's inequality.

- Mostly similar to what I covered in the lectures / sometimes the solutions are readily available by reading the AJKS book.

- I shared a document with recent RL theory papers by categories.
  - You do NOT have to pick one from there
  - Application projects are just as welcome --- e.g., applying RL to your problem / formulate your problem as an MDP.
  - I am happy to discuss with you if you have some fresh ideas.

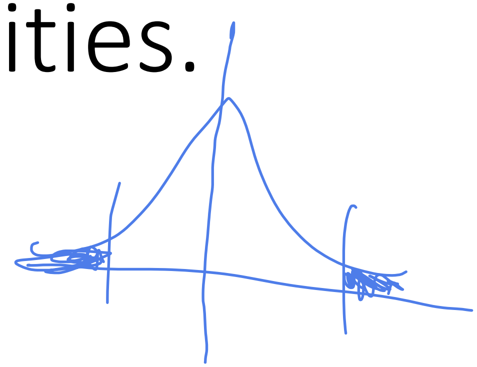
Recap: MDP planning with access to generative models

- Motivation:
  1. Solving MDP faster / approximately with randomized algs that sample
  2. Study sample complexity of RL with unknown transitions (without worrying about exploration)

$\hat{M} = (S, A, \hat{P}_{plug\text{-}in}, r, \gamma, \mu)$

- Algorithm of interest: Model-based plug-in estimator.
  - Sample all state-action pairs uniformly. Estimate the transition kernel.
  - Do VI / PI on the approximate MDP. $V^* - V^{\hat{\pi}^*} \le \varepsilon$

Recap: on a brief digression, we learned concentration inequalities.



- Hoeffding's inequality

$$\left|\bar{X} - \mathbb{E}[\bar{X}]\right| \leq \sqrt{\frac{B^2}{2n}\log(2/\delta)}$$

- Bernstein inequality

$$\left|\bar{X} - \mathbb{E}[X_1]\right| \leq \sqrt{\frac{2\mathbf{Var}[X_1]}{n}\log(2/\delta)} + \frac{2M\log(2/\delta)}{3n}$$

- McDiarmid's inequality
  - Concentration of $f(X_1,\ldots,X_n)$ when f is stable / coordinate-wise Lipschitz.

$$\left|f(X_1,\ldots,X_j,\ldots,X_n) - f(X_1,\ldots,X_j',\ldots,X_n)\right| \leq c_j \leq B$$

  - Concentration is now enough, usually we need to also compute expectation.

$$\left|f(X_1\ldots X_n) - \mathbb{E} f(X_1,\ldots,X_n)\right| \leq \text{Hoeffding's bound}$$

- Union bound: merging failure probabilities.

Recap: Sample complexity bound Attempt 1

- Simulation Lemma

$$Q^\pi - \widehat{Q}^\pi \quad = \quad \gamma(I - \gamma \widehat{P}^\pi)^{-1}(P - \widehat{P})V^\pi$$

- Uniform convergence bound for all policies
  - By Holder's inequality, McDiarmid inequality.

- Sample complexity bound it suffices that we call this many times.

$$O\left(\frac{S^2 A + SA\log(2SA/\delta))}{(1-\gamma)^4\epsilon^2}\right)$$

Recap: Sample complexity bound Attempt 2

- Show that the V* of the estimated MDP is close to the the V* function of the true MDP.

$$\|Q^\star - \widehat{Q}^\star\|_\infty \leq \frac{\gamma}{1-\gamma}\|(P - \widehat{P})V^\star\|_\infty$$

fixed not data dependent

- Use Q-value amplification lemma:

$$V^{\pi_Q} \geq V^\star - \frac{2\|Q - Q^\star\|_\infty}{1-\gamma}\mathbb{1}.$$

- Overall sample complexity bound:

$$O\left(\frac{SA\log(2SA/\delta))}{(1-\gamma)^6\epsilon^2}\right)$$

Recap: optimal sample complexity

- Optimal sample complexity:

$$\Theta\left(\frac{SA\log(2SA/\delta))}{(1-\gamma)^3\epsilon^2}\right)$$

- Ideas to achieve it:
  - Bernstein inequality. (HW1)
  - Strong variance bound. (HW1)
  - Advanced Q-value error to policy value (not covered in the class)

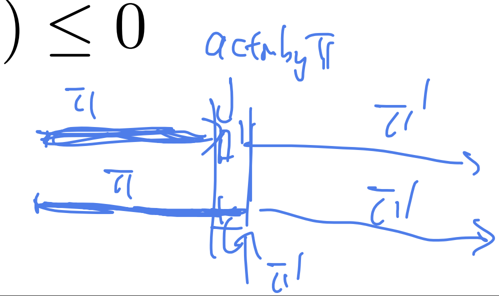
This lecture

1. Wrap up MDPs
   - Performance difference lemma and advantage decomposition (Readings: AJKS Section 1.6)
   - Remarks about **finite horizon / episodic MDPs.** (Readings: AJKS Section 1.2)

2. RL algorithms
   - Model-based vs Model-free RL algorithms
   - Temporal difference learning. (Sutton and Barto Ch 5-6)
   - TD learning with linear function approximation.

Advantage function and Performance Difference Lemma

- Advantage function: $A^\pi(s,a) := Q^\pi(s,a) - V^\pi(s)\,.$
  - The advantage of taking given action over following the policy.
  - Simple fact: $A^*(s,a) := A^{\pi^*}(s,a) \leq 0$



- Performance Difference Lemma

Lemma 1.16. *(The performance difference lemma) For all policies π, π' and distributions μ over $\mathcal{S}$,*

$$V^\pi(\mu) - V^{\pi'}(\mu) = \frac{1}{1-\gamma}\mathbb{E}_{s'\sim d_\mu^\pi}\mathbb{E}_{a'\sim\pi(\cdot|s')}\left[A^{\pi'}(s',a')\right].$$

where $d_\mu^\pi(s) = (1-\gamma)\sum_{t=1}^{\infty}\gamma^{t-1}\mathbb{P}^\pi[S_t = s] = (1-\gamma)\nu_\mu^\pi(s)$

Proof of Performance Difference Lemma

$$
\begin{aligned}
V^\pi(s) - V^{\pi'}(s) &= \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t)\right] - V^{\pi'}(s) \\
&= \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t \left(r(s_t, a_t) + V^{\pi'}(s_t) - V^{\pi'}(s_t)\right)\right] - V^{\pi'}(s) \\
&\overset{(a)}{=} \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t \left(r(s_t, a_t) + \gamma V^{\pi'}(s_{t+1}) - V^{\pi'}(s_t)\right)\right] \\
&\overset{(b)}{=} \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t \left(r(s_t, a_t) + \gamma \mathbb{E}[V^{\pi'}(s_{t+1})|s_t, a_t] - V^{\pi'}(s_t)\right)\right] \\
&\overset{(c)}{=} \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t \left(Q^{\pi'}(s_t, a_t) - V^{\pi'}(s_t)\right)\right] \\
&= \mathbb{E}_{\tau \sim \Pr^\pi(\tau|s_0=s)}\left[\sum_{t=0}^{\infty} \gamma^t A^{\pi'}(s_t, a_t)\right] = \frac{1}{1-\gamma}\mathbb{E}_{s' \sim d_s^\pi}\, \mathbb{E}_{a \sim \pi(\cdot|s)} \gamma^t A^{\pi'}(s', a),
\end{aligned}
$$

Finite horizon MDPs

- Parameterization / Setup

$$M = (\mathcal{S}, \mathcal{A}, \{P\}_h, \{r\}_h, H, \mu)$$

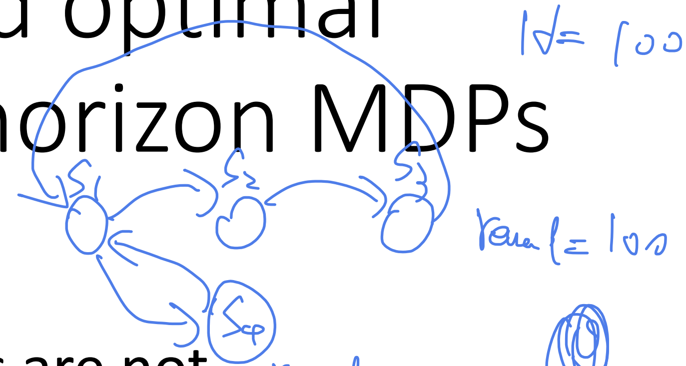
$P_h(s_h | s_{h-1}, a_{h-1})$, $r_h(s_h, a_h) = \mathbb{E}[R_h | S_h = s_h, A_h = a_h]$

goal is to find ϵ-optimal policy: $V^\pi(\mu) = \mathbb{E}^\pi[\sum_{t=1}^{H} R_h]$

- Finite horizon MDPs with stationary transitions / non-stationary transitions

if $P_h(s'|s,a) = P_{h'}(s'|s,a)$ $\forall\, 1 \le h, h' \le H$.

Bellman equations and optimal policies for the finite horizon MDPs



- Even if P and r are stationary
  - the V functions are Q functions are not.

$$V_t^\pi = r_t^\pi + P_t^\pi V_{t+1}^\pi \in \mathbb{R}^{|S|}$$

$$Q_t^\pi = r + P_t \cdot V_{t+1}^\pi = r + \tilde{P}_t^\pi \cdot Q_{t+1}^\pi \in \mathbb{R}^{|S|\cdot|A|}$$

for all $t = 1, \dots, H$

$$V_t^*(s) = \max_a \Big[r_t(s,a) + \sum_{s'} P_t(s'|s,a) \, V_{t+1}^*(s') \Big]$$

$V_{H-1}^*(s_1) = 1$

$V_{H-2}^*(s_1) = 100$

- By the Markovian property, it suffices to consider "nonstationary" but "memoryless" policies. $\pi_t(\cdot|s_t)$
  - There exists a deterministic / memoryless optimal policy.

Other aspects of finite-horizon MDPs

- Advantage function and Performance Difference Lemma

$$V^{\pi} - V^{\widetilde{\pi}} = \sum_{h=0}^{H-1} \mathbb{E}_{s,a \sim \mathbb{P}_h^{\pi}} \left[A_h^{\widetilde{\pi}}(s,a) \right]$$

$$A_h^{\pi}(s,a) = Q_h^{\pi}(s,a) - V_h^{\pi}(s)$$

- Simulation lemma (HW1, last question)
- LP-formulation and occupancy measures
- Sample complexities under a generative model setting

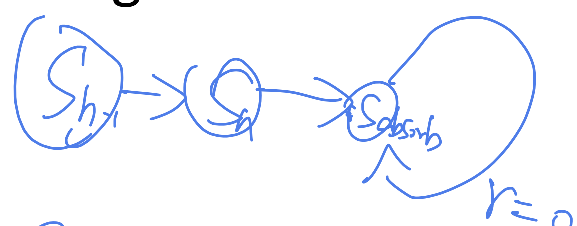
Two-way reductions between finite horizon MDPs and infinite horizon / discounted MDPs

$H = O\left(\frac{\log(1/\varepsilon)}{1-\gamma}\right)$

- Infinite horizon ➔ finite horizon
  - Clip at O(1/(1-γ)).
  - Define time-varying rewards.

$r_h(s,a) = \gamma^{h-1} r(s,a)$

- Finite horizon ➔ infinite horizon
  - The last step transitions into an absorbing state with self-loops and zero rewards.
  - Discounting factor γ set to be 1.



$S = S_1 \cup S_2 \cup \cdots \cup S_H \cup \{s_{absorb}\}$

Two-way reductions between finite-H MDPs with stationary and non-stationary transitions.

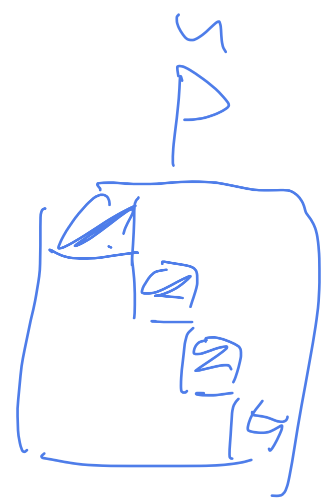
- Stationary ➔ Non-stationary

$$P_h(s'|s,a) \neq P_{h'}(s'|s,a)$$

- Non-stationary ➔ Stationary

Stationary $M = \{\tilde{S}, \tilde{A}, \tilde{P}, H, \tilde{\mu}\}$

$$\tilde{S} = S_1 \cup \cdots \cup S_H$$



H blocks

Other MDP settings that we will not consider in this course

- Infinite-horizon average reward MDPs

$$\max_{\pi} \lim_{H \to \infty} \frac{1}{H} \mathbb{E}\left[\sum_{t=1}^{H} R_t\right]$$

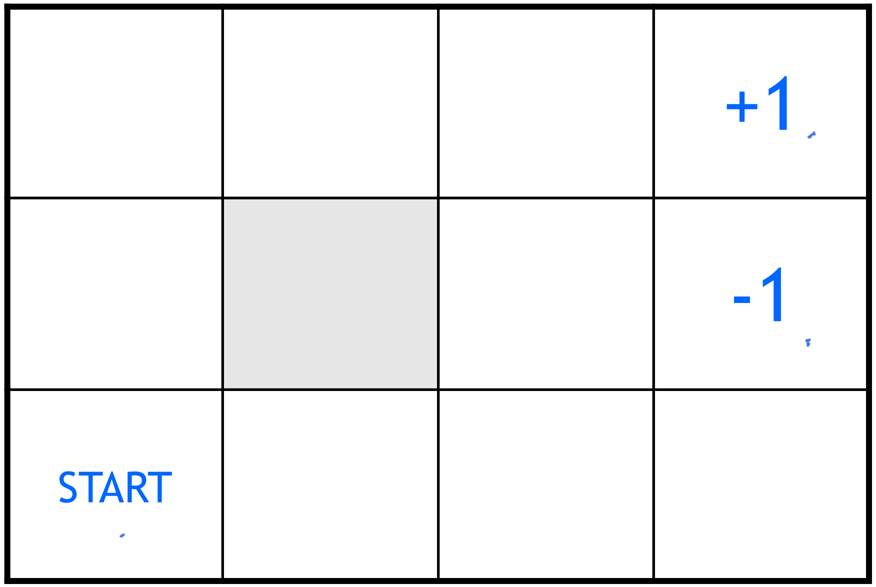
  - Usually require additional conditions for this to be well-defined.

- Indefinite-horizon setting
  - H is a random variable
  - e.g. Frozen-lake / Mountain car / other navigation tasks
  - Tricky issue: not invariant to scaling / translation of the rewards.

$$0 \le r(s,a) \le 1$$

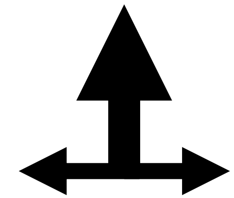
***We are not going to cover these settings in this course.**

Example: Frozen lake.



actions: UP, DOWN, LEFT, RIGHT

UP e.g.,



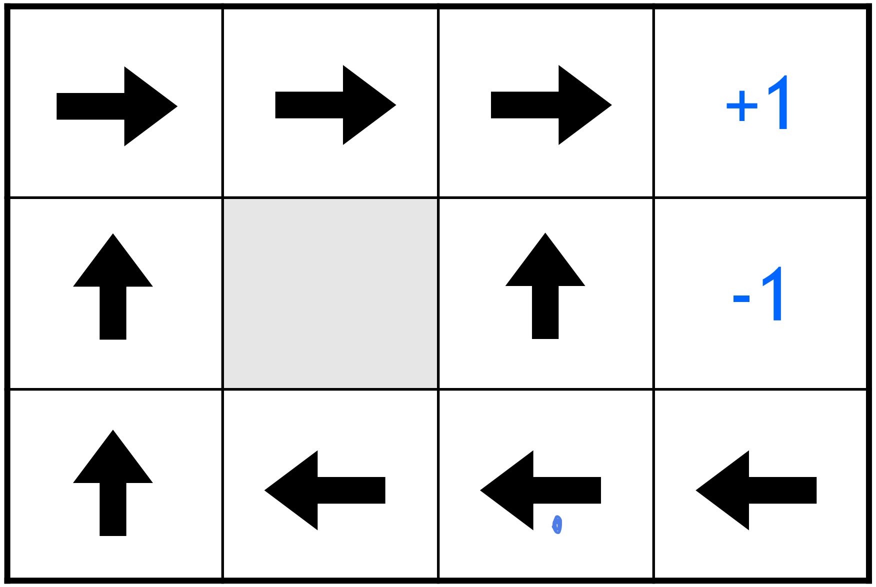
State-transitions with action **UP**:

80% move up
10% move left
10% move right

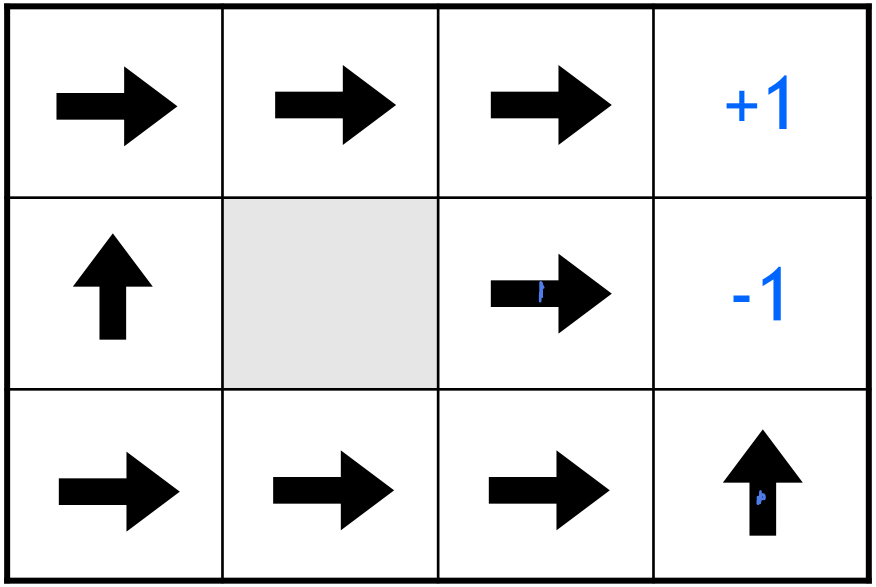
*If you bump into a wall, you stay where you are.

- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- Finite horizon or infinite horizon?
- What is a good policy?

Optimal policies in the different reward settings

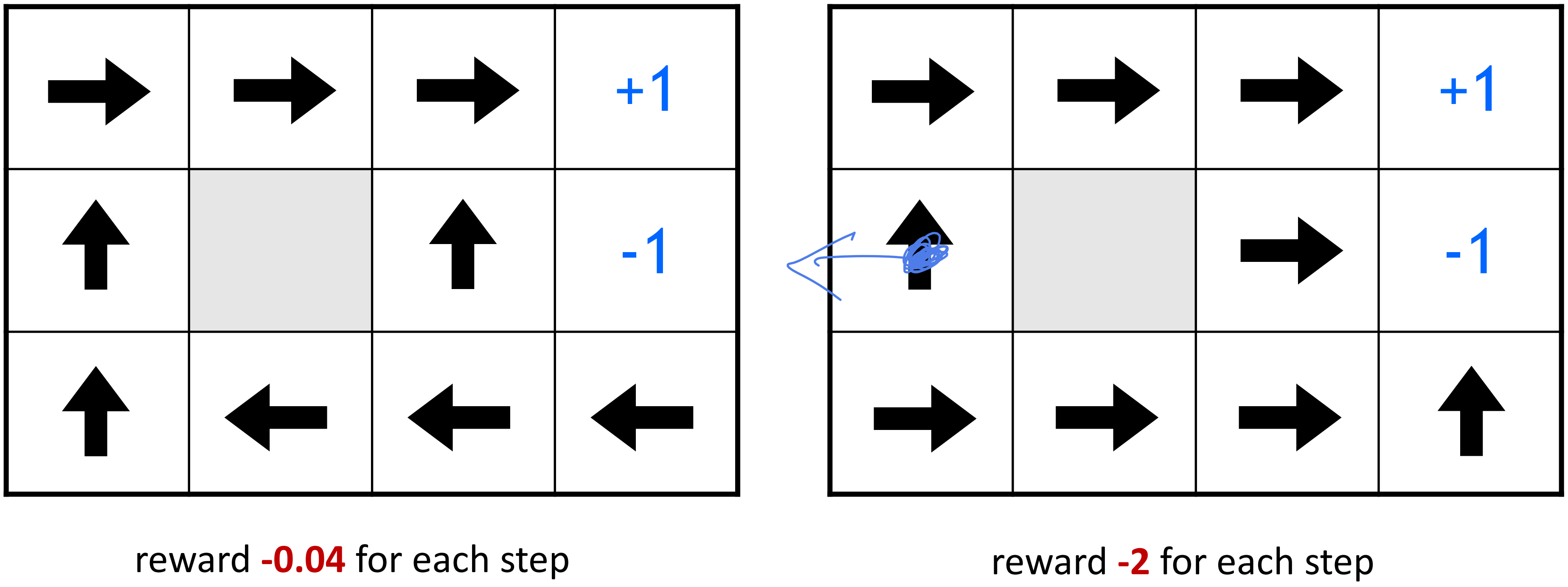


reward **-0.04** for each step



reward **-2** for each step

Optimal policies in the different reward settings



What if there is a positive reward for each step?

Partially Observed MDPs

sufficient statistic

$P(S_t \mid H_{1:t-1}, O_t)$

- POMDP:
  - Estimate belief states (posterior distribution of state given history, i.e., Kalman filter)
  - Take actions according to the belief state.

- Computational considerations
  - MDP-planning: P-complete
  - POMDP-planning: PSPACE-complete (harder than NP-complete)
  - MDP-learning: polynomial sample complexity
  - POMDP-learning: often not identifiable.

***We are not going to cover POMDP in this course, but good references are available.**

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Recap: Policy Iterations and Value Iterations

- What are these algorithms for?
  - Algorithms of computing the V* and Q* functions from MDP parameters
- Policy Iterations

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

- Value iterations

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

- How do we make sense of them?
  - Recursively applying the Bellman equations until convergence.

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*These methods are called “Dynamic Programming” approaches in Chap 4 of Sutton and Barto.

They are no longer valid in RL

- Policy Evaluation

$$V^{\pi}_{k+1}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^{\pi}_k(s')]$$

- Policy improvement

$$\pi'(s) = \arg\max_a Q^{\pi}(s,a)$$
$$= \arg\max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^{\pi}_k(s')]$$

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- Value iterations

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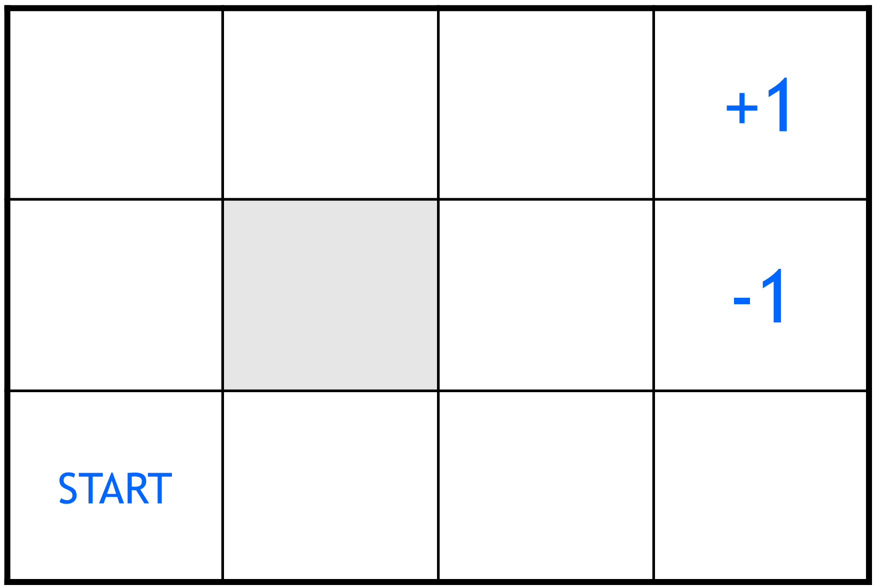
***We do not have the MDP parameters in RL!**

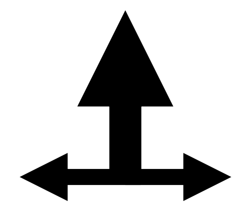
Example: Frozen lake



actions: UP, DOWN, LEFT, RIGHT

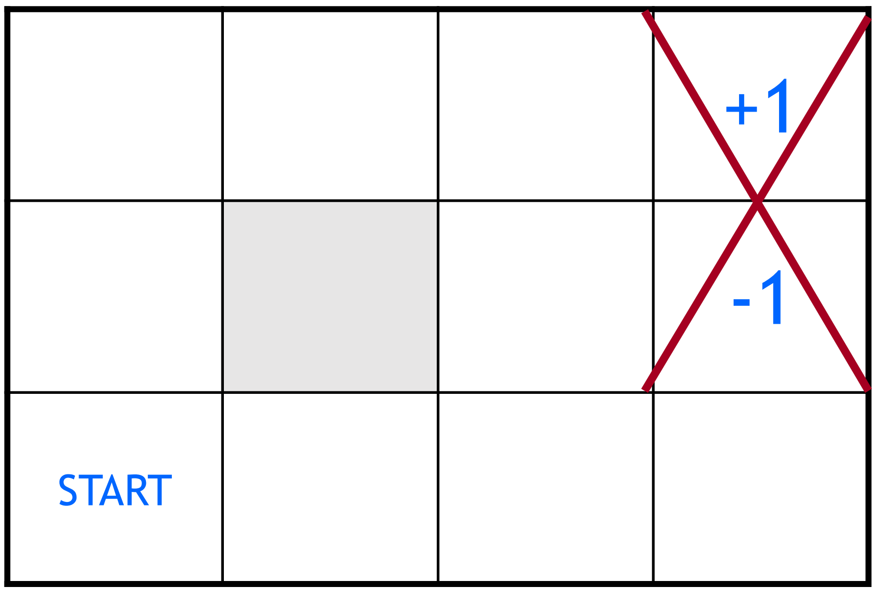
UP

80% move UP
10% move LEFT
10% move RIGHT



- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- what's the strategy to achieve max reward?

Example: Frozen lake



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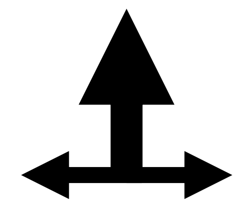

- ~~reward +1 at [4,3], -1 at [4~~,2]
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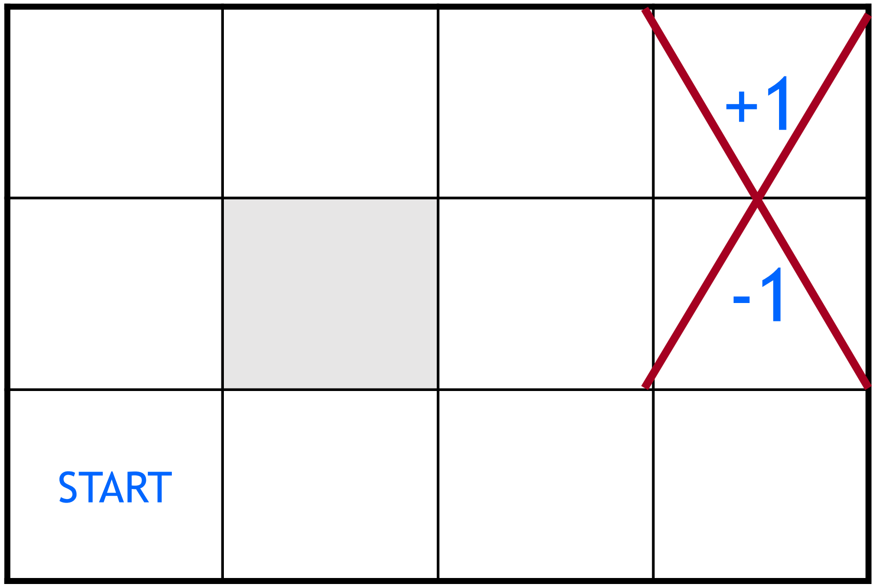
Example: Frozen lake



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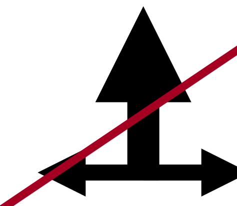

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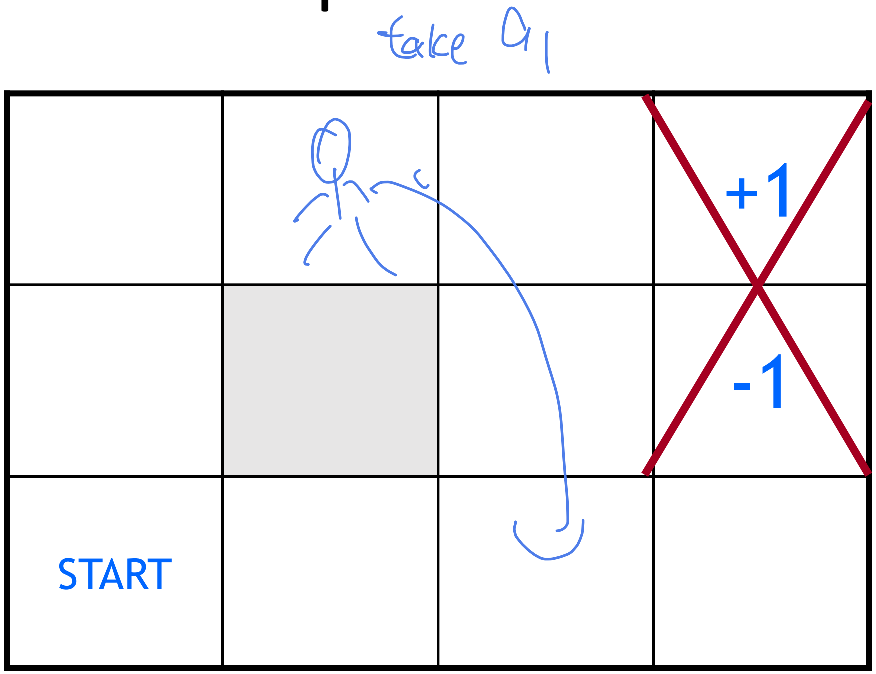

Action 1, Action 2, Action 3, Action 4

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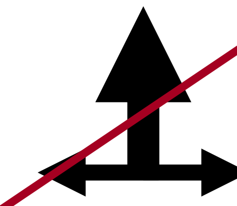

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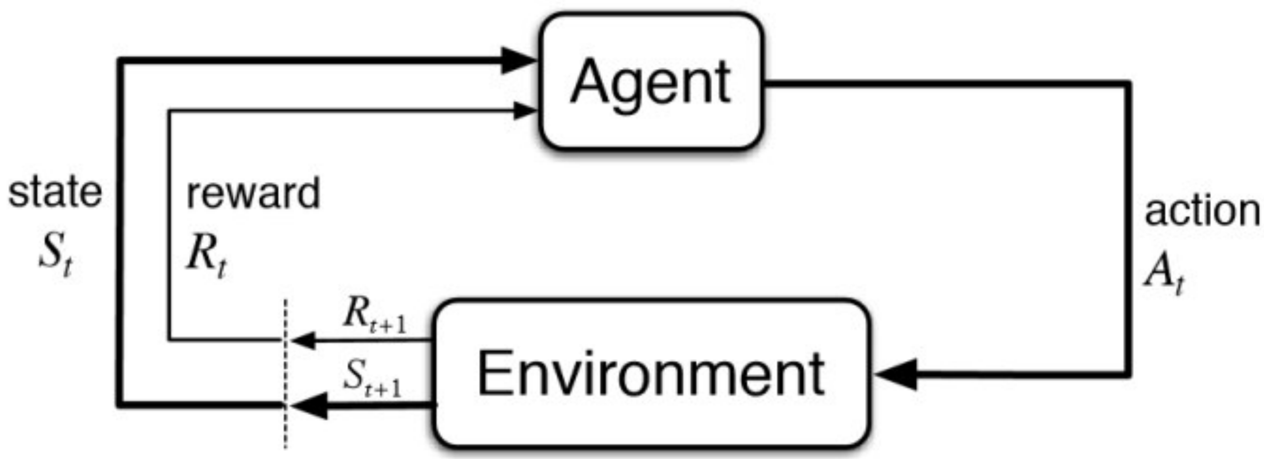
Instead, reinforcement learning agents have "online" access to an environment

- State, Action, Reward
- Unknown reward function, unknown state-transitions.
- Agents can "act" and "experiment", rather than only doing offline planning.



Idea 1: **Model-based** Reinforcement Learning

- Model-based idea
  - Let's approximate the model based on experiences
  - Then solve for the values as if the learned model were correct

- Step 1: Get data by running the agent to explore
  - Many data points of the form: $\{(s_1, a_1, s_2, r_1), \ldots, (s_N, a_N, s_{N+1}, r_N)\}$

- Step 2: Estimate the model parameters
  - $\hat{P}(s'|s,a)$ --- plug-in / MLE. We need to observe the transition many times for each s, a
  - $\hat{r}(s', a, s)$ --- this is an estimate of the empirical rewards.

Then we can plug in these estimates and then use dynamic programming for policy evaluation / improvements.

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{a} \pi(a|s) \sum_{s'} \hat{P}(s'|s,a)[\hat{r}(s,a,s') + \gamma V_k^{\pi}(s')]$$

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* These iterations will produce $\hat{V}^*$ and $\hat{Q}^*$ functions, and then $\hat{\pi}^*$

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  - The chance of randomly running into a state can be **exponentially small,** if we decide to take random actions.

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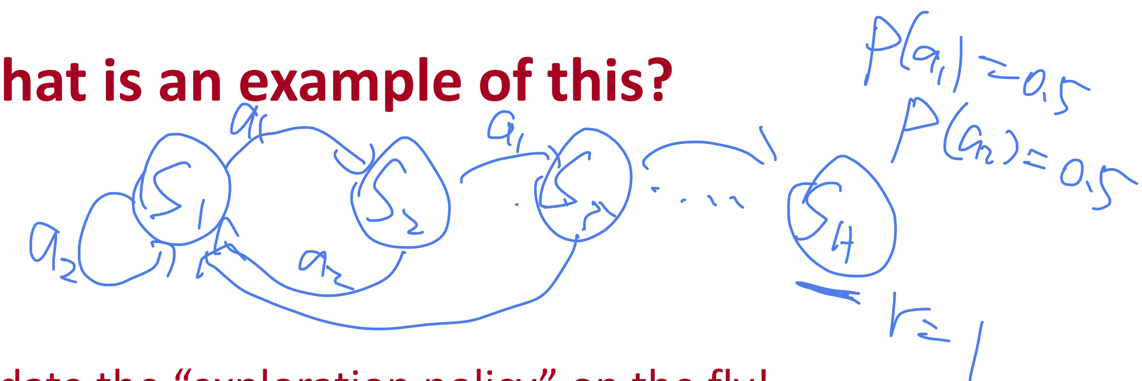

*Need to somehow update the "exploration policy" on the fly!

More generally, model-based method is a algorithm design principle.

- We use function approximation on P
- Function classes: $\hat{P} \in \mathcal{H}$ (Neural network; mixture of Gaussian) $P \notin \mathcal{H}$
- Simulation lemma still applies

$$Q^\pi - \widehat{Q}^\pi \quad = \quad \gamma(I - \gamma \widehat{P}^\pi)^{-1}(P - \widehat{P})V^\pi$$

ϵ_{approx}

$\hat{\pi}^* = \arg\max_{\hat{\pi}} \hat{Q}^{\hat{\pi}}$

  - If: $\hat{P}$ is a valid transition kernel
  - But: error propagation might be tricky

Idea 2: **Model-free** Reinforcement Learning

- Do we need the model? Can we learn the Q function directly?

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

Idea 2: **Model-free** Reinforcement Learning

P has S^2A parameters

- Do we need the model? Can we learn the Q function directly?
  - **How many free parameters are there to represent the Q-function?**

Q only has SA parameters

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Idea 2: **Model-free** Reinforcement Learning

- Do we need the model? Can we learn the Q function directly?
  - **How many free parameters are there to represent the Q-function?**

- Recall: Policy iterations

$$\pi_0 \to^E V^{\pi_0} \to^I \pi_1 \to^E V^{\pi_1} \to^I \ldots \to^I \pi^* \to^E V^*$$

  - Maybe we can do policy evaluation / value iterations without estimating the model?

Model-free method is yet another algorithm design principle

- We use function approximation on Q directly

$$Q \in \mathcal{H} : S \times A \to \mathbb{R}$$

- Function classes

- Induced policy class

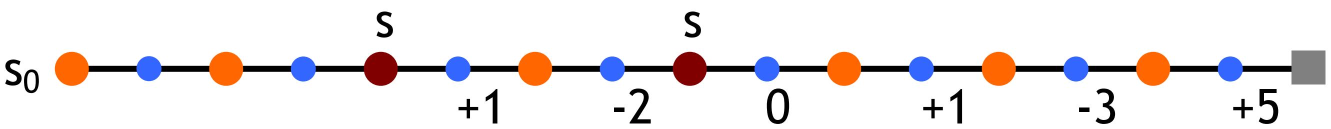
$$\Pi_{\mathcal{H}} := \{ \arg\max_{a,s} h(a,s) \mid \forall h \in \mathcal{H} \}$$

Monte Carlo Policy Evaluation (Prediction)

- want to estimate $V^{\pi}(s)$

  = expected return starting from s and following π

  - estimate as average of observed returns in state s
- We can execute the policy π
- first-visit MC
  - average returns following the first visit to state s



$G_1(s) = +2$

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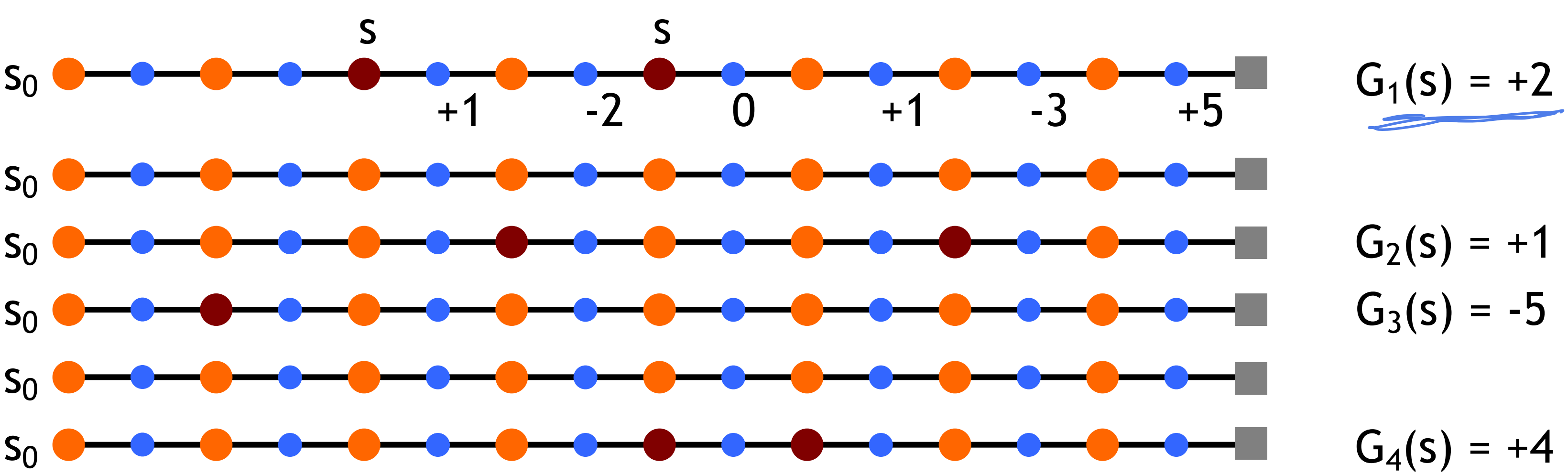
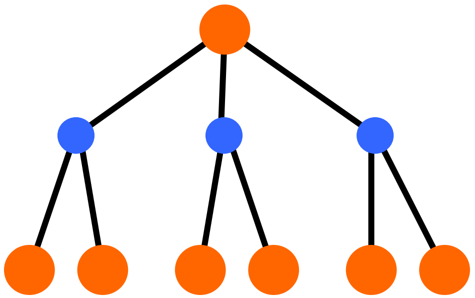
$G_1 = 0 + \gamma \cdot 1 + 0 + \gamma^3(-2) + \ldots$



$V^{\pi}(s) \approx (2 + 1 - 5 + 4)/4 = 0.5$

Monte Carlo Policy Optimization (Control)



- V^π not enough for policy improvement
  - need exact model of environment
- estimate $Q^\pi(s,a)$

$$\pi'(s) = \arg\max_a Q^\pi(s,a)$$

- MC control

$$\pi_0 \longrightarrow^E Q^{\pi_0} \longrightarrow^I \pi_1 \longrightarrow^E Q^{\pi_1} \longrightarrow^I \dots \longrightarrow^I \pi^* \longrightarrow^E Q^*$$

  - update after each episode

- Two problems
  - greedy policy won't explore all actions
  - Requires many independent episodes for the estimated value function to be accurate.

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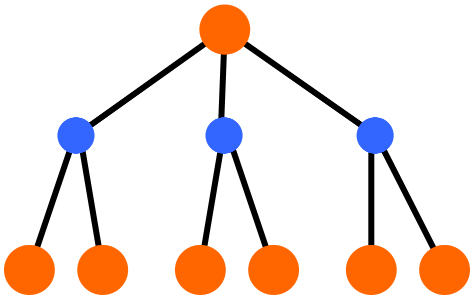

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  - update after each episode

- Two problems
  - greedy policy won't explore all actions eps-greedy, or bonus design.
  - Requires many independent episodes for the estimated value function to be accurate.

Improved Monte-Carlo Q-function estimate using Bellman equations

- Recall:

$$Q^\pi(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \sum_{a'} \pi(a'|s') Q^\pi(s',a')]$$

$$Q^\pi(s,a) = r^\pi(s,a) + \gamma \mathbb{E}_{s' \sim P(s'|s,a)}[V^\pi(s')]$$

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- We can use the empirical (Monte Carlo) estimate.

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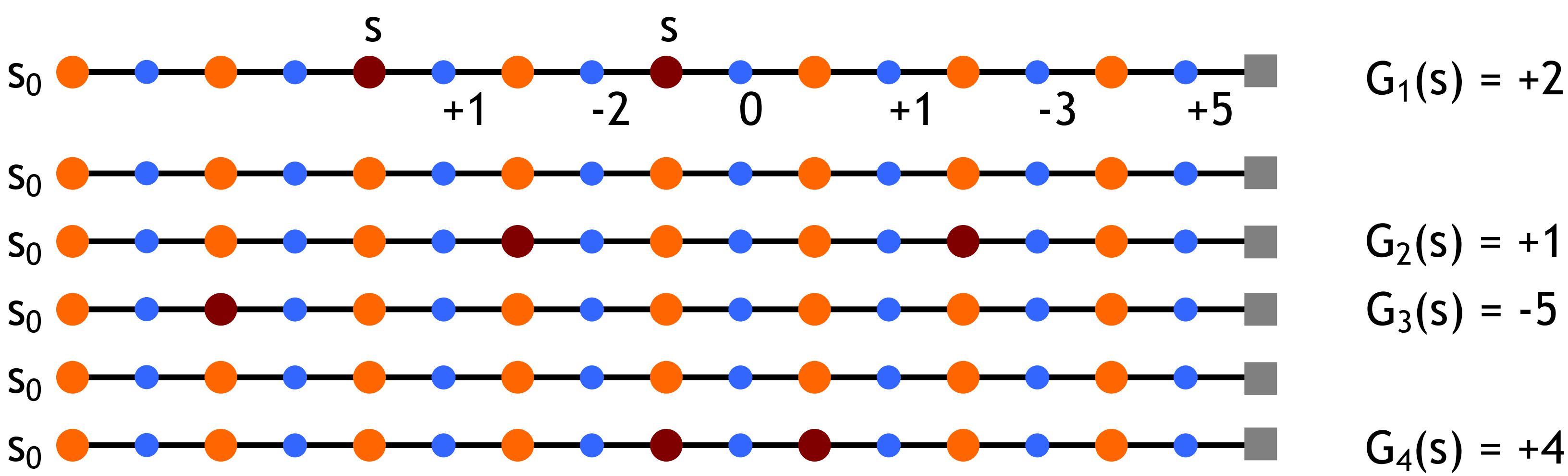
*No need to estimate P(s' | s,a) or r(s,a,s') as intermediate steps.
*Require only O(SA) space, rather than O(S^2A)

Online averaging representation of MC



$$V^{\pi}(s) \approx (2 + 1 - 5 + 4)/4 = 0.5$$

- Alternative, *online averaging* update

$$V(S_t) \leftarrow V(S_t) + \alpha \Big[G_t - V(S_t) \Big], \quad \text{where } \alpha = 1/N_{S_t}$$

DP + MC = Temporal Difference Learning

- **Monte Carlo** $V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big],$

DP + MC = Temporal Difference Learning

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$$\mathbb{E}_\pi[G_t] = \mathbb{E}_\pi[R_t|S_t] + \gamma V^\pi(S_{t+1})$$

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- TD-Policy evaluation

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\Big]$$

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- TD-Policy evaluation

Bootstrapping!

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\Big]$$

Bootstrap's origin

- "The Surprising Adventures of Baron Münchausen"
  - Rudolf Erich Raspe, 1785

- In statistics: Brad Efron's resampling methods
- In computing: Booting…
- In RL: It simply means TD learning

TD policy optimization (TD-control)

- SARSA (On-Policy TD-control)
  - Update the Q function by bootstrapping Bellman Equation

$$Q(S, A) \leftarrow Q(S, A) + \alpha\left[R + \gamma Q(S', A') - Q(S, A)\right]$$

  - Choose the next A' using Q, e.g., eps-greedy.

- Q-Learning (Off-policy TD-control)
  - Update the Q function by bootstrapping Bellman Optimality Eq.

$$Q(S, A) \leftarrow Q(S, A) + \alpha\left[R + \gamma \max_a Q(S', a) - Q(S, A)\right]$$

  - Choose the next A' using Q, e.g., eps-greedy, or any other policy.

Remarks:

- These are **proven to converge** asymptotically.
- Much more data-efficient in practice, than MC.
- Regret analysis is still active area of research.

Advantage of TD over Monte Carlo

- Given a trajectory, a roll-out, of T steps.
  - MC updates the Q function only once
  - TD updates the Q function (and the policy) T times!

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Remark: This is the same kind of improvement from Gradient Descent to Stochastic Gradient Descent (SGD).

Model-free vs Model-based RL algorithms

- Different function approximations

- Different space efficiency

- Which one is more statistically efficient?
  - More or less equivalent in the tabular case.
  - Different challenges in their analysis.