Observability of Lattice Graphs

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Abstract

We consider a graph observability problem: *how many edge colors are needed for an unlabeled graph so that an agent, walking from node to node, can uniquely determine its location from* just *the observed color sequence of the walk?* Specifically, let $G(n, d)$ be an edge-colored subgraph of d-dimensional (directed or undirected) lattice of size $n^d = n \times n \times \cdots \times n$. We say that $G(n, d)$ is t-*observable* if an agent can uniquely determine its current position in the graph from the color sequence of any t-dimensional walk, where the dimension is the number of different *directions* spanned by the edges of the walk. A walk in an undirected lattice $G(n, d)$ has dimension between 1 and d, but a directed walk can have dimension between 1 and $2d$ because of two different orientations for each axis.

We derive bounds on the number of colors needed for t-observability. Our main result is that $\Theta(n^{d/t})$ colors are both necessary and sufficient for t-observability of $G(n, d)$, where d is considered a constant. This shows an interesting dependence of graph observability on the *ratio* between the dimension of the lattice and that of the walk. In particular, the number of colors for *full-dimensional* walks is $\Theta(n^{1/2})$ in the directed case, and $\Theta(n)$ in the undirected case, *independent* of the lattice dimension. All of our results extend easily to non-square lattices: given a lattice graph of size $N = n_1 \times n_2 \times \cdots \times n_d$, the number of colors for t-observability is $\Theta(\sqrt[t]{N})$.

1 Introduction

Imagine an agent or a particle moving from node to node in an edge-colored graph. During its walk, the agent only learns the colors of the edges it traverses. If after a sufficiently long walk, the agent can uniquely determine its current node, then the graph is called observable. Namely, an edge-colored graph is *observable* if the current node of an arbitrary but sufficiently long walk in the graph can be uniquely determined simply from the color sequence of the edges in the walk [7]. A fundamental problem in its own right, graph observability also models the "localization" problem in a variety of applications including monitoring, tracking, dynamical systems and control, where only partial or local information is available for tracking [1, 2, 4, 6, 9, 10, 11, 12]. This is often the case in networks with hidden states, anonymized nodes, or information networks with minimalistic sensing: for instance, observability quantifies how little information leakage (link types) can enable precise tracking of users in anonymized networks. As another contrived but motivating scenario, consider the following problem of robot localization in minimally-sensed environment.

A low-cost autonomous robot must navigate in a physical environment using its own odometry (measuring distance and angles). The sensor measurements are noisy and inaccurate, the robot invariably accumulates errors in the estimates of its own position and pose, and it must perform periodic relocalizations. Without global coordinates (GPS), unique beacons, or other expensive navigation aids, this is a difficult problem in general. In many situations, however, an approximate relocalization is possible through inexpensive and ubiquitous sensors, such as those triggered by passing through doors (beam sensors). Privacy or cost concerns, however, may prevent use of *uniquely identifiable* sensors on all doors or entrances. Instead, sensors of only a few types (colors) are used as long as we can localize the robot from its *path history.* Formally, the robot's state space (positions and poses) is a subset of $\mathcal{R}^d$, which is partitioned into $N = n \times n \times \cdots \times n$ cells, each representing a desired level of localization accuracy in the state space. We assume access to a minimalist binary sensor that detects the robot's state transition from one cell to another. The adjacency graph of this partition is a d-dimensional lattice graph, and our robot localization problem is equivalent to observability, where edge colors correspond to different type of sensors on cell boundaries.

Problem Statement. With this general motivation, we study the observability problem for (subgraphs of) d-dimensional lattices (directed and undirected), and derive upper and lower bounds on the minimum number of colors needed for their observability. Lattices, while lacking the full power of general graphs, do provide a tractable but non-trivial setting: their uniform local structure and symmetry makes localization challenging but, as we will show, their regularity allows coding schemes to reconstruct even relatively short walks. We begin with some definitions to precisely formulate our problem and the results.

Let $G(n, d)$ denote a subgraph of d-dimensional regular square lattice of size $N = n^d$. We want to color the edges of $G(n, d)$ so that a walk in the graph can be localized based solely on the colors of the edges in the walk. The starting node of the walk is not known, neither is any other information about the walk except the sequence of colors of the edges visited by the walk. By localizing the walk, we mean that its current node can be uniquely determined.

When the graph is directed, an edge has both a natural dimension and an *orientation*: dimension is the coordinate axis parallel to the edge, and orientation is the direction along that axis (positive or negative). There are $2d$ distinct orientations in a directed lattice, two for each axis. When the graph is undirected, each edge has a dimension but not an orientation. The lack of orientation makes the observability of undirected graphs more complicated: in fact, *even if all edges have distinct colors*, the agent can create arbitrarily long walks by traversing back and forth on a single edge so that one cannot determine the current vertex from the color sequence alone. Nevertheless, we show that any undirected walk that includes *at least two distinct edges* can be observed (localized).

Our Results. We show that lattice observability depends not on the *length* of the walk, but rather on the number of different directions spanned by the walk. In order to discuss both directed and undirected graphs without unnecessary notational clutter, we use a common term *dimension* to count the number of different directions: it is the number of distinct edge orientations in a directed walk and the number of distinct axes spanned by an undirected walk. In particular, we say that a *directed* walk W has *dimension* t if it includes edges with t distinct orientations, for $t \leq 2d$. An *undirected* walk W has *dimension* t if it includes edges parallel to t distinct axes, for $t \leq d$. We say that $G(n, d)$ is *t-observable* if an agent, walking from node to node, can uniquely determine its current position from the color sequence of any t-dimensional walk.

Our main result shows that $O(\sqrt[t]{N})$ colors are always sufficient for t-observability of (directed or undirected) lattice graph $G(n, d)$, where $N = n^d$ is the size of the graph and d is assumed to be a constant. That is to say for every t, there exists a coloring scheme that uses $O(\sqrt[t]{N})$ colors such that every t dimensional walk in graph $G(n, d)$ is observable. A matching lower bound easily follows from a simple counting argument. The upper bound proof uses a combinatorial structure called *orthogonal arrays* to construct the observable color schemes, which may have other applications as well. We prove the results for subgraphs of square lattices, but the bounds are easily extended to rectangular lattices: given a lattice graph of size $N = n_1 \times n_2 \times \cdots \times n_d$, the number of colors for t-observability is $\Theta(\sqrt[t]{N})$.

An interesting implication is that for full-dimensional walks, the number of colors is independent of d: it is $O(n^{1/2})$ for directed, and $O(n)$ for undirected, graphs.

Related Work. The graph observability problem was introduced by Jungers and Blodel [7], who show that certain variations of the problem are *NP*-complete in general *directed* graphs. They also present a polynomial-time algorithm for deciding if an edge-colored graph G is observable based on the fact that the following two conditions are necessary and sufficient: (i) G does not have an *asymptotically reachable* node u with two outgoing edges of the same color, where an asymptotically reachable node is one reachable by arbitrarily long paths, and (ii) G does not have two cycles with the same edge color sequence but different node orders. These results hold only for directed graphs, and not much is known for observability of *undirected* graphs. By contrast, our results show a universal (extremal) bound that holds for all lattices of size $N = n^d$, and apply to both directed and undirected graphs.

The graph observability is related to a number of other concepts in dynamical systems, including *trackable graphs* where the goal is to detect and identify dynamical processes based on a sequence of sensor observations [4], discrete event systems where the goal is to learn the state of the process based purely on the sequence (colors) of the transitions [11], and the *local automata* where a finite state automaton is called (d, k)-local if any two paths of length k and identical color sequence pass through the same state at step d [10]. The primary objectives, however, in those papers are quite different from the combinatorial questions addressed in our paper.

2 Definitions and the Problem Statement

A *d-dimensional lattice graph* is one whose drawing in Euclidean d-space forms a regular tiling. Specifically, such a graph of size $N = n^d$ has nodes at the integer-valued points $(x_1, x_2, \ldots, x_d)$, for $x_i \in \{0, 1, \ldots, n-1\}$. Two nodes can be connected if their distance is one: in undirected graphs, there is at most one such edge, and in directed graphs, there can be two edges with opposite orientations. (Our results hold for any subgraph of the lattice, and do not require the full lattice.) In this and the following three sections, we focus on *directed* graphs only, and return to undirected graphs in Section 6. We use the notation $G(n, d)$ for a directed lattice graph of size n^d. We say a directed walk W in $G(n, d)$ is *t-dimensional* if it includes edges with t distinct orientations, for $t \leq 2d$. An edge colored graph $G(n, d)$ is called *t-observable* if an agent can uniquely determine its current position in the graph from the observed edge color sequence of *any* directed walk of dimension t, for $t \leq 2d$. Figure 1 below show a 4-observable graph (1a) and a non-observable graph 1(b). The main focus of our paper is to derive bounds on the minimum number of colors that suffice for t-observability of $G(n, d)$. We begin with a few basic definitions and preliminaries.

The embedding of the lattice graph $G(n, d)$ induces a total order $\preceq$ on the nodes. Let $u = (x_1, x_2, \ldots, x_d)$ and $u' = (x'_1, x'_2, \ldots, x'_d)$ of two nodes in the graph. Then, we say that $u \preceq u'$ if u precedes u' in the coordinate-wise lexicographic ordering. That is, $u \preceq u'$ if either $x_1 < x'_1$, or $x_1 = x'_1, \ldots, x_i = x'_i$ and $x_{i+1} < x'_{i+1}$, for some $1 \leq i < d$. Given a directed walk in G, there is a unique *minimum node* under this total order, which we call the *root* of the walk. The node-ordering also allows us to associate edges with nodes. If $e = (u, v)$ is a directed edge, then we say that e is *rooted* at u if $u \preceq v$, and at v otherwise. (We remark that the root of an edge is unrelated to its orientation: it simply allows us to associate edges to nodes in a unique way.) Thus, for any node u in G, at most $2d$ edges may be rooted at u: at most two directed edges in each dimension for which u is minimum under the $\preceq$ order. In Figure 1, for example, the edges $(1, 2)$ and $(4, 1)$ are both rooted at node 1, while both $(6, 5)$ and $(5, 8)$ are rooted at 5. The walk $(5, 4, 1, 2, 5, 8)$ is rooted at node 1.

Each edge of the graph G also has a natural orientation: it is directed either in the positive or the negative direction along its axis. To be able to refer to this directionality, we call an edge *j-up-edge*

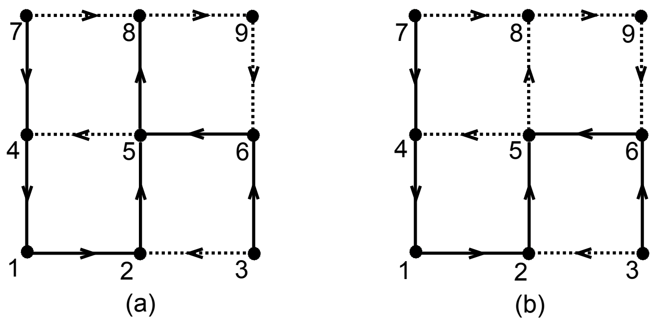

Figure 1: Two nearly identical 2-colored lattice graphs, one observable (on left), the other non-observable (on right). The two colors are shown as *solid* (S) and *dashed* (D). Only the color of edge $(5, 8)$ differs in the two graphs. In (b), the color sequence $(SSSD)^*$ can lead to either node 4 or 8, making it non-observable. In (a), any walk of dimension 4 is observable.

(resp., *j-down-edge*) if it has *positive* (resp. negative) orientation in jth dimension. In Figure 1, for instance, the edge $(5, 8)$ is the y-up-edge rooted at 5, and $(6, 5)$ is the x-down-edge at 5.

3 A Lower Bound for t-Observability

Theorem 1. *A directed lattice $G(n, d)$ requires at least $(n/2)^{d/t}$ colors for t-observability in the worst-case, for any $t \leq 2d$.*

Proof. Assume, without loss of generality, that n is even. The nodes of $G(n, d)$ have coordinates of the form $(x_1, x_2, \ldots, x_d)$, with $x_j \in \{0, 1, \ldots, n-1\}$, for all $j = 1, 2, \ldots, d$. Consider $(\frac{n}{2})^d$ unit d-cubes rooted at all the *even* nodes, namely, $(2x_1, 2x_2, \ldots, 2x_d)$, with $x_j \in \{0, 1, \ldots, n/2\}$, for $j = 1, 2, \ldots, d$. These unit cubes are pairwise edge-disjoint. We assign orientations to the some of the edges to create many t-dimensional walks, and then use a counting argument to lower bound the number of colors needed for the t-observability of this graph. See Figure 2 for an illustration. Consider a prototypical

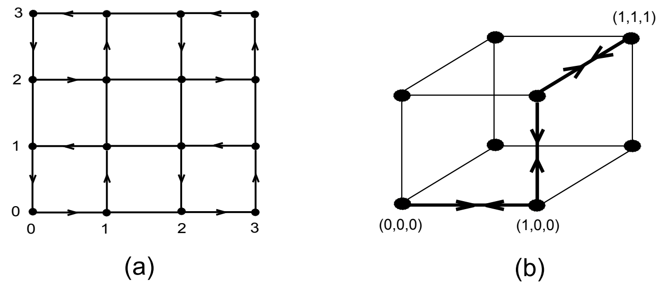

Figure 2: Illustration of the lower bound. The left figure (a) shows the *even* 2-cubes and orientation of walks. The right figure (b) shows how to orient a 6-dimensional walk for $d = 3$.

copy of a d-cube, with opposite corners at $u = (0, 0, \ldots, 0)$ and $v = (1, 1, \ldots, 1)$. We construct a t-dimensional directed walk of length t, as follows. Starting at u, for the first $\min\{t, d\}$ steps, take the j-up-edge in step $j = 1, 2, \ldots, t$; for the remaining $(t - d)$ steps, take the $(2d - t + j)$-down-edges, for $j = 1, 2, \ldots, t - d$. This construction assigns directions to $t \leq 2d$ edges of the d-cube; the remaining edges can be directed arbitrarily. (Figure 2b illustrates the construction for $d = 3$ and $t = 6$.) By repeating the construction at all d-cubes rooted at the even nodes of $G(n, d)$, we get $(n/2)^d$ disjoint t-dimensional walks, which must have pairwise distinct color sequences for t-observability. Since k

colors can produce at most k^t distinct color sequences of length t, the minimum number of colors k satisfies $k^t \geq (\frac{n}{2})^d$, from which it follows that $k \geq (n/2)^{d/t}$. It is easy to see that this argument holds for undirected walk as well, where we just have to ensure that $t \leq d$. This completes the proof. $\square$

We now describe the main result of the paper, an upper bound for t-observability. In order to build some intuition for the proof, and explain the coloring scheme, we first consider a much simpler special case: the 2-dimensional lattice $G(n, 2)$ and full-dimensional walks, namely, $t = 4$, where we show that $O(\sqrt{n})$ colors suffice. While the coloring and the decoding techniques for the general case are somewhat different, this special case is useful to explain the main ideas.

4 Observability in 2-Dimensional Lattices

In discussing the two-dimensional lattice, we name the two coordinate axes x and y, instead of x_1, x_2. Similarly, we use the more natural and visual use of left-right and up-down for the directionality of edges; i.e., we say left (resp. right) edge instead of 1-up (resp. 1-down) edge, and up (resp. down) edge instead of 2-up (resp. 2-down) edge. We begin with a discussion of our coloring scheme, and then show its correctness. We will use 4 blocks of colors, one each for *left, right, up* and *down* types of edges, where each block has $\lceil\sqrt{n}\rceil$ colors. The coloring depends on the position of the root node u associated with the edge, and uses the quotient and the remainder of u's coordinates *modulo* $\lceil\sqrt{n}\rceil$. We use the notation $x \div m$ to denote the quotient, and $x \bmod m$ to denote the remainder.

Specifically, consider a node $u = (x_u, y_u)$ in $G(n, 2)$. There are at most 4 edges rooted at u: a right outgoing edge, a left incoming edge, an up outgoing edge, and a down incoming edge. The following algorithm assigns colors to all edges *rooted* at node $u = (x_u, y_u)$.

Color2(u):

- If e is the outgoing *right* edge of u, give it color $(x_u \div \lceil\sqrt{n}\rceil)$;
  if it is outgoing *up* edge, give it color $(y_u \div \lceil\sqrt{n}\rceil) + 2\lceil\sqrt{n}\rceil$.

- If e is the incoming *left* edge of u, give it color $(x_u \bmod \lceil\sqrt{n}\rceil) + \lceil\sqrt{n}\rceil$;
  if it is incoming *down* edge, give it color $(y_u \bmod \lceil\sqrt{n}\rceil) + 3\lceil\sqrt{n}\rceil$.

The following lemma is easy.

Lemma 1. *All edges of $G(n, 2)$ are colored, using at most $4\lceil\sqrt{n}\rceil$ colors. No two outgoing edges of a node are assigned the same color.*

A simple but important fact throughout our analysis is the following *tracing lemma.*

Lemma 2. *[Tracing Lemma] Let W be a walk in $G(n, 2)$ under the coloring scheme* Color2. *Fixing the position of any node of W leads to a* unique *embedding of W in $G(n, 2)$.*

Proof. In our coloring scheme, the colors are grouped by the *direction* of edges (left, right, up and down), and so given the color of an edge, we know its direction. Once a node of the walk is fixed, all subsequent (and preceding) edges are uniquely mapped in the lattice graph. $\square$

By the Tracing Lemma, our color sequence uniquely specifies the "trace" (or, shape) of the walk's embedding, and once we localize any node of the walk, we can determine the embedding of the entire walk. Thus, the main problem, which will consume the rest of the paper, is to decode the position of one node of the walk from the color sequence. For $G(n, 2)$ and $t = 4$, we will use the following simple lemma, which the more complex coloring scheme of Section 5 *does not* need.

Lemma 3. *[Pairing Lemma] Suppose a directed walk W in $G(n,d)$ contains both up- and down-edges for some dimension j. Then W must contain a j-up and a j-down edge that are both rooted at vertices with the same jth coordinate.*

Proof. Mark each edge of W parallel to the jth axis either $+$ or $-$ depending on whether it is an up- or a down-edge. Since W contains both a j-up and a j-down edge, the sign changes at least once. Assume without loss of generality that it changes from $+$ to $-$, with e and f being the edges associated with them. None of the (unmarked) edges between e and f are parallel to the jth axis, so both e and f project to the same unit interval $(x_j, x_j + 1)$ on the jth axis. By convention, both e and f are rooted at vertices whose jth coordinate is x_j. This completes the proof. $\square$

We can now explain our decoding scheme.

Lemma 4. *Suppose $G(n,2)$ is colored using* Color2*, and W is a 4-dimensional walk in this graph. Then, the color sequence of W uniquely determines the position of the root node of W in the lattice.*

Proof. Let u be the root node of W, and let (x_u, y_u) be its coordinates. (These coordinates are precisely what we want to infer from the color sequence.) By the Pairing Lemma, since W includes all four orientations, it has two oppositely oriented edges e_1 (positive) and e'_1 (negative), both parallel to the x-axis and rooted at vertices with the same x coordinate. Let $v = (x_1, y_1)$ and $v' = (x_1, y'_1)$, respectively, be the root vertices of e_1 and e'_1. By the color assignment, e_1 and e'_1, respectively, receive colors $(x_1 \div \lceil\sqrt{n}\rceil)$ and $(x_1 \bmod \lceil\sqrt{n}\rceil) + \lceil\sqrt{n}\rceil$, which together are sufficient to uniquely calculate x_1. Since the edge colors uniquely determine the edge directions (dimension and orientation), we can trace W from e_1 to find the correct value of x_u, the x coordinate of the root node u. Similarly, since W also includes edges with both y-orientations, we can calculate u_y, thus uniquely localizing the root node u. This completes the proof. $\square$

Theorem 2. *$O(n^{1/2})$ colors suffice for 4-observability of a directed lattice $G(n,2)$.*

Proof. By Lemma 4, the color sequence of W uniquely determines the position of W's minimum node (root) in the lattice. Once u is localized, the tracing lemma can construct the unique embedding of W in $G(n,2)$, localizing all other nodes, including the current node. $\square$

5 t-Observability of Directed Lattices

We now describe the general coloring scheme for t-observability of $G(n,d)$, for any fixed d and all $t \le d$. The scheme uses a tool from combinatorial design, called *orthogonal array.* We first describe our orthogonal array construction, and then explain its application to observability.

5.1 Orthogonal Arrays

Let σ, t, d be positive integers, with $t \le d$. The parameters t and d are in fact the walk and the lattice dimensions, while σ is chosen as the smallest *prime* larger than $n^{d/t}$. (There always exists a prime between m and $2m$, for any integer m, and therefore $\sigma < 2n^{d/t}$.) Our choice of σ ensures that $\sigma^t \ge n^d$, which will be useful later because each row of the array is used to assign colors to the edges rooted at a distinct node of $G(n,d)$. An (σ,t,d)-*orthogonal array* $\mathcal{A}$ is an array of size $\sigma^t \times d$ satisfying the following two properties:

- The entries of $\mathcal{A}$ are integers from the set $\{0, 1, \ldots, \sigma - 1\}$, and

- For any choice of t columns in $\mathcal{A}$, the rows (ordered t-tuples) are unique.

Several methods for constructing orthogonal arrays are known [5, 3]. Our construction uses polynomials of degree less than t over the Galois field $GF(\sigma) = \{0, 1, \ldots, \sigma-1\}$. In particular, consider the set of all polynomials of degree less than t. Each such polynomial can be written as $P(x) : \sum_{i=1}^{t} a_i x^{t-i}$, with coefficients a_i ranging over $GF(\sigma)$. (Throughout the paper, when we use the word polynomial, we always mean these polynomials over $GF(\sigma)$.) There are exactly σ^t such polynomials, and we can order them using the lexicographic order of their coefficients. More specifically, let $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$ be two t-tuples from $GF(\sigma)$, and let P and P' denote the polynomials associated with these coefficients. Then, $P \preceq P'$ iff $(a_1, a_2, \ldots, a_t) \preceq (b_1, b_2, \ldots, b_t)$ under the lexicographic order. In particular, the polynomial for $(0, 0, \ldots, 0)$ is the first element in this order, and the polynomial for $(\sigma-1, \sigma-1, \ldots, \sigma-1)$ the last.

We will frequently need this ordering of the polynomials, and so for ease of reference, let us define *p-index*, which gives the unique position of a polynomial in the ordered list. Specifically, if a polynomial has coefficients $(a_1, a_2, \ldots, a_t)$, then its p-index equals $\sum_{i=1}^{t} a_i \sigma^{t-i}$. We can now describe how to construct our orthogonal array. The (i, j) entry of the array is defined as follows:

$$\mathcal{A}_{ij} \;=\; P_i(j) \bmod \sigma, \tag{1}$$

where $P_i(x)$ is the polynomial whose p-index is i, for $0 \le i < \sigma^t$, and the array entry is the evaluation of this polynomial at $x = j \pmod{\sigma}$, where $1 \le j \le d$. More precisely, if the ith polynomial has coefficients $(a_1, a_2, \ldots, a_t)$, then

$$\mathcal{A}_{ij} \;=\; a_1 j^{t-1} + a_2 j^{t-2} + \cdots + a_{t-1} j + a_t \pmod{\sigma}$$

In Figure 3, we show an example of the orthogonal array constructed by our scheme for $(\sigma, 2, d)$.

index	1	2	...	d
0	0	0	...	0
1	1	1	...	1
...	...	...	...	...
$\sigma-1$	$\sigma-1$	$\sigma-1$	...	$\sigma-1$
σ	1	2	...	d
$\sigma+1$	2	3	...	$d+1$
...	...	...	...	...
$2\sigma-1$	0	1	...	$d-1$
...			...	
$\sigma^2-\sigma$	$\sigma-1$	$\sigma-2$	...	$\sigma-d$
$\sigma^2-\sigma+1$	0	$\sigma-1$	...	$\sigma-d+1$
...	...	...	...	...
σ^2-1	$\sigma-2$	$\sigma-3$	...	$\sigma-d-1$

Figure 3: A $(\sigma, 2, d)$ orthogonal array. For any two columns, all rows (ordered tuples) are distinct. The ith row entries are computed from the polynomial $a_1 x + a_2$, where i is the p-index associated with (a_1, a_2). Row 0 has all zeroes because it belongs to the polynomial $0x + 0$, which evaluates to 0 for $j = 1, 2, \ldots, d$. The last row belongs to the polynomial $(\sigma-1)x + (\sigma-1)$, with coefficient vector $(\sigma-1, \sigma-1)$. Its first entry, for $j = 1$, is $(\sigma-1)1 + (\sigma-1) \equiv 2(\sigma-1) \equiv \sigma-2 \pmod{\sigma}$.

The following lemma shows that the construction is valid.

Lemma 5. *The array $\mathcal{A}$ constructed by Equation (1) is an orthogonal array.*

Proof. The array has dimensions $\sigma^t \times d$, and its entries come from the set $\{0, 1, \ldots, \sigma - 1\}$, by construction. Thus, we only need to show that within any t columns of $\mathcal{A}$, all rows are distinct. We prove this by contradiction. Let $j_1, j_2, \ldots, j_t$, for $1 \leq j_1 < j_2 < \cdots < j_t \leq d$ be any t columns of $\mathcal{A}$, and suppose that two different rows with indices i_1 and i_2, for $i_1 < i_2$ are identical over these columns. Let $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$ denote the coefficients corresponding to polynomials used for rows i_1 and i_2, respectively. Then, by Equation (1), the polynomial used to construct entries of row i_1 is $P_{i_1} : a_1 x^{t-1} + a_2 x^{t-2} + \cdots + a_t$, and the polynomial used to construct entries of row i_2 is $P_{i_2} : b_1 x^{t-1} + b_2 x^{t-2} + \cdots + b_t$. If these rows are identical, then we must have $P_{i_1}(j_k) \equiv P_{i_2}(j_k) \pmod{\sigma}$, for $k = 1, 2, \ldots, t$. This implies that $j_1, j_2, \ldots, j_t$ are t distinct roots of the equation $P_{i_1}(x) - P_{i_2}(x) \pmod{\sigma}$, which is not possible since this polynomial has degree $t - 1$ and at most $t - 1$ distinct roots.[1] Therefore, the rows i_1 and i_2 are not identical over the chosen t columns, proving that $\mathcal{A}$ is an orthogonal array. $\square$

Note that not all orthogonal arrays could help us on coloring. Here we carefully constructed $\mathcal{A}$ such that the regular structure of $\mathcal{A}$ helps us map colors to edges of the lattice graph in such a way that a small number of appropriate colors can be used to determine the position of a node in the lattice.

5.2 Color Assignment using the Orthogonal Array

We begin by indexing the nodes of $G(n, d)$ in the lexicographic rank order of their coordinates. Specifically, a node u with coordinates $(x_1, x_2, \ldots, x_d)$ has *node rank* $r(u) = \sum_{i=1}^{d} x_i n^{d-i}$, where recall that each $x_j \in \{0, 1, \ldots n - 1\}$. This ordering assigns rank 0 to the origin $(0, 0, \ldots, 0)$, and rank $n^d - 1 = (n-1) \sum_{i=1}^{d} n^{d-i}$ to the anti-origin $(n-1, n-1, \ldots, n-1)$. Our orthogonal array $\mathcal{A}$ satisfies $\sigma^t \geq n^d$, and thus we can uniquely associate the node with rank i to the ith row of $\mathcal{A}$.

Let u be a node of $G(n, d)$ whose rank (lexicographic order) is $r(u) = i$, where $0 \leq i < n^d$. Then, we use the ith row of $\mathcal{A}$ to assign to colors to the edges rooted at u. The rules for assigning colors are described in the following algorithm. The algorithm uses 2^t groups of disjoint colors $C_0, C_1, \ldots, C_{2^t - 1}$, each with $2d \times \sigma$ colors. Each edge's color depends on its orientation, so we assign integers $1, 2, \ldots, 2d$ to the $2d$ orientations. (Any such labeling will suffice but, for the sake of concreteness, we may number the j-up orientation as j and the j-down orientation as $j + d$, for $1 \leq j \leq d$.)

ColorD(u):

- Let $(a_1, a_2, \ldots, a_t)$ be the unique coefficient vector of a polynomial whose p-index i equals the rank of u, namely, $i = r(u)$.
- Let $m = \sum_{k=1}^{t} (a_k \bmod 2) 2^{t-k}$.
- If e has orientation j, then give it color $A_{ij} + (j - 1)\sigma$ from the color group C_m.

The use of disjoint color groups C_m is required to allow unique decoding of the position of a node in the lattice from the edge colors of the walk. (These groups are used critically in the proof of Lemma 10.) The following two lemmas follow easily from the color assignment.

[1] Finite fields belong to unique factorization domains, and therefore a polynomial of order r over finite fields has a unique factorization, and at most r roots.

Lemma 6. COLORD *assigns colors to all the edges of $G(n, d)$, uses $O(n^{d/t})$ colors, and no two outgoing edges of a node receive the same color.*

Lemma 7. *[d-Dimensional Tracing Lemma] Let $G(n, d)$ be colored using the scheme* COLORD*, and let W be a walk in $G(n, d)$. Fixing the position of any node of W leads to a* unique *embedding of W in $G(n, d)$.*

Lemma 8. *Let p_1, p_2, respectively, denote the p-indices of polynomials with coefficient vectors $(a_1, \ldots, a_t)$ and $(b_1, \ldots, b_t)$. Then, the quantity $A_{p_1j} - A_{p_2j}$ can be uniquely calculated, for any j, from the t coordinate* differences*, namely, $a_k - b_k$, for $k = 1, 2, \ldots, t$.*

Proof. By construction, $A_{p_1j} - A_{p_2j} \equiv P_{p_1}(j) - P_{p_2}(j) \pmod{\sigma} \equiv \sum_{k=1}^{t}(a_k - b_k)j^{t-k} \pmod{\sigma}$. □

If p_1 and p_2, where $p_1 > p_2$, are the p-indices of $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$, then by definition $p_1 = \sum_{k=1}^{t} a_k\sigma^{t-k}$ and $p_2 = \sum_{k=1}^{t} b_k\sigma^{t-k}$. Let $\ell = (p_1 - p_2)$ be the distance between these p-indices, and let $(c_1, c_2, \ldots, c_t)$, with $c_i \in GF(\sigma)$, be the coefficient vector that yields the p-index ℓ. Since $\ell = \sum_{k=1}^{t} c_k\sigma^{t-k}$, we can easily find each coefficient c_k from ℓ by modular arithmetic:

$$c_k = (\ell \bmod \sigma^{t-k+1}) \div \sigma^{t-k}.$$

The following two lemmas establish important properties of these coefficients, and are key to inferring *locations from distances*.

Lemma 9. *Let $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$ be coefficient vectors with p-indices p_1, p_2, for $p_1 > p_2$, respectively. Let $\ell = p_1 - p_2$, and suppose $(c_1, c_2, \ldots, c_t)$ is the coefficient vector of the polynomial with p-index ℓ. Then, for each $k = 1, 2, \ldots, t$, we either have $a_k - b_k \equiv c_k \pmod{\sigma}$, or we have $a_k - b_k \equiv c_k + 1 \pmod{\sigma}$.*

Proof. Because $\ell = p_1 - p_2$, we have $\ell = \sum_{k=1}^{t}(a_k - b_k)\sigma^{t-k}$. Perform a $\pmod{\sigma^{t-k+1}}$ operation on both sides of the equality: $(a_1 - b_1)\sigma^{t-1} + \cdots + (a_t - b_t) \equiv c_1\sigma^{t-1} + \cdots + c_t$. We get $(a_k - b_k)\sigma^{t-k} + \cdots + (a_t - b_t) \equiv c_k\sigma^{t-k} + \cdots + c_t \pmod{\sigma^{t-k+1}}$, which is equivalent to $(a_k - b_k - c_k)\sigma^{t-k} \equiv \sum_{i=k+1}^{t}(c_i - a_i + b_i)\sigma^{t-i} \pmod{\sigma^{t-k+1}}$. Because $a_i, b_i, c_i \in GF(\sigma)$, the right hand side clearly satisfies the following bounds:

$$-\sigma^{t-k} < \sum_{i=k+1}^{t}(c_i - a_i + b_i)\sigma^{t-i} < 2\sigma^{t-k}.$$

But since a_k, b_k, c_k on the left side are positive integers, there are only two feasible solutions: $\sum_{i=k+1}^{t}(c_i - a_i + b_i)\sigma^{t-i} \equiv 0 \pmod{\sigma^{t-k+1}}$ or $\sum_{i=k+1}^{t}(c_i - a_i + b_i)\sigma^{t-i} \equiv \sigma^{t-k} \pmod{\sigma^{t-k+1}}$. Therefore, we must have either $a_k - b_k - c_k \equiv 0 \pmod{\sigma}$, or $a_k - b_k - c_k \equiv 1 \pmod{\sigma}$. This completes the proof. □

The following lemma shows that while we cannot reconstruct the unknown coefficient vectors $(a_1, a_2, \ldots, a_t)$ and $(b_1, b_2, \ldots, b_t)$ from the edge colors, we can still compute enough information about their entries in the orthogonal array $\mathcal{A}$.

Lemma 10. *Let $G(n, d)$ be colored using the scheme* COLORD*, and let W be a walk in $G(n, d)$. Let e_1 and e_2 be two edges in this walk rooted, respectively, at u_1 and u_2 with ranks (lexicographic order) $r(u_1) = r_1$ and $r(u_2) = r_2$, with $r_1 < r_2$. Then, the difference $r_1 - r_2$ along with the colors of e_1 and e_2 are sufficient to compute $A_{r_1j} - A_{r_2j} \pmod{\sigma}$.*

Proof. Suppose that the colors of e_1 and e_2 belongs to color groups C_{m_1} and C_{m_2}, respectively. Let us consider the binary representations of m_1, m_2, namely, $m_1 = \sum_{k=1}^{t}(a_k \bmod 2)2^{t-k-1}$ and $m_2 = \sum_{k=1}^{t}(b_k \bmod 2)2^{t-k-1}$. If $(a_k - b_k) \equiv 0 \pmod 2$, then by Lemma 9 we can conclude that $(a_k - b_k) \equiv c_k \pmod \sigma$ if c_k is even; otherwise it equals $c_k + 1$. Similarly, $(a_k - b_k) \equiv 1 \pmod 2$ tells us that $(a_k - b_k) \equiv c_k \pmod \sigma$ if c_k is odd (and $c_k + 1$ otherwise). Because we can infer from the embedding whether $a_k > b_k$, we can calculate $a_k - b_k$ from $a_k - b_k \pmod \sigma$. Once we know all $a_k - b_k$, for $k = 1, 2, \ldots, t$ we can compute $A_{r_1 j} - A_{r_2 j} \pmod \sigma$ using Lemma 8. □

Lemma 11. *[Ranking Lemma] Let a lattice graph $G(n, d)$ be colored using the algorithm* ColorD*, and let u_1, u_2 be two nodes in a walk W of $G(n, d)$. We can calculate the difference of their ranks $r(u_1) - r(u_2)$ from just the color sequence of W.*

Proof. Using the colors of edges in W starting at u_1, we can trace the walk, and count the number of edges in each dimension. Suppose the number of j-up and j-down edges included in the walk from u_1 to u_2 are α_j and β_j, respectively. Then, the jth coordinate of u_2 differs from that of u_1 by precisely $\alpha_j - \beta_j$. We can, therefore, calculate $r(u_1) - r(u_2) = \sum_{j=1}^{d}(\beta_j - \alpha_j)n^{d-j}$. □

We are now ready to prove our main theorem.

Theorem 3. *$O(n^{d/t})$ colors suffice for t-observability of any directed lattice graph $G(n, d)$, for any fixed dimension d and $t \leq 2d$.*

Proof. We color the graph $G(n, d)$ using ColorD, and show how to localize a walk W of dimension t. Suppose $j_1, j_2, \ldots, j_t$, where $1 \leq j_1 < j_2 < \cdots < j_t \leq 2d$, are the t distinct edge orientations of W. Let e_1 be a j_1-oriented edge in W, rooted at a node u_1, and let $r(u_1) = r_1$ be the rank of u_1. Then, the color of e_1 can be used to uniquely determine the value $\mathcal{A}_{r_1 j_1}$.

Now, suppose the root of the walk W is the node u^*, with $r^* = r(u^*)$. We can compute the difference $r_1 - r^*$, by using the Ranking Lemma. By Lemma 10, this difference along with the color sequence of W gives us $\mathcal{A}_{r_1 j_1} - \mathcal{A}_{r^* j_1}$, from which we can calculate $\mathcal{A}_{r^* j_1}$. By repeating this argument for each of the t dimensions spanned by W, we can calculate $\mathcal{A}_{r^* j_1}, \mathcal{A}_{r^* j_2}, \ldots, \mathcal{A}_{r^*, j_t}$. By the property of orthogonal arrays, the ordered t-tuple $(\mathcal{A}_{r^* j_1}, \mathcal{A}_{r^* j_2}, \ldots, \mathcal{A}_{r^* j_t})$ is unique in $\mathcal{A}$. Therefore, the rank of the root node u^* can be uniquely determined, which in turn uniquely localizes u^* in the lattice graph $G(n, d)$. This completes the proof. □

6 Observability in Undirected Lattices

Observing a walk is more complicated in undirected graphs because edge colors fail to determine the direction of the walk. An undirected edge $e = (u, v)$ may be traversed in either direction by the walk, but reveals the same color. We say that a walk has dimensions t if it contains edges parallel to t distinct dimensions. A minor modification of the lower bound construction (Theorem 1) shows that $\Omega(n^{t/d})$ colors are necessary for t-observability of undirected lattices: there are $(n/2)^d$ undirected paths, each of length $t \leq d$, and k colors can disambiguate at most k^t paths, giving the lower bound. We now prove a matching upper bound for t-observability. As we mentioned earlier, there is one trivial walk that cannot be observed in undirected graphs: a walk that traverses a single edge back and forth. In this case, the current node cannot be determined from the color sequence. However, we show below that walks that include at least two distinct edges can always be observed.

6.1 Signs of Undirected Edges and an Auxiliary Coloring

Although the edges of $G(n,d)$ are undirected, a walk imposes an orientation on the edges it visits. To exploit this induced directionality, we introduce the notion of a *sign*. First, we consider a walk as a *sequence of nodes* $W = (u_0, u_1, \ldots, u_\ell)$, where $e_i = (u_{i-1}, u_i)$ is the ith edge in the walk. Thus, the observed sequence is the colors of edges $e_1, e_2, \ldots, e_\ell$. Next, recall that a node u with coordinates $(x_1, x_2, \ldots, x_d)$ has *node rank* $r(u) = \sum_{i=1}^{d} x_i n^{d-i}$. Now, consider an edge $e_i = (u_{i-1}, u_i)$, and assume that it is parallel to the dimension j. Then, it is easy to see that $r(u_i) - r(u_{i-1}) = \pm n^{d-j}$. We say that the *sign* of the edge e_i is *positive* if $r(u_i) - r(u_{i-1}) = +n^{d-j}$, and *negative* otherwise. (Intuitively, the sign is positive if the walk traverses the edge in the positive direction of the axis, and negative otherwise.)

We first show that the signs of the edges in a walk can be computed from a simple 3-coloring. Let $o = (0, 0, \ldots, 0)$ be the origin and $u = (x_1, x_2, \ldots, x_d)$ be a node of $G(n,d)$. Let $d_o(u) = \sum_i^d x_i$ be the length of the shortest path from the origin to u. Then, MOD3(o) colors the edges of $G(n,d)$ as follows. See Figure 4 for an example.

Assign color $m = d_o(u) \bmod 3$ *to all the edges rooted at node* u.

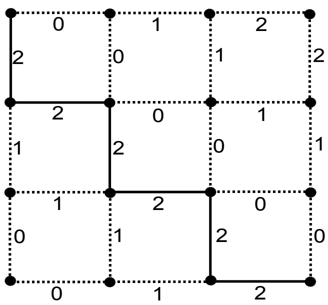

Figure 4: MOD3 coloring. A monochromatic walk using color 2 is shown as solid.

Lemma 12. *Let $G(n,d)$ be 3-colored using* MOD3(o)*, and let $W = (u_0, u_1, \ldots, u_\ell)$ be a walk with at least two distinct observed colors. Then we can compute the sign of all (u_{i-1}, u_i), for $i = 1, 2, \ldots, \ell$.*

Proof. Assume, without loss of generality, that the walk includes two consecutive edges $e_i = (u_{i-1}, u_i)$ and $e_{i+1} = (u_i, u_{i+1})$ with distinct colors c_i and c_{i+1} under MOD3(o). We observe the following relationship between the colors and signs. When e_i's sign is positive, namely, $d_o(u_{i-1}) = d_o(u_i) - 1$, we have $c_{i+1} = c_i + 1 \bmod 3$ if e_{i+1}'s sign is positive, and $c_{i+1} = c_i \bmod 3$ if e_{i+1}'s sign is negative. Similarly, when e_i's sign is negative, namely, $d_o(u_i) = d_o(u_{i-1}) + 1$, we have $c_{i+1} = c_i \bmod 3$ if e_{i+1}'s sign is positive, and $c_{i+1} = c_i - 1 \bmod 3$ otherwise. Because $c_i \neq c_{i+1} \mod 3$, we conclude that e_{i+1}'s sign is positive if $c_{i+1} = c_i + 1$, and negative otherwise. Once the sign of one edge in the walk is determined, we can repeat this process to infer the signs of all other edges. □

When the walk includes only edges of one color, we try MOD3 with $d - t + 1$ other choices of the *origin*. In particular, let MOD3(o_j) be the 3-coloring with respect to the origin $o_0 = o$ and origin $o_j = (0, 0, \ldots, n-1, \ldots, 0)$ with jth coordinate $n-1$ and zeroes everywhere else, for $j = 1, 2, \ldots, d-t+1$.

Lemma 13. *Given any t-dimensional walk $W = (u_0, u_1, \ldots, u_\ell)$ in $G(n,d)$ that visits at least two edges, there is a 3-coloring* MOD3(o_j) *for which W is* not *monochromatic, for $j = 0, 1, \ldots, d-t+1$.*

Proof. By the pigeon principle, a t-dimensional walk must contain at least one edge that parallels to one of the first $d-t+1$ axis. Assume, without loss of generality, that the walk includes two consecutive edges $e_i = (u_{i-1}, u_i)$ and $e_{i+1} = (u_i, u_{i+1})$ and edge e_i parallels to the jth dimension, $1 \le j \le d-t+1$. Let c_i^j denote the color for edge e_i using 3-coloring $\text{MOD3}(o_j)$. We show in Figure 5 the color of edges e_i and e_{i+1} are distinct under 3-coloring $\text{MOD3}(o_0)$ or $\text{MOD3}(o_j)$, namely, either $c_i^0 \ne c_{i+1}^0$ or $c_i^j \ne c_{i+1}^j$ holds. (1) When e_i roots at u_{i-1} and e_{i+1} roots at u_i, we have $c_{i+1}^0 = c_i^0 + 1 \bmod 3$. Similarly (2) when e_i roots at u_i and e_{i+1} roots at u_{i+1}, we have $c_{i+1}^0 = c_i^0 - 1 \bmod 3$. On the other hand, (3) when both e_i and e_{i+} roots at u_i, we have $c_{i+1}^j = c_i^j + 1 \bmod 3$. And at last, (4) when e_i roots at u_{i-1} and e_{i+1} roots at u_{i+1}, we have $c_{i+1}^j = c_i^j - 1 \bmod 3$. □

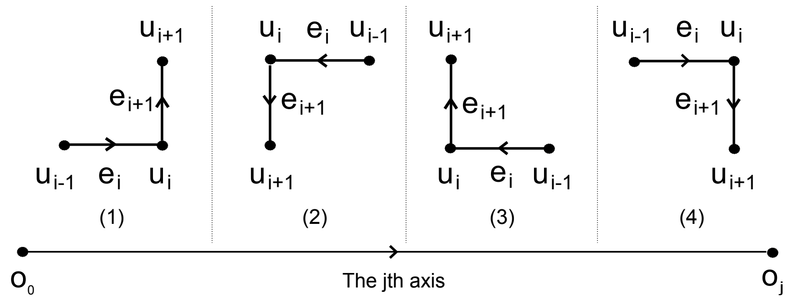

Figure 5: Demonstration of 4 situations in Lemma 13.

6.2 Coloring Scheme for Undirected t-Observability

Our final coloring scheme combines the orthogonal array based coloring with the MOD3 coloring. We use the orthogonal array $\mathcal{A}$ with $\sigma^t \ge n^d$, whose ith row is used to color edges rooted at node with rank i. The algorithm uses $2^t 3^{d-t+2}$ groups of disjoint colors, C_j, each with $d\sigma$ colors, where $0 \le j < 2^t 3^{d-t+2}$, and $\sigma = O(n^{d/t})$.

UNDIRCOLOR(u):

- Let $(a_1, a_2, \ldots, a_t)$ be the unique coefficient vector of a polynomial whose p-index i equals the rank of u, namely, $i = r(u)$. Let $(x_1, x_2, \ldots, x_d)$ be the coordinates of u.
- Let $m_1 = \sum_{k=1}^{t} (a_k \bmod 2) 2^{t-k}$.
- Let $m_2 = \sum_{j=1}^{d-t+1} ((\sum_{k=1}^{d} x_k) + n - 2x_j \bmod 3) 3^j + (\sum_{k=1}^{d} x_k \bmod 3)$.
- Let $m = 2^t m_2 + m_1$.
- An edge e rooted at u gets color $A_{ij} + (j-1)\sigma$ from the color group C_m if e is parallel to dimension j.

We can now prove the following result for undirected t-dimensional walks in $G(n, d)$.

Theorem 4. *Given an undirected lattice graph $G(n, d)$, we can color its edges with $O(n^{d/t})$ colors so that any t-dimensional walk that visits at least two distinct edges of $G(n, d)$ can be observed, where d is a constant and $t \le d$.*

Proof. We color the graph $G(n,d)$ using UNDIRCOLOR, and show how to localize a walk W of dimension t. Suppose that the color of edge e_i belongs to color group C_{m_i}. Let $m_{i1} = m_i \bmod 2^t$, $m_{i2} = m_i \div 2^t$ and consider a ternary representation of m_{i2}, namely, $m_{i2} = \sum_{j=0}^{d-t+1} c_i^j 3^j$. Then c_i^j is the color of edge e_i using 3-coloring MOD3(o_j), $j = 0, 1, \ldots, d-t+1$. By Lemma 13 there is a 3-coloring MOD3(o_j) for which W is not monochromatic. Then we can compute the sign of all edges in W by using Lemma 12. Now for any two nodes in this walk, we could using the sign of all edges on the path between them to compute their rank different. Therefore, similar argument in Theorem 3 can be applied to uniquely localize the root node of W in lattice graph $G(n,d)$. $\square$

7 Concluding Remarks and Extensions

In this paper, we explored an observability problem for lattice graphs, and presented asymptotically tight bounds for t-observability of both directed and undirected graphs. The bounds reveal an interesting dependence on the ratio between the dimension of the lattice and that of the walk, *the larger the dimension of the walk the smaller the color complexity of observing it*, as well as an unexpected conclusion that the color complexity for full-dimensional walks is independent of the lattice dimension.

Our results are easy to generalize to non-square lattice graphs, albeit at the expense of more involved calculations. In particular, given a lattice graph of size $N = n_1 \times n_2 \times \cdots \times n_d$, the number of colors for t-observability is $\Theta(\sqrt[t]{N})$. Briefly, we use an (σ, t, d) orthogonal array with σ as the smallest prime larger than $N^{1/t}$. The nodes of the lattice are ranked in the lexicographic order of their coordinates, and we can calculate the rank of a node u with coordinates $(x_1, x_2, \ldots x_d)$, $x_j \in \{0, 1, \ldots, n_j - 1\}$, as

$$r(u) \;=\; x_1 n_2 n_3 \cdots n_d \;+\; x_2 n_3 \cdots n_d + \cdots + x_{d-1} n_d + x_d \;=\; \sum_{j=1}^{d} (x_j \Pi_{k=j+1}^{d} n_k).$$

Except for these minor modifications, the coloring scheme remains unchanged. Given a walk W, and two nodes u_1, u_2 in the walk, the rank distances is calculated as follows: if the number of j-up and j-down edges in the walk from u_1 to u_2 is α_j and β_j, then $r(u_1) - r(u_2) = \Sigma_{j=1}^{d} (\beta_j - \alpha_j) \Pi_{k=j+1}^{d} n_k$. The remaining technical machinery does not depend on the square lattice, and carries over to rectangular lattices.

A number of research directions and open problems are suggested by this research. Our coloring scheme and proof techniques should extend to other regular but non-rectangular lattices; we can show this for planar hexagonal lattices but have not explored the idea fully. On the other hand, observability of general graphs, even planar graphs of bounded degree, appears to be quite challenging. It will also be interesting to explore observability under *node-coloring*.

Finally, some of the *small world graph* models are essentially lattice graphs with few random long-range neighbors at each node [8, 13]. It will be interesting to extend our results to those graphs.

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