CS 267: Automated Verification

Lectures 4: μ-calculus

Instructor: Tevfik Bultan
μ-Calculus

μ-Calculus is a temporal logic which consist of the following:

• Atomic properties AP

• Boolean connectives: \( \neg, \land, \lor \)

• Precondition operator: \( \text{EX} \)

• Least and greatest fixpoint operators: \( \mu y. \mathcal{F}y \) and \( \nu y. \mathcal{F}y \)
  
  \( \neg \mathcal{F} \) must be syntactically monotone in \( y \)
  
  • meaning that all occurrences of \( y \) in within \( \mathcal{F} \) fall under an even number of negations
\(\mu\)-Calculus

- \(\mu\)-calculus is a powerful logic
  - Any CTL* property can be expressed in \(\mu\)-calculus

- So, if you build a model checker for \(\mu\)-calculus you would handle all the temporal logics we discussed: LTL, CTL, CTL*

- One can write a \(\mu\)-calculus model checker using the basic ideas about fixpoint computations that we discussed
  - However, there is one complication
    - Nested fixpoints!
Mu-calculus Model Checking Algorithm

eval(f : mu-calculus formula) : a set of states

case: f ∈ AP \quad \text{return } \{s | L(s,f)=true\};
case: f ≡ ¬p \quad \text{return } S - \text{eval}(p);
case: f ≡ p ∧ q \quad \text{return } \text{eval}(p) \cap \text{eval}(q);
case: f ≡ p ∨ q \quad \text{return } \text{eval}(p) \cup \text{eval}(q);
case: f ≡ EX p \quad \text{return } EX(\text{eval}(p));
Mu-calculus Model Checking Algorithm

eval(f)

...  
case: \( f \equiv \mu y . g(y) \)
     y := False;
 repeat {
    y\_old := y;
    y := eval(g(y));
  } until y = y\_old
 return y;
Mu-calculus Model Checking Algorithm

eval(f)

... 
case: f ≡ ν y . g(y)
    y := True;
    repeat {
        y_{old} := y;
        y := eval(g(y));
    } until y = y_{old}
    return y;
Nested Fixpoints

• Here is a CTL property
  \[ \text{EG EF } p = \nu y . (\mu z . p \lor \text{EX } z) \land \text{EX } y \]
  – The fixpoints are not nested.
  – Inner fixpoint is computed only once and then the outer fixpoint is computed
  – Fixpoint characterizations of CTL properties do not have nested fixpoints

• Here is a CTL* property
  \[ \text{EGF } p = \nu y . \mu z . ((p \lor \text{EX } z) \land \text{EX } y) \]
  – The fixpoints are nested.
  – Inner fixpoint is recomputed for each iteration of the outer fixpoint
Nested Fixpoint Example

$0 \models EGF p$

$EGF p = \nu y . \mu z . ((p \lor EX z) \land EX y)$

$\mathcal{F}_3$

Nested Fixpoint

<table>
<thead>
<tr>
<th>$\mathcal{F}_3$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>{0,1,2}</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>0,1</td>
<td>{1}</td>
<td></td>
</tr>
<tr>
<td>0,2</td>
<td>{0,1}</td>
<td></td>
</tr>
<tr>
<td>0,3</td>
<td>{0,1}</td>
<td></td>
</tr>
<tr>
<td>1,0</td>
<td>{0,1}</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>1,1</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2,0</td>
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<td>$\emptyset$</td>
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<tr>
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<td>$\emptyset$</td>
</tr>
<tr>
<td>3,0</td>
<td>$\emptyset$</td>
<td></td>
</tr>
</tbody>
</table>

$EGE_{p} = \emptyset$

$0 \not\models EG EF p$

$EG EF p = \nu y . (\mu z . p \lor EX z) \land EX y$

$\mathcal{F}_1$

$\mathcal{F}_2$

EF $p$ fixpoint

$\emptyset$

$\mathcal{F}_1(\emptyset) = \{1\}$

$\mathcal{F}_1^2(\emptyset) = \{0,1\}$

$\mathcal{F}_1^3(\emptyset) = \{0,1\}$

EG $EF p = \{0\}$

EG $\{0,1\}$ fixpoint

S=\{0,1,2\}

$\mathcal{F}_2(S) = \{0,1\}$

$\mathcal{F}_2^2(S) = \{0\}$

$\mathcal{F}_2^3(S) = \{0\}$