# CS 267: Automated Verification

## Lecture 6: Binary Decision Diagrams

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# Binary Decision Diagrams (BDDs)

[Bryant 86]

- Reduced Ordered Binary Decision Diagrams (BDDs)
  - An efficient data structure for representing Boolean functions (or truth sets of Boolean formulas) and manipulating them

- BDDs are a canonical representation for Boolean functions
  - given two Boolean logic formulas F and G, if F and G are equivalent (i.e. if their truth sets are the same), then their BDD representations will be the same

# BDDs for Symbolic Model Checking

- BDD data structure can be used to implement the symbolic model checking algorithm we discussed earlier

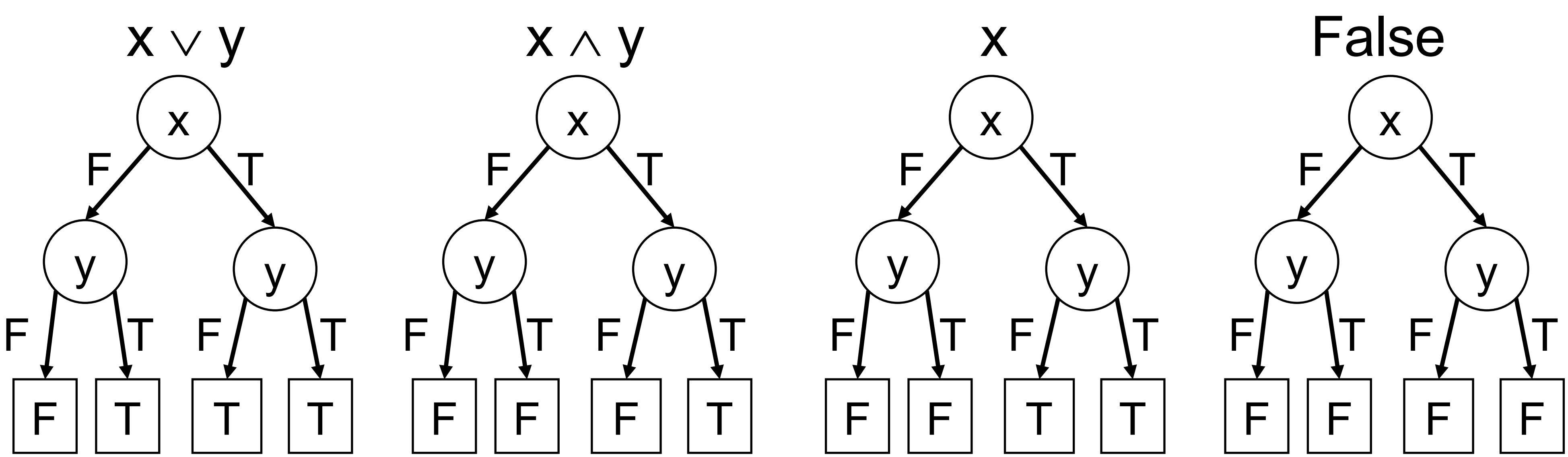
- BDDs support all the operations we need for symbolic model checking
  - take conjunction of two BDDs
  - take disjunction of two BDDs
  - test equivalence of two BDDs
  - test subsumption between two BDDs
  - negate a BDD
  - test if a BDD satisfiable
  - test if a BDD is a tautology
  - existential variable elimination

# Binary Decision Trees

Given a variable order, in each level of the tree, branch on the value of the variable in that level.

- Examples for boolean formulas on two variables

  Variable order: x, y



# Reduced and Ordered Binary Decision Diagrams

- We are interested in ***Reduced*** and ***Ordered*** Binary Decision Diagrams

- Reduced:
  - Merge all identical sub-trees in the binary decision tree (converts it to a directed-acyclic graph)
  - Remove redundant tests (if the false and true branches for a node go to the same place, remove that node)

- Ordered
  - We pick a fix order for the Boolean variables:

    $x_0 < x_1 < x_2 < \ldots$
  - The nodes in the BDD are listed based on this ordering

# BDDs

- Repeatedly apply the following transformations to a binary decision tree:

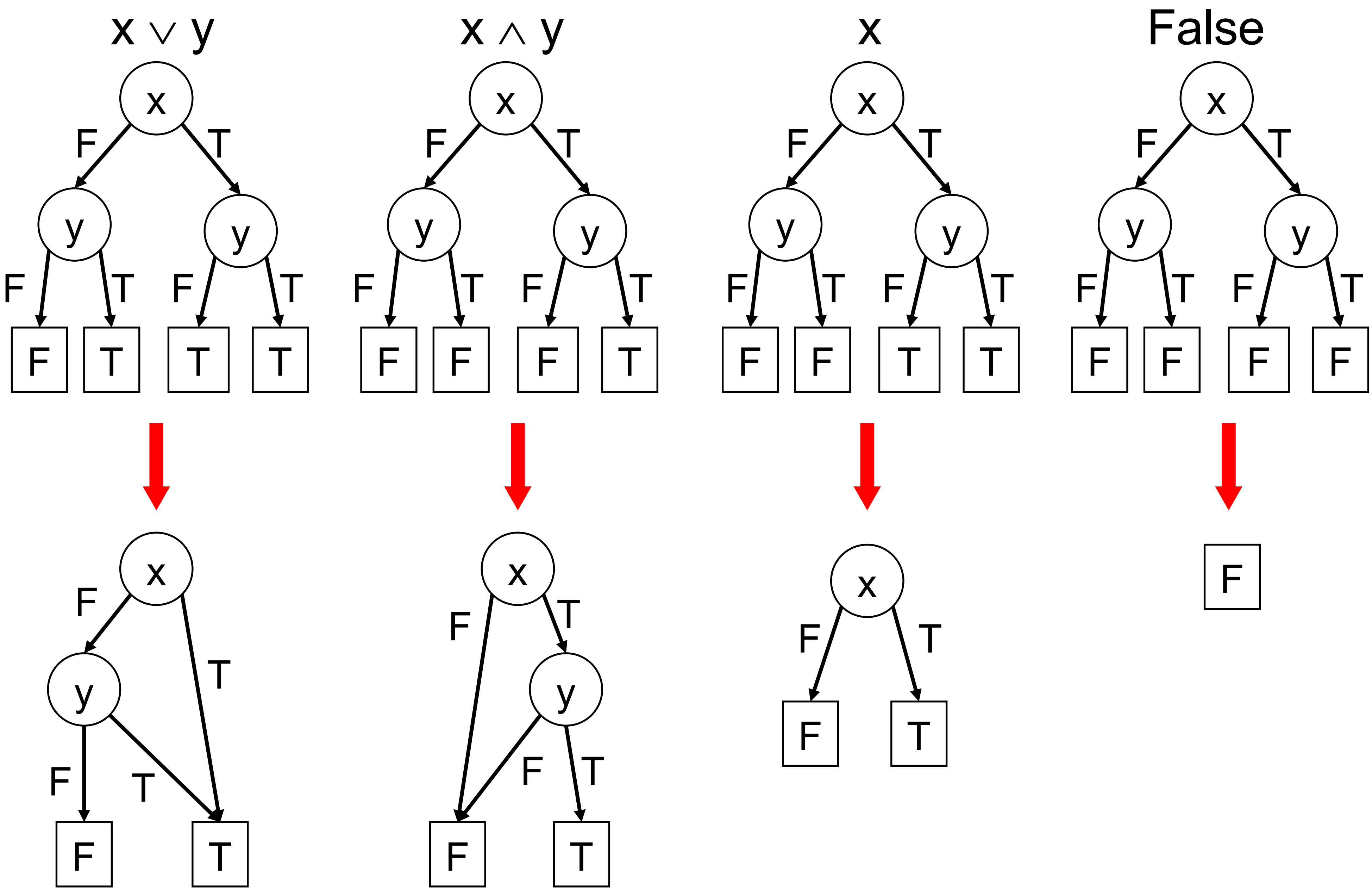
  1. Remove duplicate terminals

  2. Remove duplicate non-terminals

  3. Remove redundant tests

- These transformations transform the tree to a directed acyclic graph

# Binary Decision Trees vs. BDDs



# Good News About BDDs

- Given BDDs for two boolean logic formulas F and G
  - The BDDs for $F \wedge G$ and $F \vee G$ are of size $|F| \times |G|$ (and can be computed in that time)
  - The BDD for $\neg F$ is of size $|F|$ (and can be computed in that time)
  - $F \equiv ?\ G$ can be checked in linear time
  - Satisfiability of F can be checked in constant time
    - No, this does not mean that you can solve SAT in constant time

# Bad News About BDDs

- The size of a BDD can be exponential in the number of boolean variables

- The sizes of the BDDs are very sensitive to the variable ordering. Bad variable ordering can cause exponential increase in the size of the BDD

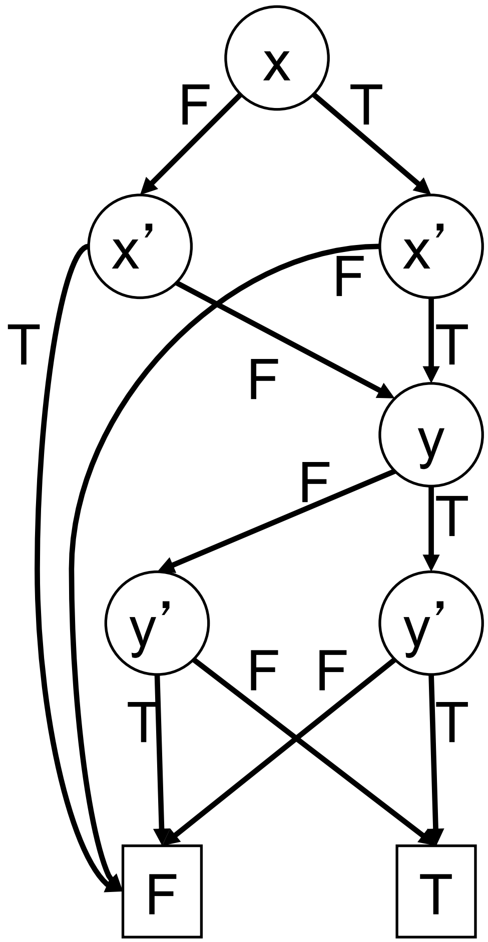
- There are functions which have BDDs that are exponential for any variable ordering (for example binary multiplication)

- Pre condition (EX) computation requires a sequence of existential variable eliminations
  - A sequence of existential variable eliminations can cause an exponential blow-up in the size of the BDD

# BDDs are Sensitive to Variable Ordering

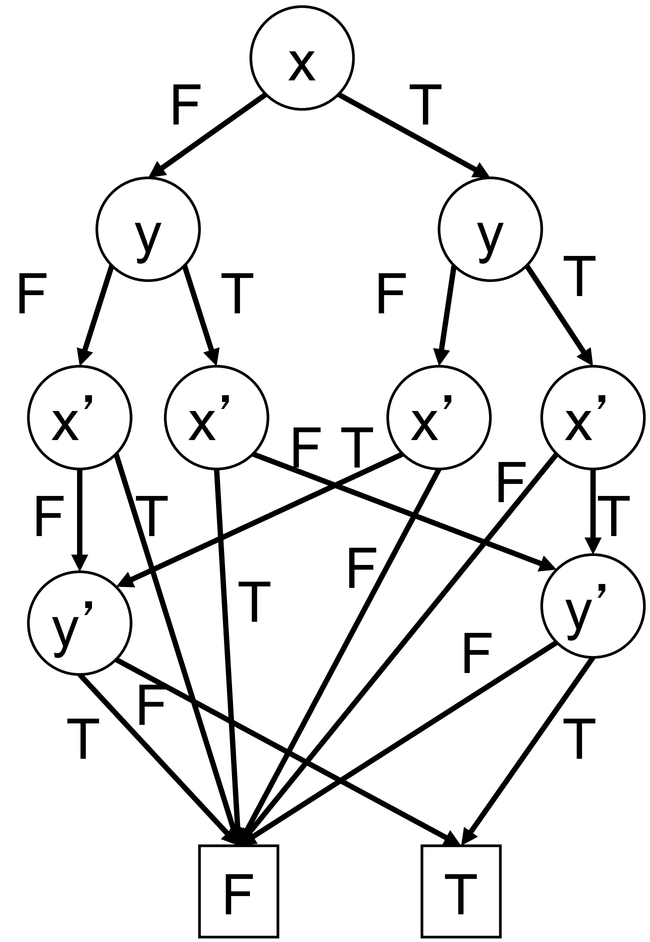
Identity relation for two variables: $(x' \leftrightarrow x) \wedge (y' \leftrightarrow y)$

Variable order: x, x’ , y, y'




*For n variables, 3n+2 nodes*

Variable order: x, y, x’ , y'




*For n variables,* $3 \times 2^n - 1$ *nodes*

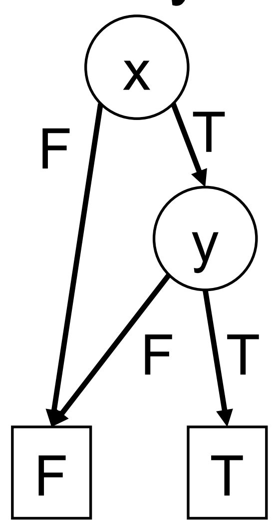
# BDDs from Another Perspective

- Any Boolean formula f on variables $x_1, x_2, \ldots, x_n$ can be written as (called Shannon expansion):

$$f = x_i \wedge f\,[True/x_i] \vee \neg\, x_i \wedge f\,[False/x_i] \quad \text{(this is an if-then-else)}$$

- BDDs use this idea

$x \wedge y$




This node corresponds to the formula False, which comes from the Shannon expansion:

$$False \equiv x \wedge y\,[False/x]$$

This node corresponds to the formula y, which comes from the Shannon expansion:

$$y \equiv x \wedge y\,[True/x]$$

# How to Implement BDDs?

- First, let's pick a data structure for a BDD node:

```
type vertex = record
     low, high: vertex;
     index: 1 .. n+1;
     val: {0, 1, X};
     id: integer;
end;
```

High is the true branch, Low is the false branch

Index from the variable ordering, each index value corresponds to one variable

n is the number of Boolean variables

0 corresponds to false,
1 corresponds to true,
X corresponds to no value

This field will be used to generate a unique id for each unique function

# How to Implement a BDD?

- A BDD is a rooted directed graph with vertex set V containing two types of vertices
  - nonterminal vertices:

    index $\in$ {1, 2, .., n}

    low(v) and high(v) are in V

    val(v) is X
  - terminal vertices:

    index(v) = n+1

    val(v) is 0 or 1

    low(v) and high(v) are null

# BDD Reduce algorithm

- Start from the terminal nodes and go towards to root
  1. Remove duplicate terminals: If two terminal vertices have the same value, remove one of them redirect all the incoming arcs to the other one
  2. Remove duplicate non-terminals: If two nonterminal vertives, v and u have the same index and v.low=u.low and  v.high=u.high then eliminate one of them and redirect all incoming arcs to the other one
  3. Remove redundant tests: If a nonterminal vertex v has v.low=v.high then remove v and redirect all incoming arcs to low(v)

# Reduce Algorithm

```
function Reduce(v: vertex): vertex;
  var subgraph: array[1 .. |G|] of vertex;
  var vlist: array[1 .. n+1] of list of vertex;

begin
  Put each vertex u reachable from v on list vlist[u.index]
  nextid := 0;

  for i := n+1 downto 1 do
  begin
    Q := empty set;

    for each u in vlist[i] do
      if u.index = n+1
      then add <key,u> to Q where key=(u.value) {terminal}
      else if u.low.id=u.high.id
           then u.id := u.low.id; {redundant vertex}
           else add <key,u> to Q where key =(u.low.id,u.high.id);
```

# Reduce Algorithm Continued

```
    Sort elements of Q by keys;

    oldkey := (-1,-1); {unmatchable key}
    for each <key,u> in Q removed in order do
      if key = oldkey
      then u.id := nextid; {matches existing vertex}
      else begin {unique vertex}
        nextid :=nextid+1;
        u.id:=nextid;
        subgraph[nextid]:=u;
        u.low := subgraph[u.low.id];
        u.high := subgraph[u.high.id];
        oldkey := key;
      end;
  end; {end of the outmost for loop}
  return(subgraph[v.id]);
end;
```

# Reduce Algorithm

- Reduce algorithm can be done in linear time
  - There is a linear time lexicographic sorting method based on bucket sorting for sorting the vertices according to their keys
  - Assuming a linear time sorting routine, processing at each level requires time proportional to the number of vertices at that level, and each level is processed once, resulting in linear time complexity

# Apply Algorithm

- We can use the Shannon expansion to recursively compute any binary Boolean operation

- Given BDDs for f and g and binary operation op

$$f \text{ op } g = x_i \wedge (f\,[T/x_i] \text{ op } g\,[T/x_i]) \vee \neg x_i \wedge (f\,[F/x_i] \text{ op } g\,[F/x_i])$$

- So the idea is to start at the root of f and g and recursively call true and false branches
  - Key idea: Use memoization!

# Apply Algorithm

```
var T: array[1 .. |G1|, 1 .. |G2|] of vertex; {init. to null}
function Apply(v1, v2: vertex; <op> operator): vertex;
begin
  u := T[v1.id, v2.id];
  if u != null then return(u); {already computed}
  u := new vertex; T[v1.id, v2.id] := u;
  u.value := v1.value <op> v2.value;
  if u.value != X
  then u.index := n+1; u.low :=null; u.high :=null; {terminal}
  else begin {nonterminal}
    u.index := Min(v1.index, v2.index);
    if v1.index = u.index
    then vlow1 := v1.low; vhigh1 := v1.high;
    else vlow1 := v1; vhigh1 := v1;
    if v2.index = u.index
    then vlow2 := v2.low; vhigh2 :=v2.high;
    else vlow2 := v2; vhigh2 := v2;
    u.low := Apply(vlow1, vlow2);
    u.high := Apply(vhigh1, vhigh2);
  end;
  return(u);
end;
```

# Apply Algorithm

- Given two input BDDs f and g the size of the resulting BDD for f <op> g is $|f| \times |g|$ and it can be computed in that time

- The important point is to use memoization so that for any pair of vertices (one from each input BDD) the computation is done exactly once

- Since there are at most $|f| \times |g|$ pairs the complexity is $|f| \times |g|$
  - We do constant amount of work for each pair of nodes