# Automata-Based String Analysis

## Model Counting

Tevfik Bultan, Abdulbaki Aydin, Lucas Bang

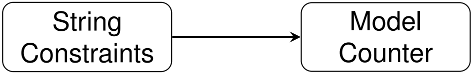
Verification Laboratory
http://vlab.cs.ucsb.edu
Department of Computer Science

# Overview

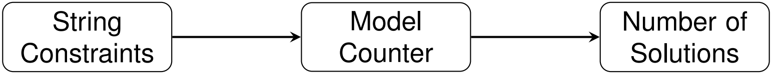
# Overview

String
Constraints

# Overview

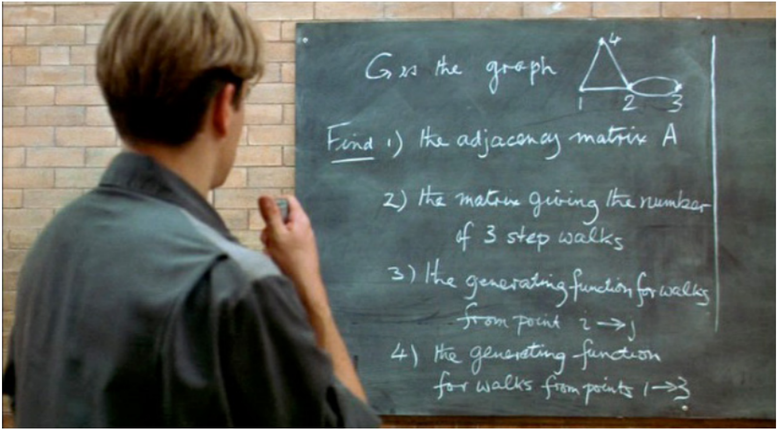


# Overview



# Can you solve it, Will Hunting?

# Can you solve it, Will Hunting?



# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

# A Motivating Example

An adversary learns a password. User must select a new password.

# A Motivating Example

An adversary learns a password. User must select a new password.

**Policy for selecting a new password.**

# A Motivating Example

An adversary learns a password. User must select a new password.

## Policy for selecting a new password.

```
public Boolean NewPWCheck(String new_p, old_p){
    if( old_p.contains(new_p)               || ...
        new_p.contains(old_p)               || ...
        old_p.reverse().contains(new_p))  || ...
        new_p.contains(old_p.reverse())     ){
        System.out.println("Too similar.");
        return false;
    } else
        return true;
}
```

# A Motivating Example

Suppose an adversary knows `old_p = "abc-16"`

# A Motivating Example

Suppose an adversary knows `old_p = "abc-16"`
*and* knows the policy.

# A Motivating Example

Suppose an adversary knows `old_p = "abc-16"`
*and* knows the policy.

## Constraints on possible values of NEW_P

```
(not (contains (toLower NEW_P) "abc-16"))
(not (contains (toLower NEW_P) "61-cba"))
(not (contains "abc-16" (toLower NEW_P)))
(not (contains "61-cba" (toLower NEW_P)))
```

# A Motivating Example

Suppose an adversary knows `old_p = "abc-16"`
*and* knows the policy.

## Constraints on possible values of NEW_P

```
(not (contains (toLower NEW_P) "abc-16"))
(not (contains (toLower NEW_P) "61-cba"))
(not (contains "abc-16" (toLower NEW_P)))
(not (contains "61-cba" (toLower NEW_P)))
```

If password length $= n$, then there are $|\Sigma|^n$ possible passwords.

# A Motivating Example

Suppose an adversary knows `old_p = "abc-16"`
*and* knows the policy.

## Constraints on possible values of NEW_P

```
(not (contains (toLower NEW_P) "abc-16"))
(not (contains (toLower NEW_P) "61-cba"))
(not (contains "abc-16" (toLower NEW_P)))
(not (contains "61-cba" (toLower NEW_P)))
```

If password length $= n$, then there are $|\Sigma|^n$ possible passwords.

If adversary knows `old_p` and the policy ...

- how much is the reduction in search space?
- what is the probability of guessing the new password?

# Motivation

**In general, we want to answer questions regarding**

- probability of program behaviors,
- number of inputs that cause an error,
- amount of information flow,
- information leakage,
- other, as yet unforeseen, applications...

# Motivation

**In general, we want to answer questions regarding**

- probability of program behaviors,
- number of inputs that cause an error,
- amount of information flow,
- information leakage,
- other, as yet unforeseen, applications...

These are **quantitative** questions which require **model counting**.

# Motivation

## Techniques for model counting for other theories

Boolean Logic Formulas

- DPLL
- Random sampling based
- Approximations

# Motivation

### Techniques for model counting for other theories

Boolean Logic Formulas

- DPLL
- Random sampling based
- Approximations

Linear Integer Arithmetic:

- LattE
- Barvinok

# Motivation

## String manipulating programs are pervasive

- security critical functions,
- server side sanitization functions,
- databases,
- dynamic code generation.

# Motivation

## String manipulating programs are pervasive

- security critical functions,
- server side sanitization functions,
- databases,
- dynamic code generation.

We need model counting for strings in order to make quantitative guarantees about these types of programs.

# Motivation

## String manipulating programs are pervasive

- security critical functions,
- server side sanitization functions,
- databases,
- dynamic code generation.

We need model counting for strings in order to make quantitative guarantees about these types of programs.

## Software for string constraint model counting

- Automata-Based Model Counter (ABC) [Aydin, et. al. CAV 2015]
- String Model Counter (SMC) [Luu, et. al. PLDI 2014 ]
- S3# [Trinh, et. al. CAV 2017]

# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

# Model Counting

## Recall the classic (boolean) SAT problem

Given a formula $\phi$ from propositional logic, is it possible to assign all variables the values $T$ (true) or $F$ (false) so that the formula is true?

# Model Counting

## Recall the classic (boolean) SAT problem

Given a formula $\phi$ from propositional logic, is it possible to assign all variables the values $T$ (true) or $F$ (false) so that the formula is true?

Example:

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

# Model Counting

## Recall the classic (boolean) SAT problem

Given a formula $\phi$ from propositional logic, is it possible to assign all variables the values $T$ (true) or $F$ (false) so that the formula is true?

Example:

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

$\phi$ is satisfiable by setting

$$(x, y, z, w, v) = (T, F, T, F, T).$$

# Model Counting

## Recall the classic (boolean) SAT problem

Given a formula $\phi$ from propositional logic, is it possible to assign all variables the values $T$ (true) or $F$ (false) so that the formula is true?

Example:

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

$\phi$ is satisfiable by setting

$$(x, y, z, w, v) = (T, F, T, F, T).$$

A satisfying assignment is called a **model** for $\phi$.

# Model Counting

### The **model counting problem**

Given a formula $\phi$ over some theory (Boolean, LIA, Strings, ...)

**how many models are there** for $\phi$?

# Model Counting

## The **model counting problem**

Given a formula $\phi$ over some theory (Boolean, LIA, Strings, . . . )

**how many models are there** for $\phi$?

## Difficulty of Model Counting

Model counting is “at least as hard” as satisfiability check.

# Model Counting

### The **model counting problem**

Given a formula $\phi$ over some theory (Boolean, LIA, Strings, ...)

**how many models are there** for $\phi$?

### Difficulty of Model Counting

Model counting is "at least as hard" as satisfiability check.

$$|\phi| > 0 \iff \phi \text{ is satisfiable}$$

# Model Counting Boolean SAT

| x | y | z | w | v | F |
|---|---|---|---|---|---|
| F | F | F | F | F | F |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| T | F | F | T | T | F |
| T | F | T | F | F | F |
| T | F | T | F | T | T |
| T | F | T | T | F | F |
| T | F | T | T | T | T |
| T | T | F | F | F | F |
| T | T | F | F | T | F |
| T | T | F | T | F | F |
| T | T | F | T | T | F |
| T | T | T | F | F | T |
| T | T | T | F | T | T |
| T | T | T | T | F | T |
| T | T | T | T | T | T |

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

# Model Counting Boolean SAT

| x | y | z | w | v | F |
|---|---|---|---|---|---|
| F | F | F | F | F | F |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| T | F | F | T | T | F |
| T | F | T | F | F | F |
| **T** | **F** | **T** | **F** | **T** | **T** |
| T | F | T | T | F | F |
| **T** | **F** | **T** | **T** | **T** | **T** |
| T | T | F | F | F | F |
| T | T | F | F | T | F |
| T | T | F | T | F | F |
| T | T | F | T | T | F |
| **T** | **T** | **T** | **F** | **F** | **T** |
| **T** | **T** | **T** | **F** | **T** | **T** |
| **T** | **T** | **T** | **T** | **F** | **T** |
| **T** | **T** | **T** | **T** | **T** | **T** |

$$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$$

$\phi$ has 6 models.

# Model Counting Boolean SAT

| x | y | z | w | v | F |
|---|---|---|---|---|---|
| F | F | F | F | F | F |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| T | F | F | T | T | F |
| T | F | T | F | F | F |
| **T** | **F** | **T** | **F** | **T** | **T** |
| T | F | T | T | F | F |
| **T** | **F** | **T** | **T** | **T** | **T** |
| T | T | F | F | F | F |
| T | T | F | F | T | F |
| T | T | F | T | F | F |
| T | T | F | T | T | F |
| **T** | **T** | **T** | **F** | **F** | **T** |
| **T** | **T** | **T** | **F** | **T** | **T** |
| **T** | **T** | **T** | **T** | **F** | **T** |
| **T** | **T** | **T** | **T** | **T** | **T** |

$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$

$\phi$ has 6 models.

Truth table method is $\theta(2^n)$.

# Model Counting Boolean SAT

| x | y | z | w | v | F |
|---|---|---|---|---|---|
| F | F | F | F | F | F |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| T | F | F | T | T | F |
| T | F | T | F | F | F |
| **T** | **F** | **T** | **F** | **T** | **T** |
| T | F | T | T | F | F |
| **T** | **F** | **T** | **T** | **T** | **T** |
| T | T | F | F | F | F |
| T | T | F | F | T | F |
| T | T | F | T | F | F |
| T | T | F | T | T | F |
| **T** | **T** | **T** | **F** | **F** | **T** |
| **T** | **T** | **T** | **F** | **T** | **T** |
| **T** | **T** | **T** | **T** | **F** | **T** |
| **T** | **T** | **T** | **T** | **T** | **T** |

$\phi = (x \vee y) \wedge (\neg x \vee z) \wedge (z \vee w) \wedge x \wedge (y \vee v)$

$\phi$ has 6 models.

Truth table method is $\theta(2^n)$.

DPLL method is $O(2^n)$, but is faster in practice.[1]

[1] Birnbaum, et. al. The good old Davis-Putnam procedure helps counting models. JAIR 1999.

# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

# Model Counting Strings

A formula over the theory of strings can involve

# Model Counting Strings

**A formula over the theory of strings can involve**

- Word Equations: $X \circ U = Y \circ Z$

# Model Counting Strings

### A formula over the theory of strings can involve

- Word Equations: $X \circ U = Y \circ Z$
- Length Constraints: $4 < \mathit{Length}(X) < 10$

# Model Counting Strings

**A formula over the theory of strings can involve**

- Word Equations: $X \circ U = Y \circ Z$
- Length Constraints: $4 < Length(X) < 10$
- Regular Language Membership: $X \in (a|b)^*$

# Model Counting Strings

## A formula over the theory of strings can involve

- Word Equations: $X \circ U = Y \circ Z$
- Length Constraints: $4 < Length(X) < 10$
- Regular Language Membership: $X \in (a|b)^*$
- and more complex constraints: $(X = \text{substring}(Y, i, j), \dots)$

# Model Counting Strings

## A formula over the theory of strings can involve

- Word Equations: $X \circ U = Y \circ Z$
- Length Constraints: $4 < \mathit{Length}(X) < 10$
- Regular Language Membership: $X \in (a|b)^*$
- and more complex constraints: ($X = \text{substring}(Y, i, j)$, ...)

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$?

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$a_0 = 1$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$a_0 = 1, a_1 = 1$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$a_0 = 1, a_1 = 1, a_2 = 1$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 3$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |
| 4 | 1001, 1100, 1111 | 3 |

# Model Counting Strings

$$X \in (0|(1(01^*0)^*1))^*$$

Q: How many solutions for $X$? A: Infinitely many!

Q: How many solutions for $X$ of length $k$?

A counting sequence for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 3, a_5 = 5, \ldots$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |
| 4 | 1001, 1100, 1111 | 3 |
| 5 | 10010, 10101, 11000, 11011, 11110 | 5 |

# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

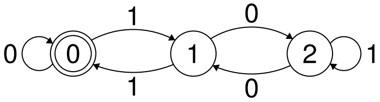
# Deterministic Finite Automata

# Deterministic Finite Automata

$$X \in (0|(1(01^*0)^*1))^*$$

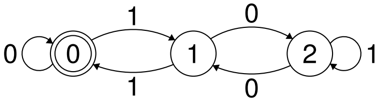
# Deterministic Finite Automata

$$X \in (0|(1(01^*0)^*1))^*$$



# Deterministic Finite Automata

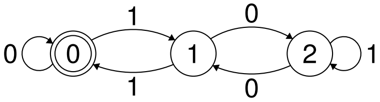
$$X \in (0|(1(01^*0)^*1))^*$$




$$|\{s : s \in \mathcal{L}, \mathrm{len}(s) = k\}| \equiv |\{\pi : \pi \text{ is accepting path of length } k\}|$$

# Deterministic Finite Automata

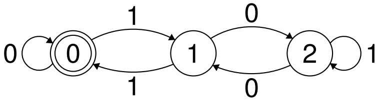
$$X \in (0|(1(01^*0)^*1))^*$$




$$|\{s : s \in \mathcal{L}, \text{len}(s) = k\}| \equiv |\{\pi : \pi \text{ is accepting path of length } k\}|$$

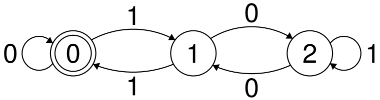
String Counting $\equiv$ Path Counting

# Deterministic Finite Automata



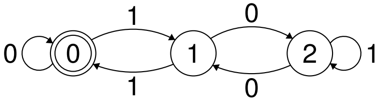

How to count paths of length $k$?

# Deterministic Finite Automata




How to count paths of length $k$?
**Dynamic Programming**

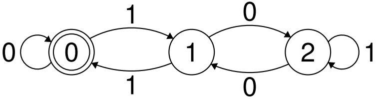
# Deterministic Finite Automata




How to count paths of length $k$?
**Dynamic Programming**

$a_k$

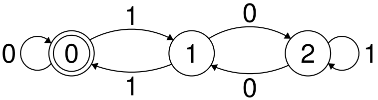
# Deterministic Finite Automata



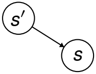

How to count paths of length $k$?
**Dynamic Programming**


$a_k(s) =$

# Deterministic Finite Automata

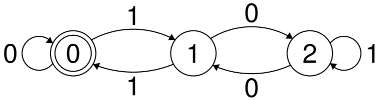



How to count paths of length $k$?
**Dynamic Programming**

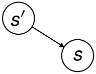



$$a_k(s) = \qquad a_{k-1}(s')$$

# Deterministic Finite Automata

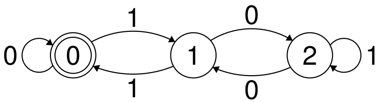



How to count paths of length $k$?
**Dynamic Programming**

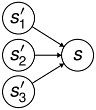



$$a_k(s) = \qquad a_{k-1}(s')$$

# Deterministic Finite Automata

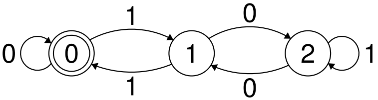


How to count paths of length $k$?
**Dynamic Programming**



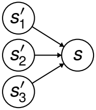
$$a_k(s) = \qquad a_{k-1}(s')$$

# Deterministic Finite Automata



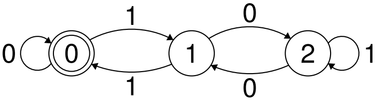

How to count paths of length $k$?

**Dynamic Programming**

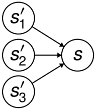



$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

# Deterministic Finite Automata



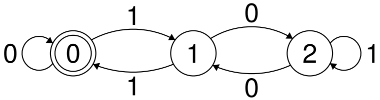

How to count paths of length $k$?
**Dynamic Programming**



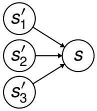

Initial Conditions

$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Dynamic Programming**

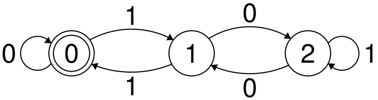



Initial Conditions

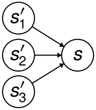
$$a_0(0) = 1$$

$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Dynamic Programming**

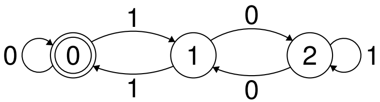



Initial Conditions

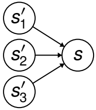
$$a_0(0) = 1, a_0(1) = 0, a_0(2) = 0$$

$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Dynamic Programming**




$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

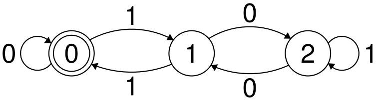
Initial Conditions

$$a_0(0) = 1, a_0(1) = 0, a_0(2) = 0$$

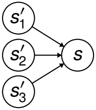
System of Recurrences

$$a_k(0) = a_{k-1}(0) + a_{k-1}(1)$$

# Deterministic Finite Automata



How to count paths of length $k$?
**Dynamic Programming**



$$a_k(s) = \sum_{s' \to s} a_{k-1}(s')$$

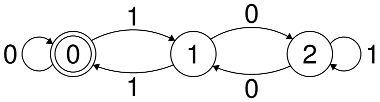
Initial Conditions

$$a_0(0) = 1, a_0(1) = 0, a_0(2) = 0$$

System of Recurrences

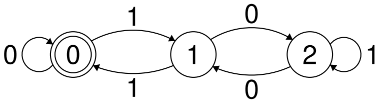
$$\begin{aligned} a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\ a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\ a_k(2) &= a_{k-1}(1) + a_{k-1}(2) \end{aligned}$$

# Deterministic Finite Automata




How to count paths of length $k$?

# Deterministic Finite Automata

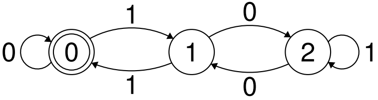



How to count paths of length $k$?

System of Recurrences

$$\begin{aligned}
a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\
a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\
a_k(2) &= a_{k-1}(1) + a_{k-1}(2)
\end{aligned}$$

# Deterministic Finite Automata



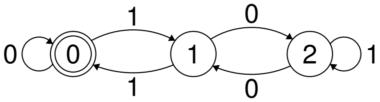

How to count paths of length $k$?

System of Recurrences

$$\begin{aligned} a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\ a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\ a_k(2) &= a_{k-1}(1) + a_{k-1}(2) \end{aligned}$$

$$\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0(k-1) \\ a_1(k-1) \\ a_2(k-1) \end{pmatrix}$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Matrix Exponentiation**

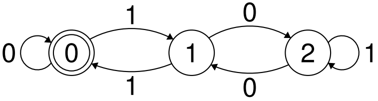
System of Recurrences

$$
\begin{aligned}
a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\
a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\
a_k(2) &= a_{k-1}(1) + a_{k-1}(2)
\end{aligned}
$$

$$
\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0(k-1) \\ a_1(k-1) \\ a_2(k-1) \end{pmatrix}
$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Matrix Exponentiation**

System of Recurrences

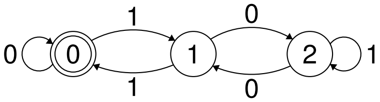
$$
\begin{aligned}
a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\
a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\
a_k(2) &= a_{k-1}(1) + a_{k-1}(2)
\end{aligned}
$$

$$\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$a_k = (A^k)_{0,F}$$

$$\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0(k-1) \\ a_1(k-1) \\ a_2(k-1) \end{pmatrix}$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Matrix Exponentiation**

System of Recurrences

$$\begin{aligned} a_k(0) &= a_{k-1}(0) + a_{k-1}(1) \\ a_k(1) &= a_{k-1}(0) + a_{k-1}(2) \\ a_k(2) &= a_{k-1}(1) + a_{k-1}(2) \end{aligned}$$

$$\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$a_k = (A^k)_{0,F}$$

$$\begin{pmatrix} a_0(k) \\ a_1(k) \\ a_2(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0(k-1) \\ a_1(k-1) \\ a_2(k-1) \end{pmatrix}$$

$$a_4 = (A^4)_{0,0} = 3$$

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

$$g(z) = \frac{1}{(1-z)^3}$$

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

$$g(z) = \frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} a_k z^k$$

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

$$g(z) = \frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} a_k z^k$$

$$g(z) = 1z^0 + 3z^1 + 6z^2 + 10z^3 + 15z^4 + \dots$$

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

$$g(z) = \frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} a_k z^k$$

$$g(z) = 1z^0 + 3z^1 + 6z^2 + 10z^3 + 15z^4 + \dots$$

$$g(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

**Generating functions** are a way to compactly represent (possibly infinite) sequences.

$$g(z) = \frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} a_k z^k$$

$$g(z) = 1z^0 + 3z^1 + 6z^2 + 10z^3 + 15z^4 + \dots$$

$$g(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

Sequence element $a_k$ is the $k^{th}$ Taylor series coefficient of $g(z)$.

The Taylor series of a function $g(z)$ that is differentiable at 0 is the power series

$$g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots .$$

The Taylor series of a function $g(z)$ that is differentiable at 0 is the power series

$$g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots .$$

which can be written in the more compact sigma notation as

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (x-a)^n$$

where $n!$ denotes the factorial of $n$ and $g(n)^{(a)}$ denotes the $n$-th derivative of f evaluated at the point a.

$$X \in (0|(1(01^*0)^*1))^*$$

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$g(z) =$$

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$g(z) = 1z^0$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$g(z) = 1z^0 + 1z^1$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \mathrm{len}(s) = k\}|$$

$$g(z) = 1z^0 + 1z^1 + 1z^2$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$g(z) = 1z^0 + 1z^1 + 1z^2 + 1z^3$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |

$$X \in (0|(1(01^*0)^*1))^*$$

A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \text{len}(s) = k\}|$$

$$g(z) = 1z^0 + 1z^1 + 1z^2 + 1z^3 + 3z^4$$

| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |
| 4 | 1001, 1100, 1111 | 3 |

$$X \in (0|(1(01^*0)^*1))^*$$

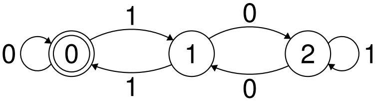
A generating function for language $\mathcal{L}$ encodes

$$a_k = |\{s : s \in \mathcal{L}, \mathrm{len}(s) = k\}|$$

$$g(z) = 1z^0 + 1z^1 + 1z^2 + 1z^3 + 3z^4 + 5z^5 + \ldots$$

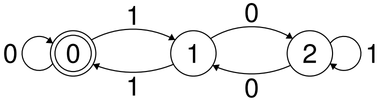
| $k$ | $X$ | $a_k$ |
|---|---|---|
| 0 | $\varepsilon$ | 1 |
| 1 | 0 | 1 |
| 2 | 11 | 1 |
| 3 | 110 | 1 |
| 4 | 1001, 1100, 1111 | 3 |
| 5 | 10010, 10101, 11000, 11011, 11110 | 5 |

# Deterministic Finite Automata




How to count paths of length $k$?

# Deterministic Finite Automata

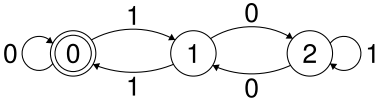



How to count paths of length $k$?
**Generating Functions**

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$g(z) = \frac{\det(I - zA : i, j)}{(-1)^n \det(I - zA)}$$

# Deterministic Finite Automata



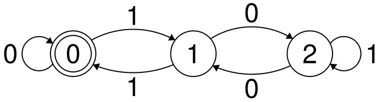

How to count paths of length $k$?
**Generating Functions**

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$g(z) = \frac{\det(I - zA : i, j)}{(-1)^n \det(I - zA)}$$

$$g(z) = \frac{1 - z - z^2}{(z-1)\,(2z^2 + z - 1)}$$

# Deterministic Finite Automata




How to count paths of length $k$?
**Generating Functions**

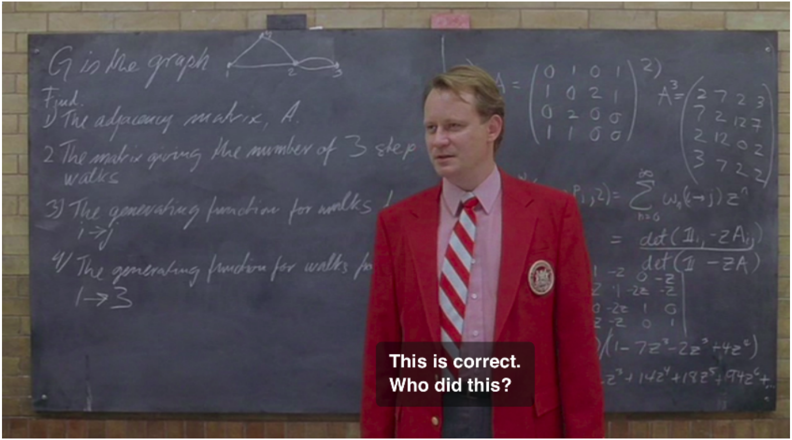
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$g(z) = \frac{\det(I - zA : i, j)}{(-1)^n \det(I - zA)}$$

$$g(z) = \frac{1 - z - z^2}{(z-1)\,(2z^2 + z - 1)}$$

$$g(z) = 1z^0 + 1z^1 + 1z^2 + 1z^3 + 3z^4 + 5z^5 + \dots$$

# Good job, Will Hunting!!!

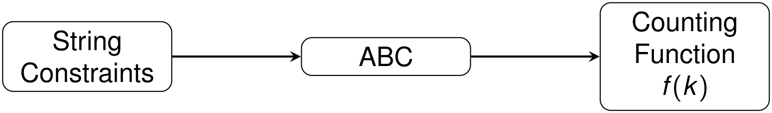


# Automata-Based Model Counter (ABC)

CAV 2015: Automata-Based Model Counting for String Constraints.
Abdulbaki Aydin, Lucas Bang, Tevfik Bultan.

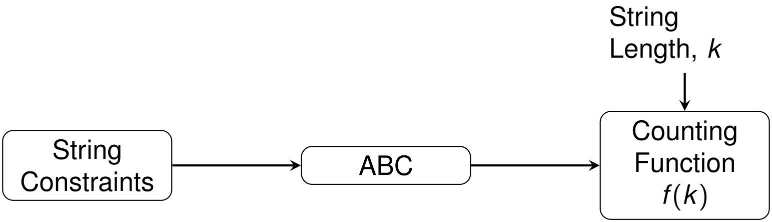
# Automata-Based Model Counter (ABC)

CAV 2015: Automata-Based Model Counting for String Constraints.
Abdulbaki Aydin, Lucas Bang, Tevfik Bultan.



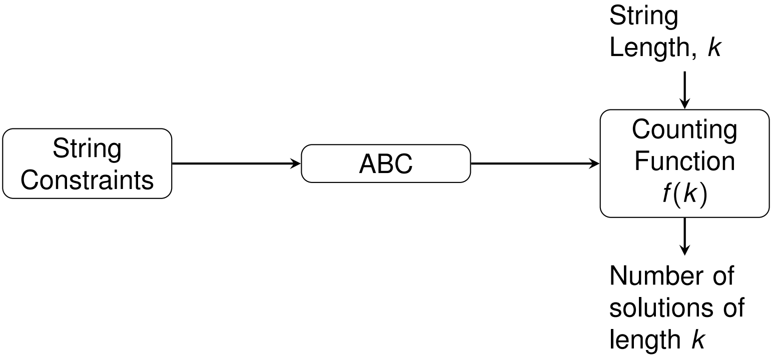
# Automata-Based Model Counter (ABC)

CAV 2015: Automata-Based Model Counting for String Constraints.
Abdulbaki Aydin, Lucas Bang, Tevfik Bultan.



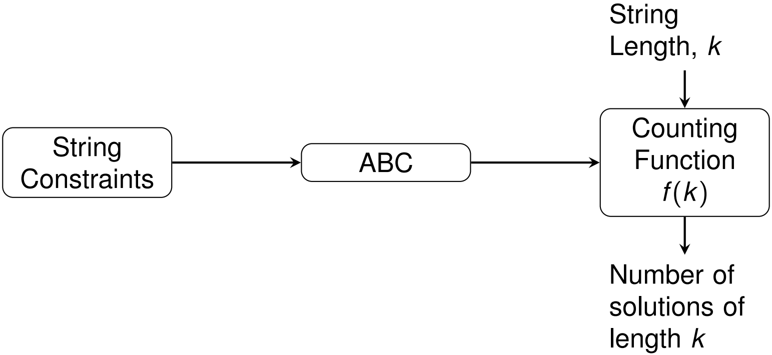
# Automata-Based Model Counter (ABC)

CAV 2015: Automata-Based Model Counting for String Constraints.
Abdulbaki Aydin, Lucas Bang, Tevfik Bultan.



# Automata-Based Model Counter (ABC)

CAV 2015: Automata-Based Model Counting for String Constraints.
Abdulbaki Aydin, Lucas Bang, Tevfik Bultan.




Idea: Convert string constraints to DFA. Count paths in DFA.

# Password Changing Policy

## Constraint on NEW_P

```
(declare-fun NEW_P () String)

(not (contains (toLower NEW_P) "abc-16"))
(not (contains "abc-16" (toLower NEW_P)))
(not (contains (toLower NEW_P) "61-cba"))
(not (contains "61-cba" (toLower NEW_P)))

(check-sat)
(model-count)
```

# Password Changing Policy

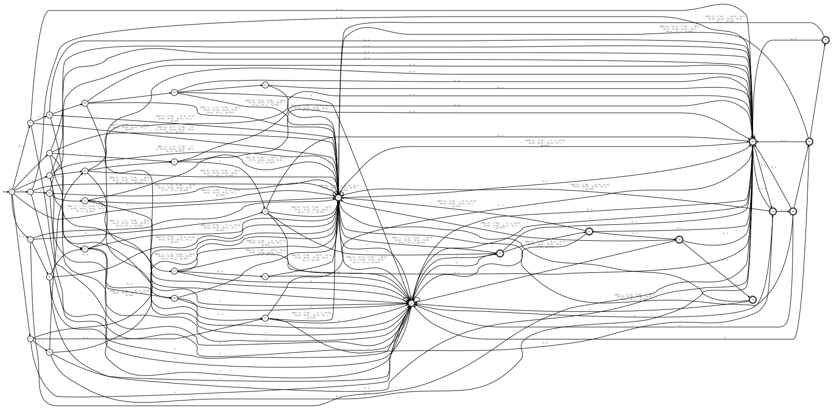


Figure: Solution DFA for all possible values of NEWP.

# Password Changing Policy

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Figure: Transition matrix for DFA for all possible values of NEWP.

# Password Changing Policy

$$
\begin{pmatrix}
0&1&1&1&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&2&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
$$

Figure: Transition matrix for DFA for all possible values of NEWP.

# Password Changing Policy

Generating function which enumerates NEW_P:

$$g(z) = \frac{8096z^{12} - 8128z^{11} + 32z^{10} + 16z^{7} - 16z^{6} - 256z^{2} + 257z - 1}{194304z^{17} + 225920z^{16} + 241984z^{15} + \ldots + z^{5} - 6114z^{4} - 2280z^{3} - 247z^{2}}$$

# Password Changing Policy

Generating function which enumerates NEW_P:

$$g(z) = \frac{8096z^{12} - 8128z^{11} + 32z^{10} + 16z^{7} - 16z^{6} - 256z^{2} + 257z - 1}{194304z^{17} + 225920z^{16} + 241984z^{15} + \ldots + z^{5} - 6114z^{4} - 2280z^{3} - 247z^{2}}$$

$$g(z) = 247z^{2} + 65759z^{3} + 16842945z^{4} + 4311810213z^{5} + 1103823437965z^{6} + \ldots$$

# Password Changing Policy

Generating function which enumerates NEW_P:

$$g(z) = \frac{8096z^{12} - 8128z^{11} + 32z^{10} + 16z^7 - 16z^6 - 256z^2 + 257z - 1}{194304z^{17} + 225920z^{16} + 241984z^{15} + \ldots + z^5 - 6114z^4 - 2280z^3 - 247z^2}$$

$$g(z) = 247z^2 + 65759z^3 + 16842945z^4 + 4311810213z^5 + 1103823437965z^6 + \ldots$$

## To answer our quantitative question:

- Brute force searching for password length = 6: $256^6 = 2^{48}$ passwords.

# Password Changing Policy

Generating function which enumerates NEW_P:

$$g(z) = \frac{8096z^{12} - 8128z^{11} + 32z^{10} + 16z^{7} - 16z^{6} - 256z^{2} + 257z - 1}{194304z^{17} + 225920z^{16} + 241984z^{15} + \ldots + z^{5} - 6114z^{4} - 2280z^{3} - 247z^{2}}$$

$$g(z) = 247z^{2} + 65759z^{3} + 16842945z^{4} + 4311810213z^{5} + 1103823437965z^{6} + \ldots$$

## To answer our quantitative question:

- Brute force searching for password length = 6: $256^{6} = 2^{48}$ passwords.
- If adversary knows `old_p` and the policy: $1103823437965 \approx 2^{40.0056}$ passwords.

# Password Changing Policy

Generating function which enumerates NEW_P:

$$g(z) = \frac{8096z^{12} - 8128z^{11} + 32z^{10} + 16z^{7} - 16z^{6} - 256z^{2} + 257z - 1}{194304z^{17} + 225920z^{16} + 241984z^{15} + \ldots + z^{5} - 6114z^{4} - 2280z^{3} - 247z^{2}}$$

$$g(z) = 247z^2 + 65759z^3 + 16842945z^4 + 4311810213z^5 + 1103823437965z^6 + \ldots$$

## To answer our quantitative question:

- Brute force searching for password length = 6: $256^6 = 2^{48}$ passwords.
- If adversary knows `old_p` and the policy: $1103823437965 \approx 2^{40.0056}$ passwords.
- Reduces search space by about factor of $2^{7.9944}$

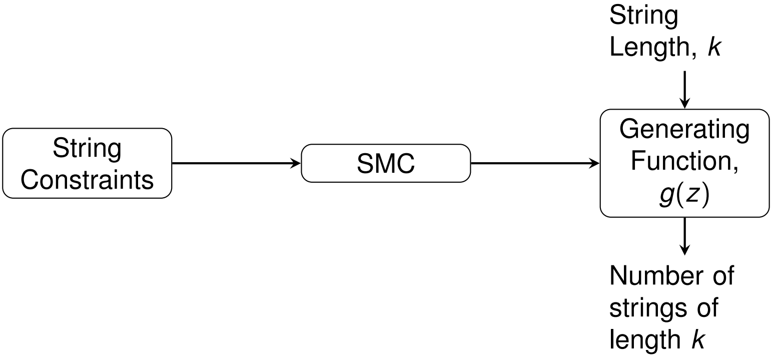
# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

# SMC Model Counting

PLDI 2014: A Model Counter For Constraints Over Unbounded Strings. Luu, Shinde, Saxena, Demsky.

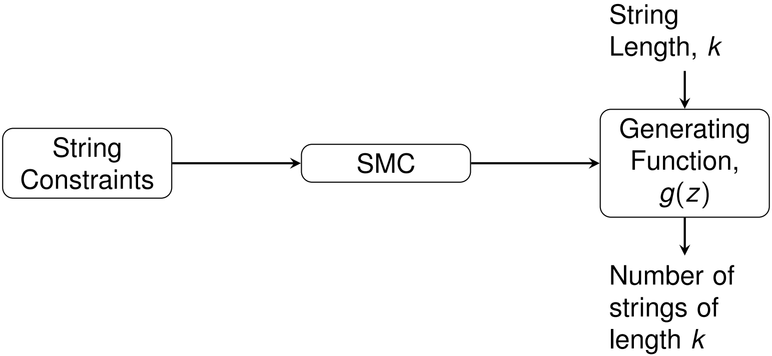
SMC Tool Online: https://github.com/loiluu/smc



# SMC Model Counting

PLDI 2014: A Model Counter For Constraints Over Unbounded Strings. Luu, Shinde, Saxena, Demsky.

SMC Tool Online: https://github.com/loiluu/smc




Idea: go directly from constraints to $g(z)$ using transformations.

# SMC Model Counting

For a regular expression constraint, generating function can be derived recursively.

# SMC Model Counting

For a regular expression constraint, generating function can be derived recursively.

$$\varepsilon \quad \mapsto \quad 1z^0$$

# SMC Model Counting

For a regular expression constraint, generating function can be derived recursively.

$$\begin{aligned} \varepsilon &\mapsto 1z^0 \\ c &\mapsto 1z^1 \end{aligned}$$

# SMC Model Counting

**For a regular expression constraint, generating function can be derived recursively.**

$$
\begin{array}{lll}
\varepsilon & \mapsto & 1z^0 \\
c & \mapsto & 1z^1 \\
A|B & \mapsto & A(z) + B(z)
\end{array}
$$

# SMC Model Counting

**For a regular expression constraint, generating function can be derived recursively.**

$$\begin{array}{lcl} \varepsilon & \mapsto & 1z^0 \\ c & \mapsto & 1z^1 \\ A|B & \mapsto & A(z) + B(z) \\ A \circ B & \mapsto & A(z) \times B(z) \end{array}$$

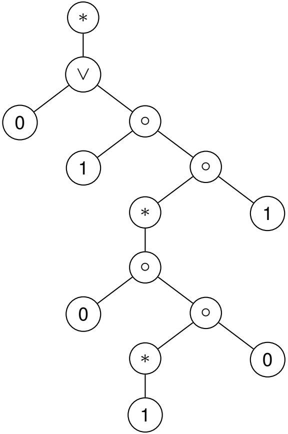
# SMC Model Counting

**For a regular expression constraint, generating function can be derived recursively.**

$$\begin{array}{lcl} \varepsilon & \mapsto & 1z^0 \\ c & \mapsto & 1z^1 \\ A|B & \mapsto & A(z) + B(z) \\ A \circ B & \mapsto & A(z) \times B(z) \\ A^* & \mapsto & 1/(1 - A(z)) \end{array}$$

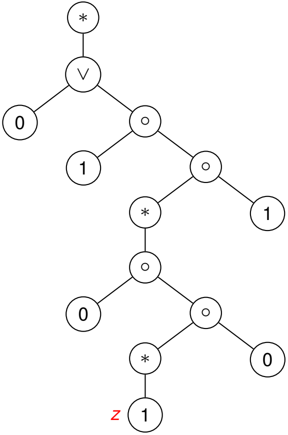
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



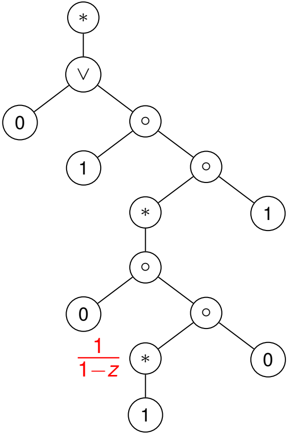
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



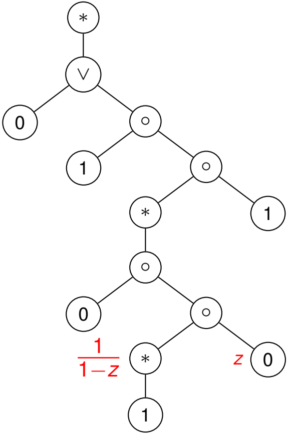
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



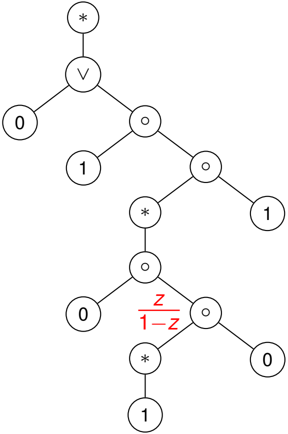
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



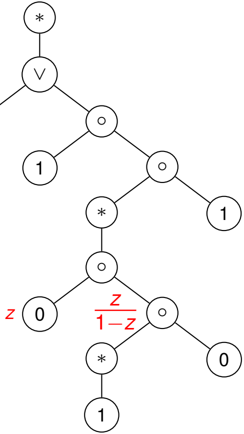
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



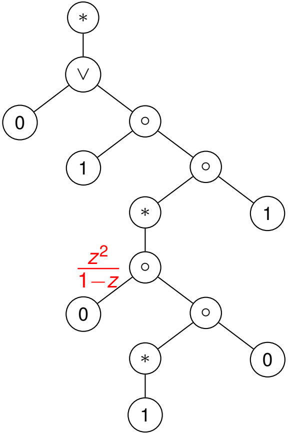
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



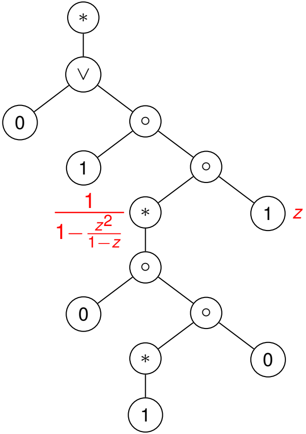
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



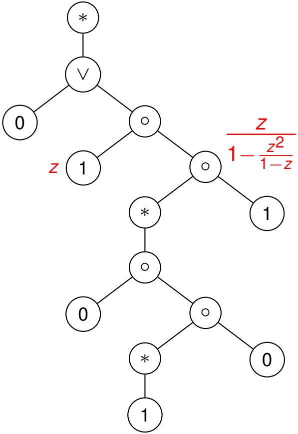
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



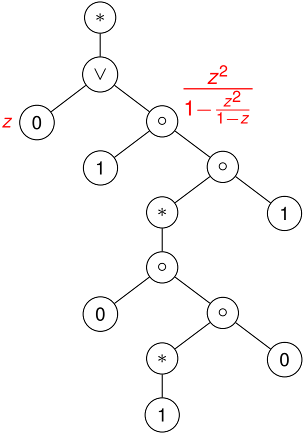
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



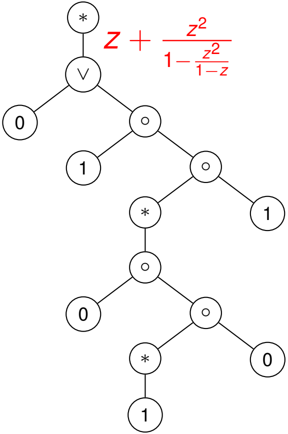
# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



# Regular Expressions

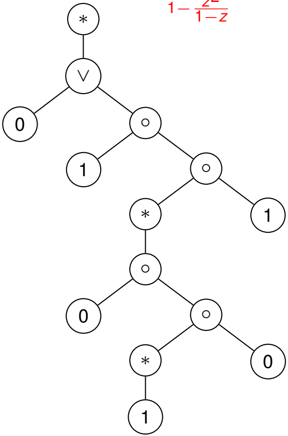
$$X \in (0|(1(01^*0)^*1))^*$$



# Regular Expressions

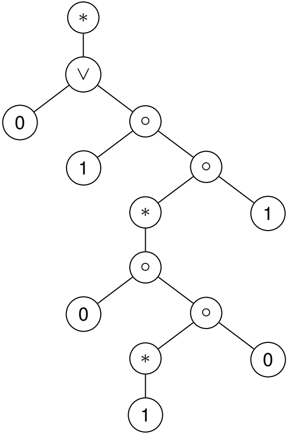
$$X \in (0|(1(01^*0)^*1))^*$$

$$\frac{1}{1-z-\frac{z^2}{1-\frac{z^2}{1-z}}}$$



# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$

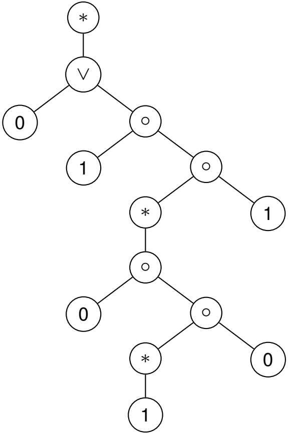



Generating Function:

$$g(z) = \frac{1}{1-z-\frac{z^2}{1-\frac{z^2}{1-z}}}$$

# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$



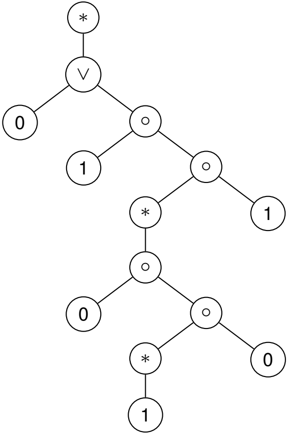

Generating Function:

$$g(z) = \frac{1}{1 - z - \frac{z^2}{1 - \frac{z^2}{1-z}}}$$

$$= \frac{1 - z - z^2}{(z-1)(2z^2 + z - 1)}$$

# Regular Expressions

$$X \in (0|(1(01^*0)^*1))^*$$




Generating Function:

$$g(z) = \frac{1}{1-z-\frac{z^2}{1-\frac{z^2}{1-z}}}$$

$$= \frac{1-z-z^2}{(z-1)(2z^2+z-1)}$$

$$g(z) = 1z^0+1z^1+1z^2+1z^3+3z^4+5z^5+\ldots$$

# Other operations in SMC

**Specialized transformations for other operations**

# Other operations in SMC

## Specialized transformations for other operations

$$contains(s_1, s_2) \quad \mapsto \quad \frac{z^n}{(1-Mz)(z^n+(1-Mz)c(z))}$$

# Other operations in SMC

## Specialized transformations for other operations

$$
\begin{aligned}
\mathit{contains}(s_1, s_2) &\mapsto \tfrac{z^n}{(1-Mz)(z^n+(1-Mz)c(z))} \\
F_1 \vee F_2 &\mapsto [\max(L_1(z), L_2(z)), \min(U_1(z) + U_2(z), G(z))]
\end{aligned}
$$

# Other operations in SMC

## Specialized transformations for other operations

$$
\begin{aligned}
\mathit{contains}(s_1, s_2) &\mapsto \frac{z^n}{(1-Mz)(z^n+(1-Mz)c(z))} \\
F_1 \vee F_2 &\mapsto [\max(L_1(z), L_2(z)), \min(U_1(z) + U_2(z), G(z))]
\end{aligned}
$$

Also handle substring, length, negation, conjunction, ...,
with upper and lower bounds.

# Outline

- Motivation and Background
- Model Counting Boolean Formulas
- String Model Counting
  - Automata-Based Methods
  - Non-Automata-Based Method
- String Model Counting Benchmarks

# Experimental Comparison

Table: Log scaled comparison between SMC and ABC

| | bound | SMC lower bound | SMC upper bound | ABC count |
|---|---|---|---|---|
| nullhttpd | 500 | 3752 | 3760 | 3760 |
| ghttpd | 620 | 4880 | 4896 | 4896 |
| csplit | 629 | 4852 | 4921 | 4921 |
| grep | 629 | 4676 | 4763 | 4763 |
| wc | 629 | 4281 | 4284 | 4281 |
| obscure | 6 | 0 | 3 | 2 |

# Experimental Comparison

## JavaScript Benchmarks

- Kaluza benchmarks, extracted from JavaScript code via DSE, [Saxena, SSP 2010]

# Experimental Comparison

## JavaScript Benchmarks

- Kaluza benchmarks, extracted from JavaScript code via DSE, [Saxena, SSP 2010]
- Small Constraints (19,731):
  - ABC: 19,731 constraints, average 0.32 seconds per constraint
  - SMC: 17,559 constraints, average 0.26 seconds per constraint.

# Experimental Comparison

## JavaScript Benchmarks

- Kaluza benchmarks, extracted from JavaScript code via DSE, [Saxena, SSP 2010]
- Small Constraints (19,731):
  - ABC: 19,731 constraints, average 0.32 seconds per constraint
  - SMC: 17,559 constraints, average 0.26 seconds per constraint.
- Big Constraints (1,587):
  - ABC: 1,587 constraints, average 0.34 seconds per constraint
  - SMC: 1,342 constraints, average 5.29 seconds per constraint

# ABC Bonus: Model Counting Linear Integer Arithmetic

# ABC Bonus: Model Counting Linear Integer Arithmetic

What is this language?

$$X \in (0|(1(01^*0)^*1))^*$$

# ABC Bonus: Model Counting Linear Integer Arithmetic

What is this language?

$$X \in (0|(1(01^*0)^*1))^*$$

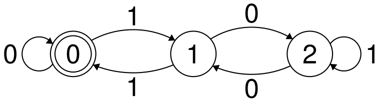
$$L(X) = \{s|s \text{ is a binary number divisible by 3}\}$$

# ABC Bonus: Model Counting Linear Integer Arithmetic

What is this language?

$$X \in (0|(1(01^*0)^*1))^*$$

$$L(X) = \{s|s \text{ is a binary number divisible by } 3\}$$

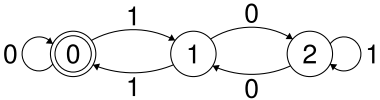


# ABC Bonus: Model Counting Linear Integer Arithmetic

What is this language?

$$X \in (0|(1(01^*0)^*1))^*$$

$$L(X) = \{s | s \text{ is a binary number divisible by } 3\}$$




**Idea:** DFA can represent (some) relations on sets of binary integers. We can use similar techniques that we used for $\#$String to solve $\#$LIA.

# Model Counting Linear Integer Arithmetic

Quantifier-Free Linear Integer Arithmetic $(\mathbb{Z}, +, <)$.

# Model Counting Linear Integer Arithmetic

Quantifier-Free Linear Integer Arithmetic $(\mathbb{Z}, +, <)$.

Constraints of the form:

$$Ax < B, x \in \mathbb{Z}^n$$

# Model Counting Linear Integer Arithmetic

Quantifier-Free Linear Integer Arithmetic $(\mathbb{Z}, +, <)$.

Constraints of the form:

$$Ax < B, x \in \mathbb{Z}^n$$

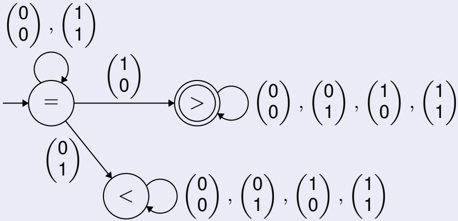
It is possible to represent the solutions to a set of LIA constraints as a binary multi-track DFA.

# Binary Multi-track DFA

## Solution DFA for LIA constraints.

- Read bits of $x$ and $y$ from most to least significant.
- Alphabet is a tuple of bits: $\begin{pmatrix} b_x \\ b_y \end{pmatrix}$

## Solution DFA for the constraint $x > y$.

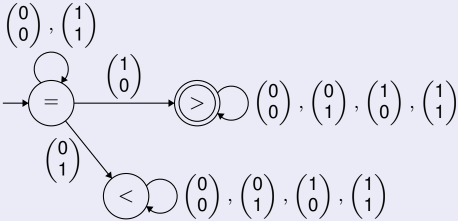


# Binary Multi-track DFA

## Solution DFA for LIA constraints.

- Read bits of $x$ and $y$ from most to least significant.
- Alphabet is a tuple of bits: $\begin{pmatrix} b_x \\ b_y \end{pmatrix}$

## Solution DFA for the constraint $x > y$.




$$
\begin{tikzcd}
\arrow[r] & {=} \arrow[loop above, "{\binom{0}{0},\binom{1}{1}}"] \arrow[r, "\binom{1}{0}"] \arrow[dr, "\binom{0}{1}"'] & {>} \arrow[loop right, "{\binom{0}{0},\binom{0}{1},\binom{1}{0},\binom{1}{1}}"] \\
& & {<} \arrow[loop right, "{\binom{0}{0},\binom{0}{1},\binom{1}{0},\binom{1}{1}}"]
\end{tikzcd}
$$

Solutions of length $n \equiv$ solutions within bound $2^n$

# Model Counting Summary

## Counting Techniques for Different Theories

- Boolean

# Model Counting Summary

## Counting Techniques for Different Theories

- Boolean
  - Truth Table (Brute Force)
  - DPLL

# Model Counting Summary

## Counting Techniques for Different Theories

- Boolean
  - Truth Table (Brute Force)
  - DPLL
- Strings
  - DFA with Dynamic Programming, Matrix Multiplication, GFs
  - Regular Expression with GFs

# Model Counting Summary

## Counting Techniques for Different Theories

- Boolean
  - Truth Table (Brute Force)
  - DPLL
- Strings
  - DFA with Dynamic Programming, Matrix Multiplication, GFs
  - Regular Expression with GFs
- Linear Integer Arithmetic
  - Binary Multi-track DFA

# Related work on model counting


- Stanley. Enumerative Combinatorics Chapter 4. 2004.
- Sedgwick. Analytic Combinatorics Chapter 5: Generating Functions. 2009
- Biere. Handbook of Satisfiability. Chapter 20: Model Counting. 2009
- Pugh. Counting Solutions to Presburger Formulas: How and Why. 1994
- Parker. An Automata-Theoretic Algorithm for Counting Solutions to Presburger Formulas. Compiler Construction 2004
- Boigelot. Counting the solutions of Presburger equations without enumerating them. TCS 2004.
- Barvinok. A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed. Mathematics of Operations Research 1994
- De Loerab. Effective lattice point counting in rational convex polytopes. JSC 2004
- Verdoolaege. Counting integer points in parametric polytopes using Barvinoks's Rational Functions. 2007
- Kopf Symbolic Polytopes for Quantitative Interpolation and Verification. CAV 2015
- Luu. A Model Counter For Constraints Over Unbounded Strings. PLDI 2014
- Ravikumara. Weak minimization of DFA - an algorithm and applications.Implementation and Application of Automata 2004
- Chomsky. The Algebraic Theory of Context-Free Languages. 1963
- Phan. Model Counting Modulo Theories. PhD Thesis 2014.
- Birnbaum. The good old Davis-Putnam procedure helps counting models. JAIR 1999

Thank you.