CS 138: Mid-quarter Examination 2

Department of Computer Science
University of California, Santa Barbara
Closed-Book, 75 minutes

Fall 2004

Instructions

- *Before* you answer any questions, print your name and perm number.
- Read each question carefully. Make sure that you clearly understand each question before answering it.
- Put your answer to each question on its own page.
- You may wish to work out an answer on scratch paper before writing it on your answer page; answers that are difficult to read may lose points for that reason.
- You may not leave the room during the examination, even to go to the bathroom.
- You may not use any personal devices, such as calculators, PDAs, or cell phones.

1. (15 points) Prove or disprove the following statement: If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language L, then $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

   Answer

   (a) Assume M is a a minimal DFA for L and $\overline{M}$ is not a minimal DFA for $\overline{L}$.

   (b) Let $M' = (Q', \Sigma, \delta', q'_0, F')$ be a minimal DFA for $\overline{L}$.

   (c) $|Q'| < |Q|$.

   (d) Let $M'' = (Q', \Sigma, \delta', q'_0, Q' - F')$.

   (e) $L(M'') = L$, contradicting the assumption that M was a minimal DFA accepting L.

2. (10 points) The **symmetric difference** of 2 sets S_1 and S_2 is defined as

$$S_1 \ominus S_2 = \{x : x \in S_1 \text{ or } x \in S_2, \text{ and } x \text{ is not in both } S_1 \text{ and } S_2\}.$$

Prove that the family of regular languages is closed under symmetric difference or give a counterexample.

Answer

It is closed under symmetric difference.

(a) Let S_1 and S_2 be regular sets.

(b) Then

$$(S_1 \text{ or } S_2) \text{ and } (\text{ not } (S_1 \text{ and } S_2)) = (S_1 \cup S_2) \cap \overline{(S_1 \cap S_2)} = S_1 \ominus S_2$$

is regular, since regular sets are closed under union, intersection, and complement.

3. (15 points) Is there an algorithm for determining if $L_1 \subseteq L_2$, for any regular languages L_1 and L_2? Prove your answer.

Answer

Yes, there is. If $L_1 \subseteq L_2$ then $L_1 - L_2 = \emptyset$. An algorithm follows.

(a) Construct regular set $L_1 - L_2 = L_1 \cap \overline{L_2} = L$. This can be done since there are constructive proofs that regular sets are closed under intersection and complement.

(b) Apply the algorithm for determining if $L = \emptyset$.

4. (15 points) Is the language $L = \{w \in \{a, b\}^* : n_a(w) = n_b(w)\}$ regular? Prove your answer.

Answer

Since

- regular languages are closed under intersection
- $L \cap a^*b^* = \{a^n b^n : n \geq 0\}$ is irregular

L is irregular.

An alternate proof that uses the Pumping Lemma follows.

(a) Assume L is regular. Then, by the Pumping Lemma, there is a natural number m such that any $w \in L$ with $|w| \geq m$ can be factored as $w = xyz$ with $|xy| \leq m$ and $|y| > 0$, and $xy^i z \in L$, for $i = 0, 1, \ldots$.

(b) Pick $w = a^m b^m$.

(c) Then, $a^m b^m = xyz$, where $y = a^k$, for $k > 0$.

(d) By the Pumping Lemma, $xz \in L$.

(e) But, $n_a(xz) \neq n_b(xz)$.

(f) The assumption that L is regular thus is false.

5. (15 points) Prove that the following statement is true or prove that it is false.

   If L_1 and $L_1 \cup L_2$ are regular languages, then L_2 is a regular language.

   Answer

   The statement is false.

   Let $L_1 = \{a, b\}^*$ and $L_2 = \{a^n b^n : n \geq 0\}$.

   Then L_1 and $L_1 \cup L_2$ are regular, but L_2 is irregular.

6. (10 points) Let $L = \{a^n b^n : n \geq 0\}$. Is L^2 context-free? Prove your answer.

Answer

Yes, it is.

A CFG that recognizes L^2 is $G_2 = (\{S_2, S\}, \{a, b\}, S_2, P)$, where P has the following productions

$$\begin{aligned} S_2 &\rightarrow SS, \\ S &\rightarrow aSb \mid \lambda. \end{aligned}$$

7. (10 points) Is the following grammar ambiguous? Prove your answer.

$$\begin{aligned} S &\rightarrow AB \mid aaB, \\ A &\rightarrow a \mid Aa, \\ B &\rightarrow b. \end{aligned}$$

Answer

Yes, it is.

The word aab has 2 different leftmost derivations:

$$\begin{aligned} S &\Rightarrow AB \Rightarrow AaB \Rightarrow aaB \Rightarrow aab \\ S &\Rightarrow aaB \Rightarrow aab \end{aligned}$$

8. (10 points) Construct a NPDA that accepts $\{a^n b^{2n} : n \geq 0\}$ over input alphabet $\{a, b, c\}$.

Answer

$M = (\{q_0, q_1, q_2\}, \{a, b, c\}, \{z, b\}, \delta, q_0, z, \{q_2\})$, where δ is given by the following diagram.