Chapter 4: Properties of Regular Languages*

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- Please read the corresponding chapter before attending this lecture.
- These notes are supplemented with figures, and material that arises during the lecture in response to questions.
- Please report any errors in these notes to `cappello@cs.ucsb.edu`. I'll fix them immediately.

*Based on **An Introduction to Formal Languages and Automata**, 3rd Ed., Peter Linz, Jones and Bartlett Publishers, Inc.

4.1 Closure Properties of Regular Languages

Closure under Simple Set Operators

Thm. 4.1: If L_1 and L_2 are regular languages, then so are $L_1 \cup L_2, L_1 \cap L_2, L_1L_2, \overline{L_1}$, and L_1^*.
Proof:

1. Assume that L_1 and L_2 are regular.
2. Let regular expression r_1 and r_2 denote L_1 and L_2, respectively.
3. Then,
    - $r_1 + r_2$ denotes $L_1 \cup L_2$,
    - r_1r_2 denotes L_1L_2,
    - r_1^* denotes L_1^*.

4. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that accepts L_1.
5. Then, $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$ accepts $\overline{L_1}$.
6. Since regular languages are closed under complement and union, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$ is a regular language.

- Let $w = s_1 s_2 \cdots s_n$ be a word over Σ. Then, w^R denotes the word $s_n \cdots s_2 s_1$, the **reverse** of w. $\lambda^R = \lambda$.
- Let L be a language. Then L^R denotes $L^R = \{w^R : w \in L\}$, called the **reverse** of L.

Thm. 4.2: The family of regular languages is closed under reversal.
Proof:

1. Let L be regular, and $M = (Q, \Sigma, \delta, q_0, \{q_f\})$ be an NFA that accepts it[1].
2. We construct $M^R = (Q, \Sigma, \delta^R, q_f, \{q_0\})$, where δ^R is δ with the orientation of the arcs reversed.
3. There is a path from q_0 to q_f in M if and only if there is a path from q_f to q_0 in M^R: $L(M^R) = L^R$.

[1] We may assume without loss of generality that $|F| = 1$

Def. 4.1: Let Σ and Γ be alphabets. Then, a function

$$h : \Sigma \mapsto \Gamma^*$$

is called a **homomorphism**.

- For each symbol in Σ, a homomorphism substitutes a word in Γ^*.
- Let $w = s_1 s_2 \cdots s_n$. Then,

$$h(s_1 s_2 \cdots s_n) = h(s_1)h(s_2) \cdots h(s_n).$$

- If L is a language on Σ, then its homomorphic image is

$$h(L) = \{h(w) : w \in L\}.$$

Example:

- Let $\Sigma = \{0, 1\}$ and $\Gamma = \{a, b, \ldots, z\}$.
- Define h as follows:

$$\begin{aligned} h(0) &= hello \\ h(1) &= goodbye \end{aligned}$$

- Then, $h(010) = hellogoodbyehello$.
- The homomorphic image of $L = \{00, 010\}$ is

$$h(L) = \{hellohello, hellogoodbyehello\}.$$

Thm. 4.3: Let h be a homomorphism. If L is a regular language, then its homomorphic image $h(L)$ is regular. The family of regular languages therefore is closed under arbitrary homomorphisms.
Proof:

1. Assume that L is regular, and let M be a DFA that accepts L.
2. Construct a generalized transition graph (GTG), based on the transition graph (TG) for M as follows:

   For each symbol, s, that labels an arc in the TG for M, label that same arc in the GTG with $h(s)$.
3. There is a path labelled w from q_0 to some final state q_f in the TG for M if and only if there is a path labelled $h(w)$ from q_0 to q_f in the GTG.

Def. 4.2: Let L_1 and L_2 be languages on the same alphabet. Then, the **right quotient** of L_1 with L_2 is defined as

$$L_1/L_2 = \{x : xy \in L_1 \text{ for some } y \in L_2\}.$$

Example: If

$$L_1 = \{a^n b^m : n \geq 1, m \geq 0\} \cup \{ba\}$$

and

$$L_2 = \{b^m : m \geq 1\},$$

then

$$L_1/L - 2 = \{a^n b^m : n \geq 1, m \geq 0\}.$$

- Draw a TG for L_1.
- Identify each state, q_i in TG_1 such that there exists a $y \in L_2$ and there is a path from q_i to a final state in TG_1.

- There are 2 such states, q_1 and q_2.

  These are the final states in L_1/L_2.

Thm. 4.4: If L_1 and L_2 are regular languages, then L_1/L_2 is regular: The family of regular languages is closed under right quotient with a regular language.

Proof:

1. Assume that L_1 and L_2 are regular, and let DFA $M = (Q, \Sigma, \delta, q_0, F)$ accept L_1.
2. We construct DFA $\widehat{M} = (Q, \Sigma, q_0, \widehat{F})$ as follows.
   (a) For each $q_i \in Q$, determine if there is a $y \in L_2$ such that
   $$\delta^*(q_i, y) \in F.$$
   (b) This can be done by the following procedure:
      i. Construct $M_i = (Q, \Sigma, \delta, q_i, F)$.
      ii. If $L_2 \cap L(\widehat{M}) \neq \emptyset$ then $q_i \in \widehat{F}$.
3. If $x \in L_1/L_2$ then $x \in L(\widehat{M})$.

4. If $x \in L_1/L_2$, there exists a $y \in L_2$ such that $xy \in L_1$.
5. If $xy \in L_1$, then:
   - $\delta(q_0, x) = q$, for some $q \in Q$
   - $\delta(q, y) \in F$
   - By construction, $q \in \widehat{F}$, so $\widehat{M}$ accepts x.
6. It similarly is easy to show that If $x \in L(\widehat{M})$ then $x \in L_1/L_2$.