Diversity in Kemeny Rank Aggregation: A Parameterized Approach

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Abstract

In its most traditional setting, the main concern of optimization theory is the search for optimal solutions for instances of a given computational problem. A recent trend of research in artificial intelligence, called *solution diversity*, has focused on the development of notions of optimality that may be more appropriate in settings where subjectivity is essential. The idea is that instead of aiming at the development of algorithms that output a *single* optimal solution, the goal is to investigate algorithms that output a small set of *sufficiently good* solutions that are *sufficiently diverse* from one another. In this way, the user has the opportunity to choose the solution that is most appropriate to the context at hand. It also displays the richness of the solution space.

When combined with techniques from parameterized complexity theory, the paradigm of diversity of solutions offers a powerful algorithmic framework to address problems of practical relevance. In this work, we investigate the impact of this combination in the field of Kemeny Rank Aggregation, a well-studied class of problems lying in the intersection of order theory and social choice theory and also in the field of order theory itself. In particular, we show that the Kemeny Rank Aggregation problem is fixed-parameter tractable with respect to natural parameters providing natural formalizations of the notions of *diversity* and of the notion of a *sufficiently good* solution. Our main results work both when considering the traditional setting of aggregation over linearly ordered votes, and in the more general setting where votes are partially ordered.

1 Introduction

Traditionally, in optimization theory, when given an instance of a computational problem, one is interested in computing *some* optimal solution for the instance in question. For certain problems of practical relevance, this framework may not be satisfactory because it precludes the user from the possibility of choosing among optimal solutions in case more than one exists, or even from choosing a slightly less optimal solution that may be a better fit for the intended application, due to subjective factors.

A recent up-coming trend of research in artificial intelligence, called *diversity of solutions* [57, 5, 45, 6, 26], has focused on the development of notions of optimality that may be more appropriate in settings where subjectivity is essential. The idea is that instead of aiming at the development of algorithms that output a *single* optimal solution, the goal is to investigate algorithms that output a small set of *sufficiently good* solutions that are *sufficiently diverse* from one another. In this way, the user has the opportunity to choose the solution that is most appropriate to the context at hand. The intuition is that the criteria employed by the user to decide what an appropriate solution is may be subjective, and therefore, impractical or even impossible to be formalized at the level of the problem specification. Examples of such criteria are aesthetic, economic, political, environmental criteria. Another motivation comes from the problem of finding several good committees such that each committee member is not overloaded with these commitments, as described in [9]; again, some diversity could be helpful.

One source of difficulty when trying to develop efficient algorithms for diverse variants of computational problems is the fact that these problems may be computationally hard. In particular, many interesting computational problems that are suitable for being studied from the perspective of diversity of solutions are already NP-hard in the usual variant in which one asks for a single solution. Additionally, it may be the case that even problems that are polynomial-time solvable in the single-solution version become NP-hard when considering diverse sets of solutions. One way to circumvent this difficulty is to combine the framework of diversity of solutions with the framework of fixed-parameter tractability theory [16]. A central notion in this framework is the one of fixed-parameter tractability. An algorithm for a given computational problem is said to be fixed-parameter tractable with respect to parameters $k_1, \ldots, k_r$ if it runs in time $f(k_1, \ldots, k_r) \cdot n^{O(1)}$, where n is the size of the input and f is a computable function that depends only on the parameters. The intuition is that if the range of the parameters is small on instances of practical relevance, then even if the function f grows relatively fast, the algorithm can be considered to be fast enough *for practical purposes*.

When studying a given computational problem from the point of veiw of solution diversity, it is crucial to have in hands a notion of distance between solutions for that problem. The *diversity* of a set of solutions S is then defined as the sum of distances between pairs of solutions in S. We denote this measure by d. Intuitively, diversity is a global measure for how representative a set of solutions is among the space of solutions. Three natural parameters can be used to quantify how good a diverse set of solutions is: the number r of solutions in the set, the maximum distance δ between the cost of a solution in the set and the cost of an optimal solution (we call this parameter the *solution imperfection* of the set), and the minimum *required* distance s between any two solutions in the set. This last parameter is also known in the literature as the *scatteredness* of S [30]. Intuitively, the parameter r is expected to be small because in practical applications we do not want to overwhelm the user with an excessive number of choices. The parameter δ is expected to be small because while we do want S to be diverse, we do not want to allow solutions of bad quality. Finally, in the context of our work, solution diversity is formalized by the parameter d, and we only use the scatteredness parameter s to enforce that one cannot achieve high diversity by copying a given solution an arbitrary number of times. For this it is enough to require $s = 1$. It is worth noting that it is possible to have s very small (say $s = 1$), but d very large, since some pairs of solutions in the set may be very far apart from one another.

In this work, we investigate the impact of the notions of diversity of solutions and of fixed parameter tractability theory in the context of social choice theory. In particular, we focus on the framework of preference list aggregation introduced by Kemeny in the late fifties [50]. Intuitively, preference lists are a formalism used in social choice theory to capture information about choice in applications involving the selection of candidates, products, etc. by a group of voters. The task is then to find a ranking of the candidates that maximizes the overall satisfaction among the voters. This problem is commonly referred to in modern terminology as the *Kemeny rank aggregation* (KRA) problem. In its most general setting, the ranking cast by each voter is a partial order on the set of candidates. The distance measure we use to define our diverse version for KRA is the Kendall-Tau distance which is widely used in the context of preference aggregation.[1] Its popularity is underlined by articles describing these issues for the interested public audience; see [22].

1.1 Our Contribution

Our main result is a multiparametric algorithm for Diverse KRA over partially ordered votes that runs in time $f(\mathrm{w}, r, \delta, s) \cdot d \cdot n \cdot \log(n^2 \cdot m)$ where n is the number of candidates, m is the number of votes, r, δ, s and d are the parameters discussed above, and w is the *unanimity width* of the votes. That is to say, the pathwidth of the cocomparability graph of the unanimity order of the input votes (Corollary 15). Intuitively, this width measure is a quantification of the amount of disagreement between the votes. Note that pathwidth and treewidth coincide for the class of cocomparability graphs [39].

On the path towards obtaining our results for Kemeny Rank Aggregation, we also make contributions to problems of independent interest arising in the theory of cocomparability graphs. First, by leveraging on classic results from [39], we show that the problem of constructing a ρ-consistent path decomposition of approximately minimum width for the cocomparability graph G_ρ of a given partial-order ρ is fixed-parameter tractable with respect to the pathwidth of G_ρ. While it was known that the pathwidth and the ρ-consistent pathwidth of G_ρ are always the same [3], and that there were fixed-parameter tractable algorithms for computing path decompositions of approximately minimum width due to structural properties of cocomparability graphs [8], the problem of computing such a decomposition satisfying the additional ρ-consistent requirement was open [3].

Second, we note that the notion of Kendall-Tau distance between partial orders (formally defined in Section 3), which is used to define our notion of diversity, can be applied equally well in the more general context of the Completion of an Ordering problem (CO), a problem of fundamental importance in order theory that unifies several problems of relevance for artificial intelligence, such as KRA, One-Sided Crossing Minimization (an important sub-routine used in the search for good hierarchical representations of graphs), and Grouping by Swapping (a relevant problem in the field of memory management) [63]. For a matter of generality, we first develop a $f(\mathrm{w}, r, \delta, s) \cdot d \cdot n \cdot \log(n^2 \cdot m)$ time algorithm for Diverse CO (Theorem 14) and then obtain our main result for Diverse KRA as a corollary. In the more general context of CO, the parameter w is the width of the cocomparability graph of the partial order given at the input.

Finally, building on recent advances in the theory of C_k-free graphs [11] we establish an upper bound for the pathwidth of a cocomparability graph in terms of the number of edges of the graph. As a by-product of this result, we obtain the first algorithm running in time $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$ (Theorem 19) for the positive completion of an ordering problem (PCO), a special case of CO which still generalizes KRA and other important combinatorial problems. Previous to our work, the best algorithm for this problem parameterized by cost had asymptotic time complexity of $\mathcal{O}^*(k^{\mathcal{O}(\sqrt{k})}) = \mathcal{O}^*(2^{\sqrt{k}\log k})$. Therefore, we remove the log-factor in the exponent. According to Theorem 18 in [3], this is optimal under the Exponential Time Hypothesis (ETH).

[1] To cite from the Stanford Encyclopedia of Philosophy [52]: *At the heart of social choice theory is the analysis of preference aggregation, understood as the aggregation of several individuals' preference rankings of two or more social alternatives into a single, collective preference ranking (or choice) over these alternatives.*

It is worth noting that in the context of KRA over totally ordered votes, the existence of diverse sets of high-quality solutions implies that any optimal solution disagrees significantly with some of the voters. More precisely, let k_{opt} be the cost of an optimal solution and suppose that there are two solutions with cost $k_{\text{opt}} + \delta_1$ and $k_{opt} + \delta_2$, respectively. It is possible to show that $\max\{k_{opt} + \delta_1, k_{opt} + \delta_2\}$ is at least half the number of votes. Therefore, if two solutions have small solution imperfection, then k_{opt} is large. In the other direction, if there is a strong consensus among the voters (k_{opt} is small) then δ_1 or δ_2 must be large. Intuitively, in the context of aggregation over totally-ordered votes, the more disagreement there is between an optimal ranking and the ranking provided by the voters, the more one can benefit from the framework of solution diversity. In the context of aggregation over partially ordered votes, such a correlation between solution imperfection and optimality does not necessarily hold even for constant unanimity width. For instance, consider a set of partially ordered votes where the unanimity order is a bucket order with buckets of size 2 (i.e., unanimity width equal to 1). Then depending on the instance, we can have diverse sets of solutions with solution imperfection 0 and optimal cost 0. Therefore, in the context of partially ordered votes, the notion of diversity makes sense even in the case where voters have small disagreement between each other.

1.2 Related Work

The framework of diversity of solutions, under distinct notions of diversity, has found applications in several subfields of artificial intelligence, such as information search and retrieval [35, 1], mixed integer programming [34, 14, 56], binary integer linear programming [37, 60], constraint programming [41, 42], SAT solving [55], recommender systems [2], routing problems [58], answer set programming [19], decision support systems [53, 40], genetic algorithms [28, 62], planning [6], and in many other fields. Recently, a general framework for addressing diversity of solutions from the perspective of parameterized complexity theory was developed [5]. This framework allows one to convert dynamic programming algorithms for finding an optimal solution for instances of a given problem into dynamic programming algorithms for finding a small set of diverse solutions.

Notice that there is also the related area of enumerating all optimal solutions, or at least encoding them all; this is known as knowledge compilation in artificial intelligence, see, e.g., [15, 21, 54]. These types of questions have also been considered from a more combinatorial viewpoint; confer [27, 36, 47, 48]. But from a practical perspective, it is not really desirable to confront a user with an exponential number of different solutions, but she wants to know what the real alternatives are.

Two measures of diversity of a set S of solutions have been particularly explored in the literature. The first one is the sum of distances between pairs of solutions in S. The second one is the minimum distance s between any two solutions in S. This last notion has been also known in the literature as *scatteredness* [30]. Both notions have been used in the context of vertex- and edge-problems on graphs using the Hamming distance of solutions as the distance measure [5, 28, 62, 26]. It is worth noting that when used alone, the diversity measure defined as the sum of Hamming distances has some weaknesses. For instance, if we take a pair $\{A, B\}$ of solutions of diversity d, then the list $A_1, A_2, ..., A_r, B_1, B_2, ..., B_r$, where each A_i is a copy of A, and each B_i is a copy of B, has high diversity ($d' > r^2 \cdot d$), while this list clearly opposes the intuitive notion of a diverse set of solutions. This weakness can be significantly mitigated by considering diversity in conjunction with scatteredness. For instance, by setting $s \geq 1$, we already guarantee that all elements in a list of solutions will be distinct from each other.

The notions of Kemeny score and Kemeny rank aggregation were fist introduced in [50]. These notions play a crucial role in fields such as social choice theory [4, 51] and found applications in diverse areas as information retrieval [18], preference learning [12], genetic map generation [46], etc. We also point to more recent publications like [13, 32] in the AI area that also allow to follow the literature on this topic. As shown by Young and Levenglick [65], this is the only aggregation method satisfying three natural requirements: *symmetry*, *consistency* and being *Condorcet*. This notion can also be regarded as a maximum-likelihood estimator [64]. While the KRA method has been originally designed to deal with the setting where the vote cast by each voter is a total order over the set of candidates, this method has been naturally generalized to the context where the ranking cast by each voter is a partial order over the set of candidates. Such partially ordered votes already generalize the notion of Bucket orderings (also known as weak orders) which has been studied in the context of aggregation and can be viewed as looking at rankings with ties, also called indifference classes [44, 66].

Most relevant to our studies is reference [66]. There, Kemeny rank aggregation is generalized both to starting with and aiming at bucket orderings, also known as weak orders. More precisely, the author considers the problem to find a bucket order with at most k buckets (indifference classes) that compromises a heap of bucket orders, each of which has at most j buckets. It is shown that this variation of KRA is NP-complete even for small constant values of j, k. This makes this parameterization useless from the viewpoint of parameterized complexity. However, one could also reverse the requirement and ask the voter to create a ranking in which no more than ℓ candidates are mutually indifferent; in other words, the buckets have a capacity limited by ℓ. As we will see, this turns out to be a useful parameter and we will study it in this paper.

The *minimum* Kendall-Tau distance over all pairs in a set is analogous to the minimum Hamming distance among pairs of solutions used in [41, 42].

2 Preliminaries

If n is a positive integer, $[n] = \{1, \ldots, n\}$ denotes the discrete interval of the first n positive integers, and $[n]_0 = [n] \cup \{0\}$. $\mathbb{N}$ denotes the non-negative integers.

Let C be a set. A *partial order* over C is a reflexive, antisymmetric and transitive binary relation $\rho \subseteq C \times C$. We say that ρ is a *linear order* if additionally, for each $(x, y) \in C \times C$, either $(x, y) \in \rho$ or $(y, x) \in \rho$. The *comparability relation* $\mathbf{sc}(\rho)$ of ρ is the symmetric closure of ρ, i.e., $(x, y) \in \mathbf{sc}(\rho)$ iff $(x, y) \in \rho$ or $(y, x) \in \rho$. For instance, $\mathbf{sc}(\rho) = C \times C$ iff ρ is a linear order. If $\rho \subseteq C \times C$ is a partial order, then $<_\rho$ denotes the corresponding strict order, which is irreflexive and transitive. Linear orders over C can be given by bijections $\pi : [|C|] \to C$. Hence, $<_\pi$ (or $\leq_\pi$) is used to denote the corresponding strict (or partial) linear order. Given a binary relation α, we denote by $\mathbf{tc}(\alpha)$ the transitive closure of α.

Definition 1 (Cocomparability graph). *Given a partial order $\rho \subseteq C \times C$, we let $G_\rho \doteq (C, C \times C \setminus \mathbf{sc}(\rho))$ be the* cocomparability graph *of ρ.*

Given an undirected graph $G = (V, E)$ and a vertex $v \in V$, we let $N(v) \doteq \{u \mid u \in V, (v, u) \in E\}$ be the neighborhood of v. A *path decomposition* of a graph $G = (V, E)$ is a sequence $\mathcal{P} = (B_1, B_2, \ldots, B_l)$ of subsets of V, such that the following conditions are satisfied.

- $\bigcup_{1 \leq i \leq l} B_i = V$.
- For each edge $(u, v) \in E$, there is an $i \in [l]$ such that $u, v \in B_i$.
- For each $i, j, k \in [l]$ with $i < j < k$, $B_i \bigcap B_k \subseteq B_j$.

For each position $p \in \{2, \ldots, l\}$, for each vertex $v \in B_p \setminus B_{p-1}$, we say that B_p *introduces* v (v is introduced by B_p) and for each vertex $v \in B_{p-1} \setminus B_p$, we say that B_p *forgets* v (v is forgotten by B_p). For a position $p \in \{1, \dots, l\}$, we write intro(p) (resp. forg(p)) for the set of all vertices introduced (resp. forgotten) by B_p, and we let $L_p = \bigcup_{1 \le i \le p} \mathrm{forg}(p)$ be the set of vertices that have been forgotten up to position p. The *width* of $\mathcal{P}$ is defined as $\mathrm{w}(\mathcal{P}) = \max_{i \in [l]} |B_i| - 1$. The *pathwidth*, $pw(G)$, of G is the minimum width of a path decomposition of G.

The pathwidth of the cocomparability graph of a partial order may be regarded as a measure of how close the order is from being a linear order. The cocomparability graph of a linear order τ on n elements is the graph with n vertices and no edges. This graph has pathwidth 0. On the other hand, if τ is a partial order where all n elements are unrelated, then the cocomparability graph of τ is the n-clique, which has pathwidth $n-1$ (the highest possible pathwidth in an n-vertex graph).

3 The Kemeny Rank Aggregation Problem

Let C be a finite set, which in this paper will denote a set of candidates, or alternatives. A *partial vote*[2] over C is a partial order over C. The KT-distance between two partial votes π_1 and π_2, denoted by KT-dist(π_1, π_2), is the number of pairs of candidates that are ordered differently in the two partial votes.

$$\text{KT-dist}(\pi_1, \pi_2) = |\{(c, c') \in C \times C \mid c <_{\pi_1} c' \wedge c' <_{\pi_2} c\}| \,.$$

Observe that when the votes are totally ordered, the Kendall-Tau distance can be seen as the 'bubble sort' distance, i.e., the number of pairwise adjacent transpositions needed to transform one linear order into the other. Given a linear order π over a set of candidates C and a set Π of votes over C, the *Kemeny score* of π with respect to Π is defined as the sum of the Kendall-Tau distances between π and each vote in Π. In this work, we consider the following problem.

Problem name: KEMENY RANK AGGREGATION (KRA)
Given: A list of partial votes Π over a set of candidates C, a non-negative integer k.
Output: Is there a linear order π on C such that the sum of the KT-distances of π from all the partial votes is $\le k$?

Hence, given partial votes $\pi_1, \dots, \pi_m$ of C and a non-negative integer k, the question is if there exists a linear order $\pi \subseteq C \times C$ such that $\sum_{i=1}^{m} \text{KT-dist}(\pi, \pi_i) \le k$.

Definition 2. *Given a set Π of partial votes, the* unanimity order *of Π is simply the partial order ρ obtained as the intersection of all partial orders in Π. In other words, a candidate c_1 has higher precedence than a candidate c_2 in ρ if and only if c_1 precedes c_2 in each vote in Π.*

As a consequence, the more disagreements there are among the voters with respect to the relative orders of pairs of candidates, the denser the cocomparability graph of ρ will be and therefore the greater its pathwidth will be. Therefore, the pathwidth of the cocomparability graph of the unanimity order of Π may be seen as a quantification of the amount of disagreement among the votes in Π.

[2]The literature is not clear about these notions. In [43], Hemaspaandra et al. call partial votes that allow ties in linear orders *preference rankings* and explain that this originally goes back to Kemeny. Allowing partial orders (as we do it here) is even more general. The authors of [43] also consider a different Kendall-Tau distance for preference rankings compared to our setting.

A notion of diversity for KRA. The notion of diversity of solutions for computationally hard problems has been considered under a variety of frameworks. In this work, we define a notion of diversity for the Kemeny Rank Aggregation problem which is analogous to the notion of diversity of vertex sets used in [5]. More precisely, if R is a set of partial orders, then we define the Kendall-Tau diversity of R as the sum of Kendall-Tau distances between votes in the set R.

$$\text{KT-Div}(R) = \sum_{\pi_1, \pi_2 \in R} \text{KT-dist}(\pi_1, \pi_2)$$

We note that the restricted version of the KRA problem where all votes are linear orders, the requirements that a set of solutions is at the same time diverse and only contains rankings with small Kemeny score are clashing. The problem is that the very existence of two distinct rankings with small Kemeny score is an impossible task. If two candidates c_1 and c_2 occur with the order (c_1, c_2) in one of the solutions and in the order (c_2, c_1) in the other solution, then at least one of these solutions will have a Kemeny score of at least half the number of votes. However this opposition between diversity and small Kemeny score is not present in the setting where votes are allowed to be partial. The generalization to partial votes is one possible way to circumvent this conflict of desiderata. Another way we will be looking at is not to consider the cost of the solutions directly but the difference between the cost of solutions and the cost of an optimal solution. In this case, we can have diversity and a small difference between the cost and the cost of an optimal solution.

Problem name: Diverse Kemeny Rank Aggregation (Diverse-KRA)
Given: A list of partial votes Π over a set of candidates C, and k, r, $d \in \mathbb{N}$.
Output: Is there a set $R = \{\pi_1, \dots, \pi_r\}$ of linear orders on C such that the Kemeny score for each order π_i is at most k and $\text{KT-Div}(R) \geq d$?

Parameterizations of KRA The problem Kemeny Rank Aggregation is known to be NP-complete [4], even if only four votes are given at the input [18].[3] For this reason, KRA has been studied from the perspective of parameterized complexity theory under a variety of parameterizations. Below, we consider two prominent parameterizations for this problem.

The first parameter we consider is the cost of a solution. Simjour [59] obtained an algorithm for the problem that runs in time $\mathcal{O}^*(1.403^k)$. There are also sub-exponential algorithms for Kemeny Rank Aggregation under this parameterization: Karpinski and Schudy [49] obtained an algorithm for Kemeny Rank Aggregation that runs in $\mathcal{O}^*(2^{O(\sqrt{k})})$ time, while the algorithm of Fernau *et al.* [24, 25], based on a different methodology, runs in $\mathcal{O}^*(k^{O(\sqrt{k})})$ time. Recently, in [3], it was shown that KRA on instances with only $m = 4$ votes on some candidate set C and some integer k bounding the sum of the Kendall-Tau distances to a solution cannot be solved neither in time $\mathcal{O}^*\left(2^{o(|C|)}\right)$ nor in time $\mathcal{O}^*\left(2^{o(\sqrt{k})}\right)$, unless ETH fails. The mentioned NP-hardness of KRA immediately translates to NP-hardness results of KRA

Hardness for KRA lead to hardness for KRA? Maybe the second one should be GKRA?

and of Diverse-KRA, in the latter case by setting $r = 1$ and $d = 0$.

[3] The proof of this fact is not contained in the conference paper [18] but only appears in Appendix B of http://www.wisdom.weizmann.ac.il/~naor/PAPERS/rank_www10.html.

The second parameter we consider is the *unanimity width* of the set of votes, which is based on the notion of unanimity order of a set of votes [10]. The *unanimity width* of Π is defined as the pathwidth of the cocomparability graph of ρ.

Is ρ define?

3.1 Some Discussion on (G)KRA

In this subsection we briefly describe some toy applications where the notion of diversity can be naturally combined with the notion of Kemeny Rank Aggregation, both in the totally ordered setting and in the partially ordered setting.

A Natural Class of Partial Orders of Low Width. A *k-bucket order* [20] is a partial order ρ where the set of vertices can be partitioned into a sequence of clusters $C_1, \ldots, C_m$, each of size at most k, where for each $i \in [m-1]$, all elements in C_i precede all elements in C_{i+1}. In the context of democratic scheduling, a unanimity order that is a k-cluster order corresponds to the situation where the tasks to be executed are split into work-packages (the clusters), each containing at most k tasks, where the order of execution of the work-packages is agreed on, but the order inside each cluster is not. It is easy to see that the cocomparability graph of a k-cluster order has pathwidth at most k. Additionally, for each m and k, there are sets of votes whose unanimity order ρ is a k-cluster order such that for each diversity Div, there is a set S containing Div linear extensions of ρ, each of which has optimal Kemeny score, and such that S has maximum diversity.

A Concrete Application of Bucket Orderings. Consider an election with 5 candidates A, B, C, D, E for 3 positions and 100 voters. If voters are forced to put (strict) linear orders, then it could be that there might be 50 votes like $A < B < C < D < E$ and 50 votes like $A < B < D < C < E$. There are two optimum Kemeny solutions, each of them coinciding with the two types of votes that were cast. But even if these two solutions are put into a diverse set of solutions, then the distance between these to votes is one. However, the sum of the Kemeny-Tau distances of any of the two optimal solutions to all given votes is 50. Hence, although the votes do agree to quite some extent, the resulting numbers are relatively big. But are these strict linear orderings really expressing the opinions of the voters? This has been discussed in the social choice literature, and there is some evidence that many people do not have a strict preference among *all* candidates, but ranking them in groups is more realistic. This is our main motivation to introduce the KRA model.

For instance, it could be that 10 voters do not care about the ranking of A versus B, but they would rank them all above C, D, E, without caring too much about their sequence, either. Another 10 voters might not care about the exact sequence of A and B, nor about the sequence of D and E, but they clearly put A and B before D which is in turn ahead of C and E. In shorter notation, we get $A = B < D < C = E$. There might be more different votes, as altogether summarized in the following table:

type I	10 voters	$A = B < C = D = E$
type II	10 voters	$A = B < D < C = E$
type III	10 voters	$A = B = C < D = E$
type IV	40 voters	$A = B = C = D < E$
type V	20 voters	$A < B < C = D = E$

This election could be turned into the first example if all voters would have been forced to commit themselves to linear orderings. Obviously, the 50 type II and type IV voters are compatible with $A < B < D < C < E$, while all but the typ II voters are compatible with $A < B < C < D < E$.

Hence, when viewing this as an instance of KRA, the ranking $A < B < C < D < E$ would clearly win, as its Kemeny score is just 10. (Hemaspaandra et al. [43] would attribute a much higher value here.) However, the diversity between $A < B < D < C < E$ and $A < B < C < D < E$ stays one. It might be also possible to consider the solution $B < A < C < D < E$ now, or even $A < B < C < E < D$. While in the model where we required all votes to be linear orderings, only the mentioned two solutions would make sense in a diverse solution that should not be too expensive in terms of costs, here it might be possible to look for three or four solutions in the diverse set. This is another aspect that makes our model interesting for election problems.

Diversity in Budget Allocation. Suppose a decision-maker wants to allocate funds for the implementation of a set $\{p_1, \ldots, p_n\}$ of projects, which would be executed sequentially. In order to determine the priority in which the funds will be allocated, the decision-maker asks each voter to rank the projects in the desired order of execution. The goal of the decision-maker is to come up with an order that maximizes satisfaction among the voters. Nevertheless, instead of having a unique final ranking computed completely automatically, the decision-maker may prefer to have in hands a diverse set of sufficiently good rankings. This would allow the decision-maker to also take into consideration important external factors before taking a final decision for the allocation of funding. Such external factors may be budgetary constraints, environmental constraints, compliance with regulations, etc.

Diversity in Search Rankings. Search engines are part of our everyday live. But who is really following the links presented by the search engine beyond the first few pages that are displayed on the user's screen? Therefore, it is crucial that important and interesting information is put on the very first pages. Usually, search engines consider some sort of relevance measure to rank the answers. Clearly, the search engine knows a bit more. For instance, is it important to display different hits from the same domain? Rather, it would be better for the user to see "really different" hits on the first page. Our concept of diversity could be implemented on two levels here: Either, we build a meta-search engine that collects the rankings of answers from different search engines (on the same question) and tries to come up with a diverse set of rankings that could help build the first couple of pages. Or, we consider different ranking functions within a search engine itself; in this second scenario, there could be also ties, so that our more general framework would apply.

Diversity in Team Formation. An organization wants to form a team/committee to perform some task and there are several candidates. But the committee will have only a few members (say three). To choose the committee, the organization will pick a rank with sufficiently good score and select the first three candidates of that particular rank. The intuition is that candidates that appear in the first three positions of some rank with good score have enough legitimacy to take the role. On the other hand, it may be important for the organization to have some liberty to choose which rank they will be using due to external factors, such as political, social, or affinity factors. For instance, if in one of the sufficiently good rankings the first three candidates are male, and in another sufficiently good ranking we have two female and one male, then it may be better to pick the latter one for gender equality reasons.

Diversity in Planning of Menus. Suppose a pizza delivery service service wants to optimize their menu. Each of the current customers is asked to rank his/her four preferred toppings. All other toppings are equally ranked below the fourth one. The goal is to obtain a small, sufficiently diverse, set of rankings (say, containing 10 rankings). Each of these rankings will correspond to a pizza on the menu. In order to cover many customer tastes, high diversity would be very welcome, so that each customer could find her favorite among the (only) 10 pizzas on the menu.

4 Completion of an Ordering

In this section, we will introduce the Completion of an Ordering problem, a generalization of the Positive Completion of an Ordering (PCO) problem originally considered in [17, Sec. 8] and [23, Sec. 6.4].

Problem name: Completion of an Ordering (CO)
Given: A partial order $\rho \subseteq C \times C$ over a set C, a cost function $\mathfrak{c} : C \times C \to \mathbb{N}$, and some $k \in \mathbb{N}$.
Output: Is there a linear order $\tau \supseteq \rho$ with $\mathfrak{c}(\tau \setminus \rho) = \sum_{(x,y)\in\tau\setminus\rho} \mathfrak{c}(x,y) \leq k$?

Intuitively, given a partial order ρ and a cost function $\mathfrak{c}$, the goal is to find a linear extension of ρ incurring a cost of at most k. The only difference between CO and the original PCO problem introduced in [17, 23] is that, in the latter, the cost function needs to satisfy the following condition: for every pair $(x, y) \in C \times C$ such that x and y are incomparable in ρ, the cost of (x, y) is strictly positive ($\mathfrak{c}(x, y) > 0$). In CO, for such pair the cost can be zero ($\mathfrak{c}(x, y) > 0$).

A Diversity Measure for CO. We note that the notion of Kendall-Tau diversity introduced in Section 3 can also be used as a notion of diversity for CO, i.e., given a set R of (not necessarily optimal) solutions for a given instance $(\rho, \mathfrak{c})$ of CO, we let KT-Div(R) be the diversity of this set.

Problem name: Diverse Completion of an Ordering (Diverse-CO)
Given: A partial order $\rho \subseteq C \times C$ over a set C, a cost function $\mathfrak{c} : C \times C \to \mathbb{N}$, and non-negative integers $k, r, d \in \mathbb{N}$.
Output: Is there a set $R = \{\tau_1, \ldots, \tau_r\}$ of linear extensions of ρ such that $\mathfrak{c}(\tau_i \setminus \rho) \leq k$ for each $i \in [r]$, and KT-Div(R) $\geq d$?

Reducing KRA to CO. Next, we give a rather straightforward reduction from KRA to CO. Given an instance (Π, C) of KRA with partial votes $\Pi = (\pi_1, \ldots, \pi_m)$ and candidates $C = \{c_1, \ldots, c_n\}$, we construct an instance $(\rho, \mathfrak{c})$ of CO by letting ρ be the unanimity order of Π, and by defining the cost function $\mathfrak{c} : C \times C \to \mathbb{N}$ as follows. For every pair of candidates (c, c'), we define its *cost*, $\mathfrak{c}(c, c')$, as the number of votes that order c' before c. More formally, $\mathfrak{c}(c, c') = |\{i \in [m] \mid c' <_{\pi_i} c\}|$. With this reduction, it is straightforward to check that a given linear order π of the candidates has Kemeny score

$$
\begin{aligned}
\sum_{i=1}^{m} \text{KT-dist}(\pi, \pi_i) &= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} [c_j <_{\pi_i} c_k \wedge c_k <_{\pi} c_j] \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \mathfrak{c}(c_k, c_j)[c_k <_{\pi} c_j].
\end{aligned}
$$

Here, for a logical proposition p, we use the bracket notation $[p]$ to denote the integer 1 if p is true the integer 0 if p is false. In other words, $\sum_{i=1}^{m} \text{KT-dist}(\pi, \pi_i)$ is the cost of π as a linear extension of the ordering ρ.

It is important to note that if all votes in Π are linear orders then $(\rho, \mathfrak{c})$ is actually an instance of PCO. In other words, if two candidates c and c' are incomparable in the unanimity order, then the cost assigned by $\mathfrak{c}$ to both pairs (c, c') and (c', c) are strictly positive. This property will be used crucially in the development of our sub-exponential time algorithm for KRA parameterized by cost.

We also note that since our reduction is solution preserving, it is also immediate that it is diversity preserving. In other words, R is a set of solutions of diversity d for an instance of KRA if and only if it is also a set of solutions of diversity d for the corresponding instance of CO.

5 Diverse CO Parameterized by Pathwidth

In this section, we devise a fixed parameter tractable algorithm for Diverse CO parameterized by solution imperfection, number of solutions, scatteredness, and pathwidth of the cocomparability graph of the input instance. Given our reduction that preserves solution and parameters from KRA to CO introduced in Section 4, this algorithm immediately implies that Diverse KRA is fixed parameter tractable when parameterized by solution imperfection, number of solutions, scatteredness, and unanimity width.

We start by defining a suitable notion of consistency between a path decomposition and a given partial order. Let $G = (C, E)$ be a graph and $\rho \subseteq C \times C$ be a partial order on the vertices of G. We say that a path decomposition $D = (B_1, \dots, B_l)$ is *ρ-consistent* if there is no pair of vertices $(x, y) \in \rho$ such that

$$\max(\{i \in [l] \mid y \in B_i\}) < \min(\{i \in [l] \mid x \in B_i\}).$$

In other words, if x is smaller than y in ρ, then y cannot be forgotten in D before x is introduced in D. The *ρ-consistent pathwidth* of G, denoted by $cpw(G, \rho)$, is the minimum width of a ρ-consistent path decomposition of G.

It has been shown recently that for any partial order $\rho \subseteq C \times C$, the pathwidth of the cocomparability graph G_ρ is equal to the consistent pathwidth of G_ρ [3]. The proof of this result was based on the fact that the consistent pathwidth of a cocomparability graph of a partial order is equal to the interval width of the order [39]. Nevertheless, the problem of constructing, or even approximating, a minimum-width consistent path decomposition in FPT time was left open in [3].

By taking a closer look at the theory of cocomparability graphs, we solve this open problem in a constructive way. More precisely, in Lemma 3 we show that for any partial order ρ, one can construct a ρ-consistent path decomposition of the cocomparability graph G_ρ in fixed-parameter tractable time parameterized by the pathwidth of the graph G_ρ.

Lemma 3. *Let $\rho \subseteq C \times C$ be a partial order and G_ρ be the cocomparability graph of ρ. Then one can construct a nice ρ-consistent path decomposition $\mathcal{P}$ of G_ρ of width $\mathcal{O}(pw(G_\rho))$ in time $2^{\mathcal{O}(pw(G_\rho))} \cdot |C|$.*

Proof. Let ρ be a partial order over a set C and G_ρ be the cocomparability graph of ρ. It has been shown in Theorem 2.1 of [39] that any minimal triangulation H of G_ρ is not only a cocomparability graph, but also an interval graph. This result allows us to compute a ρ-consistent path decomposition of G_ρ as follows.

We start by computing a tree decomposition $\mathcal{T}$ of G_ρ of width at most $5 \cdot pw(G) + 4$ in time $2^{\mathcal{O}(pw(G))} \cdot |C|$ using the algorithm from [7]. Subsequently we construct a triangulation $H_\mathcal{T}$ of G_ρ by transforming each bag of the decomposition $\mathcal{T}$ into a clique. More precisely, we add an edge to vertices u and v in G_ρ if and only if u and v occur together in some bag. This operation clearly preserves treewidth, since the size of the bags do not increase. Therefore, the graph $H_\mathcal{T}$ is a triangulation of G_ρ of treewidth at most $5 \cdot pw(G)+4$. Now, we successively delete edges from $H_\mathcal{T}$ until we get a minimal triangulation H of G_ρ. In other words, by removing any additional edge from H, we either get a graph that is not triangulated or that is not a supergraph of G_ρ. We have $E(G_\rho) \subseteq E(H) \subseteq E(H_\mathcal{T})$ and $E(H)$ is minimal with respect to inclusion. Therefore, $tw(H) \leq tw(H_\mathcal{T}) = \mathrm{w}$. By [39], we know that H is an interval graph.

Now, adding an edge $\{u, v\}$ to G_ρ is equivalent to removing the edge constructing the DAG $\rho\backslash\{(u, v)\}$. What is shown in [39] is that the DAG $\rho\backslash\{(u, v) \mid \{u, v\} \in E(H)\backslash E(G_\rho)\}$ is actually a partial order ι. Note that when deleting edges from a partial order, the only axiom that can be broken is transitivity. So what this result is really saying is that by deleting the pairs corresponding to edges that are in H but not in G, we can indeed preserve transitivity. The crucial fact about this construction is that the partial order ι is actually an interval order, and therefore H is an interval supergraph of G_ρ.

Now, from the interval graph H, we derive a ρ-consistent path decomposition $\mathcal{P}$ of G_ρ. This construction is as follows. Given two maximal cliques X and Y of H, we say that X is smaller than Y if there exist vertices $x \in X$ and $y \in Y$ such that $(x, y) \in \iota$. This relation defines a linear order on the maximal cliques [38]. It follows from [38] that the sequence of maximal cliques obtained by ordering the maximal cliques of H according to the order above is a path decomposition of H. As the bags of the path decomposition follow the order above, this path decomposition is consistent with ι. Now, since any path decomposition of H is also a path decomposition of G_ρ, and as $\iota \subseteq \rho$, this path decomposition is also ρ-consistent.

We note that the process of finding all maximal cliques of an interval graph H can be realized in time linear in the size of H [38]. Note that since H has pathwidth $\mathcal{O}(pw(G_\rho))$, the number of edges of H is bounded by $\mathcal{O}(pw(G_\rho)^2 \cdot |C|)$. So the process of finding the maximal cliques in H takes time $\mathcal{O}(pw(G_\rho)^2 \cdot |C|)$. Finally, since the most time-expensive part of the process described above is the construction of the tree decomposition $\mathcal{T}$, we have that the whole process takes time $2^{\mathcal{O}(pw(G_\rho))} \cdot |C|$. □

Let ρ be a partial order and $\mathcal{P} = (B_1, B_2, \ldots, B_l)$ be a ρ-consistent path decomposition of G_ρ of width w. For each $p \in [l]$, let $\mathbb{P}_p$ be the set of pairs of the form (S, τ) where S is a subset of B_p that contains vertices introduced by B_p ($\mathrm{intro}(p) \subseteq S \subseteq B_p$), $\tau \supseteq \rho|_S$ is a linear extension of the restriction $\rho|_S \doteq S \times S \cap \rho$ of ρ to S.

Definition 4. *Let $p \in [l]$, $\delta \in \mathbb{N}$, and $f : \mathbb{P}_p \to \mathbb{N}$. Then, we let $\mathcal{T}_p(f, \delta)$ be the set of all triples of the form (S, τ, γ), where $(S, \tau) \in \mathbb{P}_p$ and $f(S, \tau) \leq \gamma \leq f(S, \tau) + \delta$.*

Intuitively, the function f will be used by our dynamic programming algorithm to record the optimal values of partial solutions at each bag B_p when processing the path decomposition from left to right (see Theorem 9 and Theorem 14) and δ will be the allowed solution imperfection. In the case of a unique solution, this value will be 0 but this parameter will be useful in the diverse case as we allow sub-optimal linear extensions. A partial solution up to the p-th bag is a linear order σ of $\bigcup_{j \leq p} B_j$. Vertices that will be introduced in future bags can be inserted in σ, to extend it, only after vertices already forgotten in the p-th bag. If u will be introduced in a future bag and v is in some B_j but not in B_p, then by consistency of the path decomposition with respect to ρ, we have $v <_\rho u$. Therefore, in B_p, we only need to remember the "last" part of σ, which are the vertices that are in B_p and after all forgotten vertices in σ.

Remark 5. *For each $p \in [l]$, $f : \mathbb{P}_p \to \mathbb{N}$ and $\delta \in \mathbb{N}$, the size of $\mathcal{T}_p(f, \delta)$ is bounded by $e \cdot (\delta + 1) \cdot (\mathrm{w} + 1)!$.*

Proof. Given a bag B_p at position p, the size of $\mathcal{T}_p(f, \delta)$ is bounded by:

$$(\delta + 1) \cdot \sum_{0 \leq i \leq |B_p|} \binom{|B_p|}{i} \cdot i! \leq e \cdot (\delta + 1) \cdot |B_p|! \leq e \cdot (\delta + 1) \cdot (\mathrm{w} + 1)!$$

where $\binom{|B_p|}{i}$ is the number of subsets of B_p of size i and $i!$ is the number of possible ordering of a set of size i. □

For each $p \in [l - 1]$, $f : \mathbb{P}_p \to \mathbb{N}$ and $\delta \in \mathbb{N}$, we say that a triple $(S, \tau, \gamma) \in \mathcal{T}_p(f, \delta)$ is *compatible* with a triple $(S', \tau', \gamma') \in \mathcal{T}_{p+1}(f, \delta)$ if the following conditions are satisfied.

- C1 Let $v = \max_\tau(S \cap \mathrm{forg}(p+1))$ be the maximum vertex of S forgotten by B_{p+1} according to the linear order τ. Then, we have the following equality $S' = \mathrm{intro}(p+1) \cup \{u \in S \mid v <_\tau u\}$. This means that one can build S' from S by removing all vertices that are either forgotten by B_{p+1} or smaller than some vertex forgotten by B_{p+1}, and subsequently, by adding all vertices that have been introduced by B_{p+1}.
- C2 $\tau|_{S \cap S'} = \tau'|_{S \cap S'}$, i.e., τ and τ' agree on $S \cap S'$.
- C3 $\gamma' = \gamma + \sum_{v \in \mathrm{intro}(p+1)} (\sum_{u \in S', u <_{\tau'} v} \mathfrak{c}(u, v) + \sum_{u \in S \cap S', v <_{\tau'} u} \mathfrak{c}(v, u) + \sum_{u \in B_{p+1} \setminus S'} \mathfrak{c}(u, v))$. To compute γ', we add to γ the cost of adding the introduced vertices in the order. The first two terms compute the cost of each new vertex in τ' and the last one computes the cost of placing the new vertices after all vertices in $B_{p+1} \setminus S'$.

A *compatible sequence* for $\mathcal{P}$ is a sequence of triples $\gamma = (S_1, \tau_1, \gamma_1) \dots (S_l, \tau_l, \gamma_l)$ such that for each $p \in [l]$, (S_p, τ_p, γ_p) is compatible with $(S_{p-1}, \tau_{p-1}, \gamma_{p-1})$.

Our interest in compatible sequences stems from the two following lemmas.

Lemma 6. *Let $\rho \subseteq C \times C$ be a partial order over C, $\mathfrak{c} : C \times C \to \mathbb{N}$ be a cost function, and $\mathcal{P}$ be a ρ-consistent path decomposition of the graph G_ρ. Let*

$$\gamma = \mathfrak{t}_1 \dots \mathfrak{t}_l = (S_1, \tau_1, \gamma_1) \dots (S_l, \tau_l, \gamma_l)$$

be a compatible sequence for $\mathcal{P}$. Then, the linear order $\pi = \mathbf{tc}(\rho \cup \tau_1 \cup \cdots \cup \tau_l)$ is a linear extension of ρ of cost γ_l.

Proof. Lemma 6 follows straightforwardly by the following claim, which can be proved by induction on p.

Claim 7. *For each position $p \in [l]$, $\pi_p = \mathbf{tc}(\rho|_{L_p \cup B_p} \cup \tau_1 \cup \cdots \cup \tau_p)$ is a linear extension of $\rho|_{L_p \cup B_p}$ of cost γ_p.*

In the base case, $S_1 = B_1 = \mathrm{intro}(1)$ and by definition $\tau_1 = \pi_1$ is a linear extension of $\rho|_{B_1}$ and $\gamma_1 = \mathfrak{c}(\tau_1)$.

Now, let $p \in [l]$ be a position in $\mathcal{P}$, and suppose that the claim holds for p. We show that it also holds for $p+1$. For that, we need to check that the transitive closure of $\rho|_{L_p \cup B_p} \cup \tau_1 \cup \cdots \cup \tau_{p+1}$

defines a linear extension of $\rho|_{L_{p+1}\cup B_{p+1}}$. This means that π_{p+1} does not contain loops and each pair $u, v \in L_{p+1} \cup B_{p+1}$ is ordered by π_{p+1}. By C2, we have that $\mathbf{tc}(\rho \cup \tau_1 \cup \cdots \cup \tau_p)$ is compatible with τ_{p+1} and ρ and so there is no loop in π_{p+1}. Let $u, v \in L_{p+1} \cup B_{p+1}$. If $u, v \in S_{p+1}$, then u and v are ordered by τ_{p+1} and thus by π_{p+1}. If $u, v \notin S_{p+1}$, then $u, v \in L_p \cup B_p$, therefore they are ordered by π_p and thus by π_{p+1}. If $u \in S_{p+1}$ and $v \notin S_{p+1}$, then, by C1, we have $v <_{\tau_p} \max_{\tau_p}(S_p \cup \mathrm{forg}(p+1))$ and by ρ-consistency of $\mathcal{P}$, we have $u >_\rho \max_{\tau_p}(S_p \cup \mathrm{forg}(p+1))$. Therefore, by transitivity we have $v <_{\pi_{p+1}} u$. By C3, we have $\gamma_{p+1} = \mathfrak{c}(\pi_{p+1})$. □

Lemma 8. *Let $\rho \subseteq C \times C$ be a partial order over C, $\mathfrak{c} : C \times C \to \mathbb{N}$ be a cost function, and $\mathcal{P}$ be a ρ-consistent path decomposition of the graph G_ρ. Let π be a linear extension of ρ, and $\gamma = (S_1, \tau_1, \gamma_1) \dots (S_l, \tau_l, \gamma_l)$ be a sequence such that for each position $p \in [l]$, $S_p = \{v \in B_p \mid v >_\pi \max_\pi(L_p)\}$, $\tau_p = \pi|_{S_p}$, and $\gamma_p = \mathfrak{c}(\pi|_{L_p \cup B_p})$. Then, γ is a compatible sequence for $\mathcal{P}$.*

Proof. One can see that this construction satisfies conditions C1, C2 and C3 for each position in the path decomposition. □

Lemma 6 and Lemma 8 immediately yield an FPT dynamic programming algorithm for computing a linear extension of ρ. To define the algorithm more precisely, we first need to define the set of functions f_p that we will use to define a set of triples with $\mathcal{T}_p$. For each $p \in [l]$, we define $f_p : \mathbb{P}_p \to \mathbb{N}$ as follows. For each $(S, \tau) \in \mathbb{P}_p$, we let γ be the minimum cost of a partial solution π up to bag p such that $S = \{v \in B_p \mid v >_\pi \max_\pi(L_p)\}$ and $\tau = \pi|_{S_p}$; then we let $f_p(S, \tau) = \gamma$. Intuitively, f_p associate to each linear order τ the cost of an optimal partial solution "ending" by τ. Now, we will describe the algorithm, we process the path decomposition from left to right in l time steps, where at each time step p, we construct the value of f_p that we need and a subset $\mathcal{Q}_p \subseteq \mathcal{T}_p(f_p, 0)$ of *promising triples*, which are, intuitively, triples that have a potential to lead to an optimal solution. At time step 1, we let $\mathcal{Q}_1 = \{(B_1, \tau, \mathfrak{c}(\tau))\}_{\tau \text{ is a linear extension of } \rho|_{B_1}}$. At each time step $p \geq 2$, $\mathcal{Q}_p$ is the set of all triples in $\mathcal{T}_p(f_p, 0)$ that are compatible with some triple in $\mathcal{Q}_{p-1}$. At the end of the process, assuming that $\mathcal{Q}_p$ is non-empty for each $p \in [l]$, we can reconstruct a compatible sequence by backtracking. First, by selecting an arbitrary triple $\mathfrak{t}_l$ in $\mathcal{Q}_l$, then by selecting an arbitrary triple $\mathfrak{t}_{l-1}$ in $\mathcal{Q}_{l-1}$ compatible with $\mathfrak{t}_l$, and so on. Once we have constructed a compatible sequence $\mathfrak{t}_1 \dots \mathfrak{t}_l$, we can extract a linear extension π of cost γ_l by setting $\pi = \mathbf{tc}(\rho \cup \tau_1 \cup \dots \tau_l)$. This description gives rise to the following theorem.

Theorem 9. *Let $\rho \subseteq C \times C$, let $n = |C|$, let w be the pathwidth of the cocomparability graph of ρ, and $\mathfrak{c} : C \times C \to [m]_0$ be a cost function. Then, one can compute an optimal solution in time $\mathcal{O}\left(\mathrm{w}!^{\mathcal{O}(1)} \cdot n \cdot \log(n \cdot m)\right)$.*

Proof. By Lemma 3, one can construct a nice ρ-consistent path decomposition $\mathcal{P}$ of G_ρ of width $\mathcal{O}(\mathrm{w})$ in time $2^{\mathcal{O}(\mathrm{w})} \cdot n$.

Lemma 8 shows that, if a solution π exists, then there exist a compatible sequence associated to it. Now we will show that this sequence is actually build by the algorithm. Let γ be the sequence define in Lemma 8. We recall that the sequence is define as follow $\gamma = (S_1, \tau_1, \gamma_1) \dots (S_l, \tau_l, \gamma_l)$, where for each position $p \in [l]$, $S_p = \{v \in B_p \mid v >_\pi \max_\pi(L_p)\}$, $\tau_p = \pi|_{S_p}$ and $\gamma_p = \mathfrak{c}(\pi|_{L_p \cup B_p})$. First, because π is optimal, one can easily check that for each $p \in [l]$, $(S_p, \tau_p, \gamma_p) \in \mathcal{T}_p(f_p, 0)$. We will prove that this sequence is built by the algorithm by recurrence on the bags. By definition, $S_1 = B_1$ and $\gamma_1 = \mathfrak{c}(\tau_1)$, therefore the first triple is build by the algorithm. As $(S_{p+1}, \tau_{p+1}, \gamma_{p+1})$ is compatible with (S_p, τ_p, γ_p), then, by definition of compatibility and how the algorithm proceeds, if the algorithm builds (S_p, τ_p, γ_p) it will build $(S_{p+1}, \tau_{p+1}, \gamma_{p+1})$ in the next step. This proves the correctness of the algorithm.

Now we will prove the running time. Without loss of generality, we can assume that the path decomposition is a nice path decomposition. In a path decomposition with no duplicate bags, there are at most $2n$ bags. For each position $p \in [2n]$, if B_p forgets a vertex v, then $\mathcal{Q}_p$ can be computed by removing v in each triple in $\mathcal{Q}_{p-1}$ and keeping triples with minimum cost for each fixed pair (S, τ). This can be done in time $\mathcal{O}(|\mathcal{Q}_{p-1}|^2)$. If B_p introduces a vertex v, then $\mathcal{Q}_p$ can be computed from $\mathcal{Q}_{p-1}$ by taking each triple $(S, \tau, \mathfrak{c})$ and adding v in τ at every possible position. Computing the new cost can be done in time $\mathcal{O}(\mathrm{w} \cdot \log(n^2 \cdot m))$ and there are $|\mathcal{Q}_{p+1}|$ triples to compute. The log factor $\log(n^2 \cdot m)$ is the time of performing an addition on the costs, as the cost of a solution can be at most $n^2 \cdot m$. By Remark 5, we can bound the size of each $\mathcal{Q}_p$. $\square$

Now, leveraging on Lemma 6 and Lemma 8, we will devise a fixed-parameter tractable algorithm for Diverse-CO parameterized by solution imperfection, number of solutions, scatteredness, and pathwidth of the cocomparability graph of the input partial order. Let ρ be a partial order and $\mathcal{P} = (B_1, B_2, \ldots, B_l)$ be a ρ-consistent path decomposition of G_ρ of width w.

Definition 10. *Let $p \in [l]$, and $f : \mathbb{P}_p \to \mathbb{N}$. Then, we let $\mathcal{I}_p(f, \delta)$ be the set of all tuples of the form*

$$((S^1, \tau^1, \gamma^1), \ldots, (S^r, \tau^r, \gamma^r), \partial, (\xi_{\{i,j\}})_{1 \le i < j \le r})$$

where $\partial \in [d+1]_0$, for each $1 \le i < j \le r$, $\xi_{\{i,j\}} \in [s]_0$, and for each $i \in [r]$, (S^i, τ^i, γ^i) is a triple in $\mathcal{T}_p(f, \delta)$.

Intuitively, $((S^i, \tau^i, \gamma^i))_{i \in [r]}$ are r partial linear extensions, ∂ will be the diversity of the r partial linear extensions and ξ will be the distance between all pair of the r partial linear extensions.

Remark 11. *For each $p \in [l]$, $f : \mathbb{P}_p \to \mathbb{N}$ and $\delta \in \mathbb{N}$, the size of $\mathcal{I}_p(f, \delta)$ is bounded by $(e \cdot (\delta+1) \cdot (\mathrm{w}+1)!)^r \cdot s^{r^2} \cdot d$.*

For each $p \in [l-1]$, each tuple

$$\mathfrak{u}_p = ((S_p^1, \tau_p^1, \gamma_p^1), \ldots, (S_p^r, \tau_p^r, \gamma_p^r), \partial_p, (\xi_{\{i,j\}}^p)_{1 \le i < j \le r})$$

in $\mathcal{I}_p$ and each tuple

$$\mathfrak{u}_{p+1} = ((S_{p+1}^1, \tau_{p+1}^1, \gamma_{p+1}^1), \ldots, (S_{p+1}^r, \tau_{p+1}^r, \gamma_{p+1}^r), \partial_{p+1}, (\xi_{\{i,j\}}^{p+1})_{1 \le i < j \le r})$$

in $\mathcal{I}_{p+1}$, we define the *scatteredness increase table* of the pair $(\mathfrak{u}_p, \mathfrak{u}_{p+1})$, denoted by $\Delta\xi(\mathfrak{u}_p, \mathfrak{u}_{p+1})$, as the table holding the distance increase between each pair of partial linear extensions due to the elements of $\mathrm{intro}(p+1)$. To compute this increase for each pair $1 \le i < j \le r$, we will compute the increase of adding one vertex and repeat this operation for each element in $\mathrm{intro}(p+1)$. Let $v \in \mathrm{intro}(p+1)$ be a vertex introduced by B_{p+1} and let $\mathfrak{u}_v$ be a tuple obtained by extending $\mathfrak{u}_p$ to include v, we have

$$\begin{aligned}\Delta\xi_{\{i,j\}}(\mathfrak{u}_p, \mathfrak{u}_v) = {} & |\{u \in B_p \mid (u \notin S_{p+1}^i \vee u <_{\tau_{p+1}^i} v) \wedge v <_{\tau_{p+1}^j} u\}| + \\ & |\{u \in B_p \mid (u \notin S_{p+1}^j \vee u <_{\tau_{p+1}^j} v) \wedge v <_{\tau_{p+1}^i} u\}|\end{aligned}$$

where $u \notin S_{p+1}^i \vee u <_{\tau_{p+1}^i} v$ means that u is smaller than v in the i^{th} tuple, $v <_{\tau_{p+1}^j} u$ means that u is bigger than v in the j^{th} tuple, $u \notin S_{p+1}^j \vee u <_{\tau_{p+1}^j} v$ means that u is smaller than v in the j^{th} tuple, and $v <_{\tau_{p+1}^i} u$ means that u is bigger than v in the i^{th} tuple. Now, we define the increase between two bags, given an arbitrary ordering $v_1, \ldots, v_{|\mathrm{intro}(p+1)|}$ of $\mathrm{intro}(p+1)$:

we let $\mathfrak{u}_{v_1}$ be the tuple obtained by extending $\mathfrak{u}_p$ to include v_1 and $\mathfrak{u}_{v_t}$ be the tuple obtained by extending $\mathfrak{u}_{v_{t-1}}$ to include v_t and we have, for each pair $\{i,j\}$,

$$\Delta\xi_{\{i,j\}}(\mathfrak{u}_p, \mathfrak{u}_{p+1}) = \Delta\xi_{\{i,j\}}(\mathfrak{u}_p, \mathfrak{u}_{v_1}) + \sum_{t=2}^{|\text{intro}(p+1)|} \Delta\xi_{\{i,j\}}(\mathfrak{u}_{v_{t-1}}, \mathfrak{u}_{v_t}). \tag{1}$$

Intuitively, this measures the increase of the distance between each pair of the r partial solutions from the bag p to the bag $p+1$.

We define, in a similar way, the *diversity increase* of the pair $(\mathfrak{u}_p, \mathfrak{u}_{p+1})$, denoted by writing $\Delta\partial(\mathfrak{u}_p, \mathfrak{u}_{p+1})$, as the amount of diversity due to the elements of $\text{intro}(p+1)$. As the diversity is the sum of all pairwise distances between the r partial solutions, we will use the scatteredness increase table $(\mathfrak{u}_p, \mathfrak{u}_{p+1})$ to compute the diversity increase. We define the increase between two bags as

$$\Delta\partial(\mathfrak{u}_p, \mathfrak{u}_{p+1}) = \sum_{1\le i<j\le r} \Delta\xi_{\{i,j\}}(\mathfrak{u}_p, \mathfrak{u}_{p+1}). \tag{2}$$

Intuitively, this measures the increase of the diversity between the r partial solutions up to the bag p and the r partial solutions up to bag $p+1$.

We say that $\mathfrak{u}_p$ is compatible with $\mathfrak{u}_{p+1}$ if $\xi_{p+1} = \min(\xi_p + \Delta\xi(\mathfrak{u}_p, \mathfrak{u}_{p+1}), s)$, $\partial_{p+1} = \min(\partial_p + \Delta\partial(\mathfrak{u}_p, \mathfrak{u}_{p+1}), d)$, and for each $i \in [r]$, the triple $(S_p^i, \tau_p^i, \gamma_p^i)$ is compatible with the triple $(S_{p+1}^i, \tau_{p+1}^i, \gamma_{p+1}^i)$.

A *diversity-compatible sequence* is a sequence of the form

$$\{((S_p^1, \tau_p^1, \gamma_p^1), \dots, (S_p^r, \tau_p^r, \gamma_p^r), \partial_p, (\xi_{\{i,j\}}^p)_{1\le i<j\le r})\}_{p\in[l]},$$

where for each $p \in [l-1]$, the tuples at positions p and $p+1$ are compatible.

The next lemma is an analogue of Lemma 6 in the context of solution diversity.

Lemma 12. *Let $\rho \subseteq C \times C$ be a partial order over C, $\mathfrak{c} : C \times C \to \mathbb{N}$ be a cost function, and $\mathcal{P}$ be a ρ-consistent path decomposition of the graph G_ρ. Let*

$$\hat{\gamma} = \{((S_p^1, \tau_p^1, \gamma_p^1), \dots, (S_p^r, \tau_p^r, \gamma_p^r, \partial_p), (\xi_{\{i,j\}}^p)_{1\le i<j\le r})\}_{p\in[l]}$$

be a diversity-compatible sequence for $\mathcal{P}$. Then, the following properties can be verified.

1. *For each $i \in [r]$, the order $\pi_i = \mathbf{tc}(\rho \cup \tau_1^i \cup \cdots \cup \tau_l^i)$ is a linear extension of ρ of cost γ_l^i.*

2. *For each i, j with $1 \le i < j \le r$, $\xi_{\{i,j\}}^l = \min(\text{KT-dist}(\pi_i, \pi_j), s)$.*

3. *$\partial_l = \min(\text{KT-Div}(\{\pi_1, \dots, \pi_r\}), d)$.*

We note that property 1 of Lemma 12 follows from Lemma 6, property 2 follows from Equation 2, and property 3 from Equation 1.

The next lemma is an analogue of Lemma 13 in the context of solution diversity.

Lemma 13. *Let $\rho \subseteq C \times C$ be a partial order over C, $\mathfrak{c} : C \times C \to \mathbb{N}$ be a cost function, and $\mathcal{P}$ be a ρ-consistent path decomposition of the graph G_ρ. Let $\pi_1, \dots, \pi_r$ be r linear extensions of ρ, and*

$$\hat{\gamma} = \{((S_p^1, \tau_p^1, \gamma_p^1), \dots, (S_p^r, \tau_p^r, \gamma_p^r), \partial_p, (\xi_{\{i,j\}}^p)_{1\le i<j\le r})\}_{p\in[l]}$$

be a sequence satisfying the following conditions.

1. *For each position* $p \in [l]$, *and each* $i \in [r]$, $S_p^i = \{v \in B_p \mid v >_{\pi_i} \max_{\pi_i}(L_p)\}$, $\tau_p = \pi_i|_{S_p}$, $\gamma_p = \mathfrak{c}(\pi_i|_{L_p \cup B_p})$.

2. *For each* i, j *with* $1 \leq i < j \leq r$, $\xi^p_{\{i,j\}} = \min(\text{KT-dist}(\pi_i|_{L_p}, \pi_j|_{L_p}), s)$.

3. $\partial_p = \min(\text{KT-Div}(\{\pi_1|_{L_p \cup B_p}, \ldots, \pi_r|_{L_p \cup B_p}\}), d)$.

Then, $\hat{\gamma}$ *is a diversity-compatible sequence for* $\mathcal{P}$.

Intuitively, what those lemmas say is that in order to construct a set of r solutions for an instance $(\rho, \mathfrak{c}, r, \delta, d, s)$ of Diverse-CO, all one needs to do is to construct r compatible sequences in parallel, by processing the given path decomposition from left to right, while using an additional register to keep track of the overall diversity at each time step and all the pairwise distances. In the same way that Lemma 6 and Lemma 8 yield an FPT dynamic programming algorithm parameterized by pathwidth for computing a single solution of an instance of CO (Theorem 9), Lemma 12 and Lemma 13 yield a dynamic programming algorithm to compute a diverse set of solutions, in case it exists, parameterized by cost of solution, number of solutions and pathwidth (Theorem 14).

Theorem 14. *Let* $\rho \subseteq C \times C$, *let* $n = |C|$ *and* w *be the pathwidth of the cocomparability graph of* ρ, *and* $\mathfrak{c} : C \times C \to [m]_0$ *be a cost function. Then, one can determine whether* ρ *admits* r *linear extensions* $\pi_1, \ldots, \pi_r$ *at distance at most* δ *from the optimum, diversity at least* d, *and scatteredness at least* s *in time* $\mathcal{O}\left((\mathrm{w}! \cdot \delta)^{\mathcal{O}(r)} \cdot s^{r^2} \cdot d \cdot n \cdot \log(n^2 \cdot m)\right)$.

Proof. This theorem is similar to Theorem 9. Here we build r linear extension in parallel and we incrementally compute the diversity between the r solutions. The main difference is the computation of the diversity. Using Equation 1 and 2, one can compute the increase of the pairwise distances and diversity in time $\mathcal{O}(r^2 \cdot \mathrm{w} \cdot \log(n^2 \cdot m))$ for each vertex and each tuple. The log factor $\log(n^2 \cdot m)$ again comes from performing an addition of costs, as the cost of a solution can be at most $n^2 \cdot m$. $\square$

By combining Theorem 9 with our reduction from KRA to CO, we have an FPT algorithm for KRA, parameterized by solution imperfection, number of solutions, scatteredness, and unanimity width (Corollary 15).

Corollary 15. *Let* Π *be a list of* m *partial votes over a set of* n *candidates* C. *Let* w *be the unanimity width of* Π. *Given* Π *and non-negative integers* δ, r, s *and* d, *one can determine in time*

$$\mathcal{O}\left((\mathrm{w}! \cdot \delta)^{\mathcal{O}(r)} \cdot s^{r^2} \cdot d \cdot n \cdot \log(n^2 \cdot m)\right)$$

whether there is a set $R = \{\pi_1, \ldots, \pi_r\}$ *of* r *linear orders on* C *such that the Kemeny score for each order* π_i *is at distance at most* δ *of the optimum, and we find that* $\text{KT-Div}(R) \geq d$ *and that scatteredness is at least* s.

Corollary 15 is our most general result that combines all the parameters, but not all applications need all parameters. Therefore, we will now derive some special cases.

Corollary 16. *Let* Π *be a list of* m *partial votes over a set of* n *candidates* C. *Let* w *be the unanimity width of* Π. *Given* Π *and non-negative integers* δ, r *and* d, *one can determine in time*

$$\mathcal{O}\left((\mathrm{w}! \cdot \delta)^{\mathcal{O}(r)} \cdot d \cdot n \cdot \log(n^2 \cdot m)\right)$$

whether there is a set $R = \{\pi_1, \ldots, \pi_r\}$ *of* r *linear orders on* C *such that the Kemeny score for each order* π_i *is at distance at most* δ *of the optimum, and we find that* $\text{KT-Div}(R) \geq d$.

Namely, by our formulation of R as a set, we implicitly have the requirement $s \geq 1$.

With our algorithm, it is possible to check if there exists r different optimal solutions. We can do this by setting $\delta = 0$, $s = 1$ and $d = 0$ and we get the following corollary.

Corollary 17. *Let Π be a list of m partial votes over a set of n candidates C. Let w be the unanimity width of Π. Given Π and non-negative integers δ, r, s and d, one can determine in time*

$$\mathcal{O}\left((\mathrm{w}!)^{\mathcal{O}(r)} \cdot 2^{r^2} \cdot n \cdot \log(n^2 \cdot m)\right)$$

whether there is a set $R = \{\pi_1, \ldots, \pi_r\}$ of r different optimal linear orders on C.

To get some insights of the structure of the solution space, one can ask for a set of r solutions of cost at most δ from the minimum cost with maximum diversity. This is not strictly a consequence of Corollary 15 but the same algorithm can be used to answer this question. In our algorithm, the value ∂ is used to compute the diversity of a partial solution up to d, but if the diversity is bigger than d, we just remember d. Then if we set $d = r \cdot n^2$, which is an upper bound of the maximum diversity of r solutions, at the end of the algorithm, we will have a set of possible solutions with their exact diversities. From this, we can select the one with the biggest diversity. Therefore, we have the following result.

Corollary 18. *Let Π be a list of m partial votes over a set of n candidates C. Let w be the unanimity width of Π. Given Π and non-negative integers δ and r, one can compute in time*

$$\mathcal{O}\left((\mathrm{w}! \cdot \delta)^{\mathcal{O}(r)} \cdot r \cdot n^3 \cdot \log(n^2 \cdot m)\right)$$

a set $R = \{\pi_1, \ldots, \pi_r\}$ of r linear orders on C such that the Kemeny score for each order π_i is at distance at most δ of the optimum and such that KT-Div(R) *is maximal.*

6 Sub-Exponential Time Algorithm for PCO

For special cases of PCO, such as those arising from KRA or from the graph-drawing problem OSCM, single-exponential sub-exponential time algorithms have been known, i.e., algorithms with running times of the form $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$. In contrast, for the more general problem of PCO, only algorithms with running time $\mathcal{O}^*(k^{\sqrt{k}})$ were known before, where k is the cost parameter [25]. Here, we prove that PCO also admits algorithms of the form $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$, by making use of several structural insights for cocomparability graphs. More precisely, we prove the following theorem.

Single and sub?

Theorem 19. *Given a partial order $\rho \subseteq C \times C$ and a cost function $\mathfrak{c} : C \times C \to \mathbb{N}$, one can solve a* PCO *instance $(\rho, \mathfrak{c}, k)$ in time $|C| \cdot 2^{\mathcal{O}(\sqrt{k})} + \mathcal{O}(|C|^2 \cdot \log(k))$.*

The remainder of this section is dedicated to the proof of Theorem 19.

The treewidth of a graph is another structural parameter that quantifies how close the graph is to being a forest (i.e., a graph without cycles). In general graphs, this parameter is more expressive than pathwidth in the sense that any graph of pathwidth k has also treewidth k, but there exist graphs of treewidth k whose pathwidth is unbounded. Nevertheless, in the class of cocomparability graphs, treewidth and pathwidth coincide.

Lemma 20. *[[39, Theorem 1.2]] Let G be a cocomparability graph. Then, $pw(G) = tw(G)$.*

Let $G = (V, E)$ be a graph and $S \subseteq V$ be a set of vertices. We call $G_S = (S, E \cap (S \times S))$ the subgraph of G *induced* by S. Let H be a graph, we say that G contains H as an induced subgraph if there exist a subset of vertices $S \subseteq V$ of G such that H is isomorphic to G_S. We say that G is H-free if G does not contain H as an induced subgraph.

Let $t \geq 3$ be an integer, we write C_t for the cycle on t vertices and $C_{\geq t} = \{C_k \mid k \geq t\}$. We say that G is $C_{\geq t}$-free if G excludes all C_k as an induced subgraph for any $k \geq t$.

Lemma 21. *Cocomparability graphs are $C_{\geq 5}$-free.*

This fact is known but not that easy to track down in the literature. We refer to [31, 33, 29], which contain corresponding results on comparability graphs. We also refer to the textbook of Trotter [61]. To keep the paper self-contained, we present a short self-contained proof of the fact that cocomparability graphs are C_k free for $k \geq 5$. As a key notion, we consider bad triples in cocomparability orders. A *cocomparability order* of a cocomparability graph $G = (V, E)$ is a bijection $\sigma : V \to \{1, \dots, |V|\}$ that linearly extends a transitive orientation ρ of G, meaning that $(x, y) \in \rho$ implies $\sigma(x) < \sigma(y)$. Given a graph G and a linear order σ of its vertices, a *bad triple* is three vertices x, y, z so that $\sigma(x) < \sigma(y) < \sigma(z)$, $xy \notin E$, $yz \notin E$, and $xz \in E$. Notice that if G is a cocomparability graph and σ a cocomparability order then (G, σ) has no bad triple. Namely, if x, y and y, z are comparable in some partial order ρ with $\sigma(x) < \sigma(y) < \sigma(z)$, then (as σ extends ρ) whenever $x <_\rho y$ and $y <_\rho z$ then $x <_\rho z$ is enforced by transitivity, ruling out a bad triple.

Lemma 22. *A cycle C_k on $k \geq 5$ vertices is not a cocomparability graph.*

Proof. Suppose for contradiction that $\sigma : V(C_k) \to \{1, \dots, k\}$ is a cocomparability order of C_k with edge set E_k. Define $\sigma^{-1}(i)$ to be the vertex x in C_k so that $\sigma(x) = i$. Let $u = \sigma^{-1}(1)$ and $v = \sigma^{-1}(k)$. If $uv \in E_k$, let x be any vertex non-adjacent to both u and v (such a vertex exists since $k \geq 5$), we have that u, x, v is a bad triple. We conclude that $uv \notin E_k$. Let P and Q be the two paths that connect u and v in C_k, excluding u, v. Without loss of generality, $|V(P)| \geq |V(Q)|$. Since $k \geq 5$, we have that the path P contains an edge pq so that $\{p, q\} \cap \{u, v\} = \emptyset$. Without loss of generality, $\sigma(p) < \sigma(q)$.

First, suppose that there is a vertex $\ell \in V(Q)$ so that $\sigma(p) < \sigma(\ell) < \sigma(q)$. Then p, ℓ, q is a bad triple. So such a vertex cannot exist. It follows that some edge ab of $E(Q \cup \{u, v\})$ is such that $\sigma(a) < \sigma(p)$ and $\sigma(q) < \sigma(b)$. Since the path Q has at least one internal vertex, we have that $|\{a, b\} \cap \{u, v\}| \leq 1$, and therefore (since each of u and v has at most one neighbor in $\{p, q\}$) we have $|N(\{a, b\}) \cap \{p, q\}| \leq 1$. However, if $p \notin N(\{a, b\})$ then a, p, b is a bad triple. If $q \notin N(\{a, b\})$ then a, q, b is a bad triple. But $|N(\{a, b\}) \cap \{p, q\}| \leq 1$ implies that one of the two former cases must hold, yielding the desired contradiction. $\square$

Finally, as the proof is based on the notion of bad triples and as bad triples in induced subgraphs are also bad triples in the whole graph, this proves Lemma 21.

The following statement is put forward in [11, Theorem 1.5].

Lemma 23 ([11] (Theorem 1.5))**.** *A $C_{\geq t}$-free graph with maximum degree Δ has treewidth bounded by $\mathcal{O}(t \cdot \Delta)$. Furthermore, a tree decomposition of this width can be computed in polynomial time.*

From Lemma 23, we can prove the following lemma.

Lemma 24. *Let G be a $C_{\geq 5}$-free graph, let m be the number of edges of G. Then, we have $m = \Omega(tw(G)^2)$.*

Proof. A graph on m edges has at most $2\sqrt{m}$ vertices of degree at least $\sqrt{m}$. Let G' be the graph obtained from G by removing all vertices of degree at least $\sqrt{m}$. Since G is $C_{\geq 5}$-free, so is G'. Therefore, by applying Lemma 23, we get that $tw(G') = \mathcal{O}(\sqrt{m})$. The removal of a vertex reduces the treewidth by at most 1. Hence, we have $tw(G) - 2\sqrt{m} \leq tw(G') = \mathcal{O}(\sqrt{m})$. This implies that $tw(G) = \mathcal{O}(\sqrt{m})$, or conversely, that $m = \Omega(tw(G)^2)$. □

By combining Lemma 24 with Lemma 20, we get:

Lemma 25. *Let G be a cocomparability graph and let m be the number of edges of G. Then, $m = \Omega(pw(G)^2)$.*

Now, in a PCO instance, each edge contributes at least 1 to the cost of any solution. Therefore, if a solution has cost at most k, then the cocomparability graph of the input partial order can have at most k edges. Therefore, this observation, together with Lemma 25 yields the following lemma.

Lemma 26. *Let $(\rho, \mathfrak{c}, k)$ be an YES-instance of* PCO. *Then, $pw(G_\rho) = \mathcal{O}(\sqrt{k})$.*

To get the running time of Theorem 19, we need to either compute a ρ-consistent path decomposition of width at most $\mathcal{O}(\sqrt{k})$, or to trigger an early rejection. For this, we will use the following lemma which is based on Lemma 3.

Lemma 27. *There is a polynomial-time algorithm that takes an instance of $(\rho, \mathfrak{c}, k)$ of* PCO *as input, and either constructs a ρ-consistent path decomposition of the graph G_ρ of width $\mathcal{O}(\sqrt{k})$, or determines that this instance is a NO-instance.*

In Section 5, in order to define our algorithm for Diverse-CO, we first devised a simpler algorithm for the single-solution version of CO, that could be used as a building block for the diverse algorithm. It turns out that if our only goal is to solve the single-solution version of CO, then the basic algorithm developed in Section 5 can be optimized, to become a single-exponential time algorithm parameterized by the pathwidth of the cocomparability graph of the input order. More precisely, we have the following lemma.

Lemma 28 ([3] (Theorem 1))**.** *Given an instances $(\rho, \mathfrak{c}, k)$ of* CO *and a ρ-consistent path decomposition $\mathcal{P}$ of the graph G_ρ, one can solve this instance in time $|C| \cdot 2^{\mathcal{O}(pw(G_\rho))} \cdot \log(k) + \mathcal{O}(|C|^2 \cdot \log(k))$.*

Now we are ready to prove the statement of Theorem 19. Given an instance $(\rho, \mathfrak{c}, k)$ of PCO, we apply the algorithm stated in Lemma 27. This algorithm either determines that the instance is a NO-instance, or constructs a ρ-consistent path decomposition $\mathcal{P}$ of G_ρ of width $\mathcal{O}(\sqrt{k})$. In the first case, we are done and simply answer NO. Otherwise, we give both the instance $(\rho, \mathfrak{c}, k)$ and the decomposition $\mathcal{P}$ to the algorithm stated in Lemma 28 to determine in time $|C| \cdot 2^{\mathcal{O}(\sqrt{k})} + \mathcal{O}(|C|^2 \cdot \log(k))$ whether $(\rho, \mathfrak{c}, k)$ is a YES- or a NO-instance of PCO. In case this is a YES-instance, the algorithm also constructs a linear extension of ρ of cost at most k. This concludes the proof of Theorem 19.

7 Conclusion

In this work, we have addressed the Kemeny Rank Aggregation problem, one of the most central problems in the theory of social choice, from the perspective of diversity of solutions and parameterized complexity theory. We have devised a fixed parameter tractable algorithm for the diverse version of KRA with partially ordered votes where parameters are the solution imperfection, the number of solutions, the scatteredness and the unanimity width of the set of votes. As a by-product of our work, we have introduced new parameterized algorithms for problems in order theory that are of independent interest. In particular, we developed parameterized algorithms for the diverse version of the Completion of an Ordering problem (CO). Furthermore, we have developed a new sub-exponential time algorithm for the Positive Completion of an Ordering problem (PCO), a restriction of CO which can be used as a base to solve combinatorial problems in a wide variety of fields, such as artificial intelligence, graph drawing, computational biology, etc. We believe that both our new sub-exponential time algorithm for finding single solutions for PCO and our algorithm for finding diverse solutions in CO have a very positive impact on the study of these and related computational problems in neighboring fields.

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