# On the maximum number of edges in chordal graphs of bounded degree and matching number


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**Abstract.** We determine the maximum number of edges that a chordal graph $G$ can have if its degree, $\Delta(G)$, and its matching number, $\nu(G)$, are bounded. To do so, we show that for every $d, \nu \in \mathbb{N}$, there exists a chordal graph $G$ with $\Delta(G) < d$ and $\nu(G) < \nu$ whose number of edges matches the upper bound, while having a simple structure: it is a disjoint union of cliques and stars.


## 1 Introduction

A problem that dates back to 1960 is to determine the maximum number of edges that a graph can have if its maximum degree and matching number are each bounded. It is important to note that this problem does not impose any constraint on the number of vertices of the graph. Because of that, in general, if one of the two parameters is not bounded, there is no upper bound on the number of edges that a graph can have. One can simply construct graphs formed by stars (trees that have only a single vertex of degree greater than one) or single edges. A star with unbounded number of leaves has matching number one but unbounded degree, while a graph that is a disjoint union of an unbounded number of edges has bounded degree and unbounded matching number. By Vizing's Theorem, every graph can have its edge set partitioned into a family of at most $\Delta(G) + 1$ matchings, where $\Delta(G)$ denotes the degree of the graph $G$. Thus, bounding both the maximum degree and the matching number is actually enough to bound the number of edges that a graph can have. Chvátal and Hanson [4] gave a tight upper bound on this value, in the case where no further restrictions are imposed to the graphs considered. Later on, Balachandran and Khare [1] gave a constructive proof of the same result, which made it possible to identify the structure of the graphs achieving the given bound on the number of edges. Such graphs are called edge-extremal graphs. They contain collections of stars and, in some cases, induced cycles of length four.

An interesting problem that arises from these results is to investigate how the number of edges in the edge-extremal graphs is affected if we impose some additional structural property on the graphs considered. More specifically, what happens if we restrict the question to graph classes in which cycles of length four or stars are forbidden induced subgraphs? Natural candidates for such graph classes are chordal graphs and claw-free graphs. In the past few years, bounds for this problem have indeed been established for claw-free graphs in the work of Dibek et al. [5]. Furthermore, the problem is resolved on other graph classes, such as bipartite graphs, split graphs, disjoint unions of split graphs and unit interval graphs in the work of Måland [9]. However, on chordal graphs, the problem had so far remained unresolved. Chordal graphs form an extremely well-studied graph class, both from a structural and from an algorithmic point of view, with many and various applications. Hence, a large number of computer science papers are published every year on chordal graphs and their subclasses.

In this work, we determine the maximum number of edges that a chordal graph can have, given the constraints on its maximum degree and matching number. Given $d, \nu \in \mathbb{N}$, we denote by $\mathcal{M}_{chordal}(d,\nu)$ the set of chordal graphs such that $\Delta(G) < d$ and $\nu(G) < \nu$. A graph in $\mathcal{M}_{chordal}(d,\nu)$ achieving this maximum number of edges is called an edge-extremal graph. In order to establish the upper bound on the number of edges of an edge-extremal graph of $\mathcal{M}_{chordal}(d,\nu)$ we show that, among them, there is one that has a very simple structure: it is a disjoint union of cliques and stars of a given size.

**Theorem 1.** *There exists an edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars.*

Section 3 is entirely devoted to the proof of Theorem 1[1]. Once the structure of this special edge-extremal graph is known, we are able to establish the following upper bound on the number of edges of a graph is $\mathcal{M}_{chordal}(d,\nu)$.

**Theorem 2.** *Given $d, \nu \in \mathbb{N}$, the maximum number of edges of a graph in $\mathcal{M}_{chordal}(d,\nu)$ is given by:*

$$\begin{cases} (d-1)(\nu-1), & \text{if } d \text{ is even} \\ (d-1)(\nu-1) + \lfloor \frac{d-1}{2} \rfloor \lfloor \frac{\nu-1}{\lceil \frac{d-1}{2} \rceil} \rceil, & \text{if } d \text{ is odd} \end{cases}$$

*Moreover, a graph achieving this number of edges is*

$$\begin{cases} (\nu-1)K_{1,d-1}, & \text{if } d \text{ is even} \\ rK_{1,d-1} + qK_d, & \text{if } d \text{ is odd,} \end{cases}$$

*where $\nu - 1 = q\lceil \frac{d-1}{2} \rceil + r$, with $r \geq 0$.*

We also show that this result is tight in the sense that the same bound does not hold for any superclass of chordal graphs that is defined by a finite collection of forbidden induced cycles. It is also worth mentioning that this problem is related to the famous problem of computing Ramsey numbers, being the general case equivalent to determining Ramsey numbers for line graphs [2].

[1] Statements marked with ♠ had their proofs moved to the appendix.

## 2 Preliminaries

The graphs considered are simple and undirected. We denote by $V_G$ and $E_G$ the vertex set and edge set of $G$, respectively. Given $x \in V_G$, we denote by $N_G(x)$ the neighborhood of $x$, that is, the set of vertices that are adjacent to $x$. For a set $X \subset V_G$, $N_G(X)$ denotes the set of vertices in $V_G \setminus X$ that have at least one neighbor in $X$. The *degree* of $x$ is denoted by $\deg_G(x)$ and is defined as $|N_G(x)|$. The *degree of a graph* $G$ is the maximum degree of a vertex in $G$ and it is denoted by $\Delta(G)$. A vertex $x$ is a *leaf* of $G$ if $\deg_G(x) = 1$.

Given $S \subseteq V_G$, the *subgraph induced by* $S$ is denoted by $G[S]$, and has $S$ as its vertex set and $\{uv \mid u, v \in S \text{ and } uv \in E_G\}$ as its edge set. A *clique* is a set $K \subseteq V_G$ such that $G[K]$ is a complete graph. A clique is *maximal* if it is not properly contained in another clique. An *independent set* is a set $S$ such that $G[S]$ has no edges. A vertex $v \in V_G$ is a *simplicial vertex* if $N_G(v)$ is a clique. Given a set $S \subseteq V_G$, we denote the graph $G[V_G \setminus S]$ by $G \setminus S$. If $S = \{v\}$, we denote the graph $G[V_G \setminus \{v\}]$ simply by $G \setminus v$. The set $S$ is a *separator* if $G \setminus S$ has a larger number of connected components than $G$.

A set $M \subseteq E_G$ is a *matching* if no two edges in $M$ share a common vertex and $M$ is a *perfect matching* if every vertex of $V_G$ is the endpoint of an edge in $M$. The *matching number of* $G$, denoted by $\nu(G)$, is the largest size of a matching in $G$. A graph $G$ is a *factor-critical graph* if for every $v \in V_G$, $G \setminus v$ has a perfect matching.

Given a family $\mathcal{H}$ of graphs, we say that $G$ is an *$\mathcal{H}$-free graph* if $G$ does not contain an induced subgraph that is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, we say $G$ is an $H$-free graph. A *tree* is a connected acyclic graph. A *star* is a tree with at most one vertex that is not a leaf, and for $k \in \mathbb{N}$, a *$k$-star*, denoted by $K_{1,k}$, is a star with $k$ leaves. A graph is a *complete graph* on $n$ vertices, denoted by $K_n$, if there is an edge between every pair of its vertices. Given two graphs $G$ and $H$, the *disjoint union of $G$ and $H$*, denoted by $G + H$ is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote by $rH$ the graph that is the disjoint union of $r$ copies of a graph $H$. A graph $G$ is a *bipartite graph* if $V_G$ can be partitioned into two independent sets. A bipartite graph with bipartition $(A, B)$ is a *chain graph* if there exists an ordering $v_1 v_2 \ldots v_r$ of the vertices of $A$ such that $N_G(v_r) \subseteq \ldots \subseteq N_G(v_1)$. This property of the vertices of $A$ is called the *nested neighborhood* property. Bipartite chain graphs are also known to be the bipartite $2K_2$-free graphs.

A graph is a *chordal graph* if it has no induced cycle of length at least four. Chordal graphs constitute a widely studied graph class, with many different characterisations. Given a graph $G$, let $\mathcal{T}$ be a tree such that every vertex of $\mathcal{T}$ is a maximal clique of $G$. We denote the vertices of $\mathcal{T}$ with capital letters and, for simplicity, we denote the set of vertices of $G$ associated with a vertex of $\mathcal{T}$ with the same capital letter. Let $T_v = \{A \in V_{\mathcal{T}} \mid v \in A\}$. The tree $\mathcal{T}$ is a *clique tree* of $G$ if for every $v \in V_G$, $T_v$ is a subtree of $\mathcal{T}$. A characterisation of chordal graphs due to Gavril [7] states that a graph is chordal if and only if it has a clique tree. If $\mathcal{T}$ is a clique tree of a chordal graph $G$ and $AB \in E_{\mathcal{T}}$, then $A \cap B$ is a separator for the graph $G$. Another important characterisation of chordal graphs in terms of vertex orderings and simplicial vertices. An ordering $v_1v_2\dots v_n$ of the vertices of $G$ is a *perfect elimination ordering* for $G$ if for every $i$, the vertex $v_i$ is simplicial in the graph $G[\{v_{i+1},\dots,v_n\}]$. A characterisation of chordal graphs due to Fulkerson and Gross [6] states that a graph is chordal if and only if it has a perfect elimination ordering. See [3] for an overview of the properties of chordal graphs and clique trees.

Given two integers $d$ and $\nu$ and a graph class $\mathcal{C}$, we denote by $\mathcal{M}_{\mathcal{C}}(d,\nu)$ the set of all graphs $G$ in $\mathcal{C}$ such that $\Delta(G) < d$ and $\nu(G) < \nu$. A graph in $\mathcal{M}_{\mathcal{C}}(d,\nu)$ that has the maximum number of edges is called an *edge-extremal graph*. When the graph class considered is the class of all graphs, we write simply $\mathcal{M}(d,\nu)$. The following lemma establishes a connection between edge-extremal graphs and factor-critical graphs in some graph classes. Even though the statement we present here is different from the one stated in [1], the proof in [1] suffices to prove the result as stated below.

**Lemma 1 ([1]).** *Let $\mathcal{C}$ be a graph class that is closed under vertex deletion and closed under taking disjoint union with stars. Let $G$ be an edge-extremal graph in $\mathcal{M}_{\mathcal{C}}(d,\nu)$ with maximum number of connected components that are $(d-1)$-stars. Then every connected component of $G$ that is not a $(d-1)$-star is factor-critical.*

The following statement gives a summary of the results obtained by Balachandran and Khare [1].

**Theorem 3 ([1]).** *Given $d,\nu \in \mathbb{N}$, the maximum number of edges of a graph in $\mathcal{M}(d,\nu)$ is given by $(d-1)(\nu-1) + \lfloor \frac{d-1}{2} \rfloor \lfloor \frac{\nu-1}{\lceil \frac{d-1}{2} \rceil} \rfloor$. Moreover, a graph achieving this number of edges is*

$$\begin{cases} rK_{1,d-1} + qK'_d, & \text{if } d \text{ is even} \\ rK_{1,d-1} + qK_d, & \text{if } d \text{ is odd,} \end{cases}$$

*where $\nu - 1 = q\lceil \frac{d-1}{2} \rceil + r$, with $r \geq 0$, and $K'_d$ is the graph obtained from $K_d$ by the removal of the edges of a perfect matching and addition of a new vertex adjacent to $d-1$ vertices.*

In Section 3, we show the corresponding bounds for $\mathcal{M}_{chordal}(d,\nu)$ and obtain graphs that achieve these bounds. We remark that, in Theorem 3, the graph $rK_{1,d-1} + qK_d$, obtained when $d$ is odd, is already a chordal graph. Thus, for odd $d$, the edge-extremal chordal graphs have the same number of edges as the edge-extremal general graphs. Our proof, however, does not rely on this fact and has a unified approach, that works regardless of the parity of $d$.

## 3 Chordal graphs

In this section we present our main result. The strategy to determine the maximum number of edges that a graph in $\mathcal{M}_{chordal}(d,\nu)$ can have is to show that among the edge-extremal graphs in $\mathcal{M}_{chordal}(d,\nu)$, there is one that has a very simple structure: it is a disjoint union of cliques and stars of a given size.

**Theorem 1 (restated).** *There exists an edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars.*

*Overview of the proof.* The proof is by contradiction. We start with an edge-extremal graph of $\mathcal{M}_{chordal}(d,\nu)$ that is, in some sense, closest to being a disjoint union of cliques and stars. From that, we will perform a series of modifications in the graph in order to obtain another graph of $\mathcal{M}_{chordal}(d,\nu)$ that has at least as many edges as the one we started with, but that is closer to being a disjoint union of cliques and stars, which will be a contradiction with our initial choice. To perform the modifications, we will consider a specific clique tree of our edge-extremal graph and exploit the structure of this graph around one of its cliques, given by a carefully chosen node of the tree. A crucial part of the proof is to ensure that, after each modification, the obtained graph still belongs to $\mathcal{M}_{chordal}(d,\nu)$. In this vein, Lemmas 3 and 4 will precisely show that the two modifications we describe can indeed be performed without disrupting membership in $\mathcal{M}_{chordal}(d,\nu)$. In this way, we obtain a new edge-extremal graph that, as a result, has several structural properties that will be exploited to conclude the proof.

*Proof of Theorem 1.* Assume for a contradiction that there is no edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars. Let $H$ be an edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu)$ with maximum number of $(d-1)$-stars and subject to that, with maximum number of connected components. Let $W$ be a connected component of $H$ that is not a clique nor a star and let $\nu_1 = \nu(W)+1$. By Lemma 1, $W$ is a factor-critical graph and therefore $|V_W| = 2\nu_1 - 1$. Note that $W \in \mathcal{M}_{chordal}(d,\nu_1)$ and, in fact, $W$ is edge-extremal in $\mathcal{M}_{chordal}(d,\nu_1)$. Among all the edge-extremal graphs in $\mathcal{M}_{chordal}(d,\nu_1)$ with $2\nu_1-1$ vertices, let $G$ be the one that has a clique tree with minimum number of leaves. Note that, in particular, $G$ is connected, by the maximality of the number of connected components of the graph $H$.

Let $\mathcal{T}$ be a clique tree of $G$ achieving the minimum number of leaves. We consider $\mathcal{T}$ rooted in an arbitrary bag $R$. Let $X$ be a node of $\mathcal{T}$. We denote by $T_X$ the subtree of $\mathcal{T}$ rooted at the node $X$. We define a subgraph $G_X$ associated with each node $X$ of $\mathcal{T}$ in the following way. If $X = R$, then $G_X = G$. Otherwise, let $S$ be the separator of $G$ given by the intersection between $X$ and its parent in $\mathcal{T}$ and let $V_{T_X}$ be the set of vertices appearing in the bags of $T_X$. The subgraph $G_X$ associated with the node $X$ is given by $G[V_{T_X} \setminus S]$. Observe that if $X$ is a leaf of $\mathcal{T}$, then $G_X$ is a complete graph. Let $B$ be a bottommost bag in $\mathcal{T}$ such that $G_B$ is not a complete graph. Note that such a node indeed exists since $G$ is not a complete graph itself. Let $B_1, \ldots, B_k$ be the children of $B$ in $\mathcal{T}$ and let $S_i = B \cap B_i$. For simplicity, from now on we denote $C_i = V_{T_{B_i}} \setminus S_i$. Note that $G[C_i] = G_{B_i}$ is a complete graph for every $i$.

*Observation 1 (♠).* For every $i$, the subgraph of $G$ induced by the edges $E_i = \{xy \mid x \in S_i \text{ and } y \in C_i\}$ is a chain graph.

*Observation 2 (♠).* For every $i$, the subtree $T_{B_i}$ is a path.

In what follows, we want to modify the graph $G$ in such a way to obtain a graph that is still chordal, has the same number of vertices of $G$ and belongs to $\mathcal{M}_{chordal}(d, \nu_1)$, but either has more edges than $G$, or is disconnected or has a clique tree with smaller number of leaves. Either one of these outcomes will contradict the choice of $G$. Note that since $G$ is a factor-critical graph, the addition of edges to $G$ does not increase its matching number. Moreover, for any $k \in \mathbb{N}$, the removal of $k$ vertices from $G$ and addition of $k$ new vertices does not increase its matching number either. Therefore, all the modifications that are to be performed in what follows will not lead to a graph with larger matching number than $G$.

For every $v \in B$ let $f_G(v,i)$ denote the number of neighbors that vertex $v$ has in the clique $C_i$, that is, $f_G(v,i) = |N_G(v) \cap C_i|$ and let $u_{i,1}, \ldots, u_{i,|C_i|}$ be an ordering of the vertices of $C_i$ such that $\deg_G(u_{i,1}) \geq \deg_G(u_{i,2}) \geq \ldots \geq \deg_G(u_{i,|C_i|})$.

We first state and prove the following lemma that can be understood as the converse of Observation 1 and that will be useful throughout the paper to show that a graph is chordal.

**Lemma 2 (♠).** *Let $H$ be any graph and $B, C_1, \ldots, C_k$ be cliques of $H$ such that*

- $N_H(C_i) \subseteq B$, *for every* $1 \leq i \leq k$;
- $H[V_H \setminus (\cup_{i=1}^{k} C_i)]$ *is a chordal graph.*

*If the subgraph of $H$ induced by the edges $E_i = \{xy \mid x \in B \text{ and } y \in C_i\}$ is a chain graph for every $1 \leq i \leq k$, then $H$ is a chordal graph.*

We are now ready to state the two modifications that will be used repeatedly throughout the proof of Theorem 1.

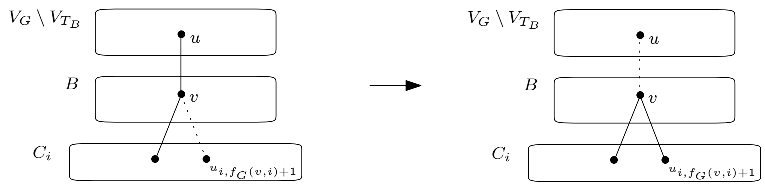
**Modification 1.** Let $B, C_1, \ldots, C_k$ be subsets of the vertex set of the chordal graph $G$ as previously described and let $v \in B$. For $1 \leq i \leq k$, if $0 < f_G(v,i) < |C_i|$ and $v$ has a neighbor that does not belong to $G[V_{T_B}]$, we do the following (see Figure 1a):

(i) Add an edge between $v$ and the vertex $u_{i,f_G(v,i)+1}$;

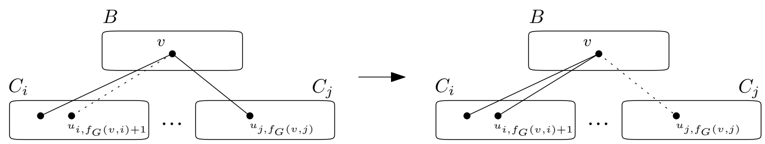
(ii) Delete the edge from $v$ to one of its neighbors outside $G[V_{T_B}]$. This neighbor is chosen in the following way: consider the subtree $T_v$ of $\mathcal{T}$ formed by the bags that contain the vertex $v$. Let $L$ be a leaf of $T_v$ that is not in the subtree rooted in $B$. Such a leaf exists since $v$ has a neighbor outside $G[V_{T_B}]$. Let $L'$ be the bag that is adjacent to $L$ in $T_v$. Since $L \nsubseteq L'$, there exists $u \in L \setminus L'$. Let $u$ be the chosen neighbor of $v$ and delete the edge $uv$.

**Lemma 3.** *Modification 1 preserves both membership in $\mathcal{M}_{chordal}(d, \nu)$ and number of edges.*

*Proof.* Let $G'$ be the graph obtained with the application of Modification 1. First, note that the addition of the edge $vu_{i,f_G(v,i)+1}$ preserves the nested neighborhood property in the bipartite graph induced by the edges between $B$ and $C_i$. Thus,




(a) Modification 2




(b) Modification 3

Fig. 1: The dotted lines between two vertices indicate non-edges.

since $G$ is chordal and by Lemma 2, the addition of this edge does not disrupt membership in the class of chordal graphs. Therefore, to show that $G'$ is chordal it suffices to show that the removal of the edge $uv$ preserves chordality. We do so by providing a clique tree to $G - uv$. This clique tree is obtained from $\mathcal{T}$ as follows. Let $L'' = L \setminus \{u\}$. If $L'' \neq L'$, add $L''$ between $L$ and $L'$ in the tree $\mathcal{T}$ and delete $v$ from $L$. If $L'' = L'$, just delete $v$ from $L$ in $\mathcal{T}$. Also, note that this operation does not change the number of leaves in $\mathcal{T}$. Hence, we obtain that the graph $G'$ is chordal. Note that the degree of $v$ does not change with this modification. The only vertex whose degree was increased by Modification 1 is $u_{i,f_G(v,i)+1}$. However, note that since $vu_{i,f_G(v,i)} \in E_G$ and $vu_{i,f_G(v,i)+1} \notin E_G$, we have that $\deg_G(u_{i,f_G(v,i)+1}) < \deg_G(u_{i,f_G(v,i)})$. Thus, $\deg_{G'}(u_{i,f_G(v,i)+1}) \leq \deg_{G'}(u_{i,f_G(v,i)}) = \deg_G(u_{i,f_G(v,i)}) < d$. We conclude the proof by observing that $|E_{G'}| = |E_G|$, since exactly one edge was deleted and exactly one edge was added by this modification. $\Diamond$

**Modification 2.** Let $B, C_1, \ldots, C_k$ be subsets of the vertex set of the chordal graph $G$ as previously described and let $v \in B$. For $1 \leq i \leq k$, if $0 < f_G(v,i) < |C_i|$ and $f_G(v,j) > 0$ with $j > i$, we do the following (see Figure 1b):

(i) Delete the edge $vu_{j,f_G(v,j)}$;
(ii) Add the edge $vu_{i,f_G(v,i)+1}$.

**Lemma 4 (♠).** *Modification 2 preserves both membership in $\mathcal{M}_{chordal}(d,\nu)$ and number of edges.*

Recall that our graph $G$ is an edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu_1)$, since it is a connected component of an edge-extremal graph $H \in \mathcal{M}_{chordal}(d,\nu)$, where $H$ has maximum number of connected components among the edge-extremal graphs of $\mathcal{M}_{chordal}(d,\nu)$. Let $G^*$ be the graph obtained from $G$ by exhaustive applications of Modification 2 followed by exhaustive applications of Modification 1. It follows immediately from Lemmas 3 and 4 that $G^* \in \mathcal{M}_{chordal}(d, \nu_1)$ and that $G^*$ is edge-extremal in this set. Moreover, if the graph obtained after the application of any modification is disconnected, we reach a contradiction with the maximality of the number of components of $H$. Therefore, we can assume $G^*$ is connected. The following lemma describes the major structural property of $G^*$ that will be exploited in the remainder of the proof.

**Lemma 5.** *Let $G^*$ be the graph obtained from $G$ by exhaustive applications of Modification 2 followed by exhaustive applications of Modification 1. Then, for every $v \in V_{G^*} \cap B$ and every $i$, if $v$ has at least one neighbor in $C_i$, one of the following conditions hold:*

(a) $C_i \subseteq N_{G^*}(v)$;
(b) $\deg_{G^*}(v) = \Delta(G^*)$ *and* $N_{G^*}(v) \subseteq B \cup C_1 \cup \ldots \cup C_i$.

*Proof.* First, let $G'$ be the graph obtained from $G$ by exhaustive applications of Modification 2. Since this modification can no longer be applied, then for every $v \in B$ and every $i$ such that $f(v, i) > 0$, we have that either $f(v, i) = |C_i|$ or $f(v, j) = 0$ for every $j > i$. Thus, for every $v \in B$, there exists at most one index $\ell$ such that $0 < f(v, \ell) < |C_\ell|$. Now we apply Modification 1 exhaustively to $G'$ and obtain the graph $G^*$. Recall that $f_{G'}(v, i) = |N_{G'}(v) \cap C_i|$. Observe that for every $v \in B$, if $f_{G'}(v, i) = 0$, then $f_{G^*}(v, i) = 0$ and if $f_{G'}(v, i) = |C_i|$, then $f_{G^*}(v, i) = |C_i|$. Furthermore, since Modification 1 can no longer be applied, if a vertex $v$ is such that $0 < f_{G^*}(v, i) < |C_i|$, then $v$ has no neighbors outside $B \cup C_1 \cup \ldots \cup C_i$. That is, if condition (a) does not hold, then $N_{G^*}(v) \subseteq B \cup C_1 \cup \ldots \cup C_i$. It remains to show that, in this case, $\deg_{G^*}(v) = \Delta(G^*)$. Indeed, if $\deg_{G^*}(v) < \Delta(G^*)$, we can add to $G^*$ the edge $vu_{i, f_{G^*}(v,i)+1}$. The addition of this edge does not change the maximum degree of $G^*$ by assumption. Moreover, by Lemma 2, it also preserves chordality. Since by Lemmas 3 and 4, we have that $|E_{G^*}| = |E_G|$ and that $G^* \in \mathcal{M}_{chordal}(d, \nu_1)$, this leads to a contradiction with the fact that $G$ is edge-extremal in $\mathcal{M}_{chordal}(d, \nu_1)$. $\Diamond$

Since the graph $G^*$ is such that $\Delta(G^*) < d$ and $|E_{G^*}| = |E_G|$, we can replace the connected component $G$ in our edge-extremal graph $H$ by $G^*$. This replacement will be convenient since Lemma 5 provides useful information on the structure of $G^*$. More concretely, *in the rest of the proof we shall assume that $B, C_1, \ldots, C_k$ satisfy the conclusion of Lemma 5.*

Let $b$ be the size of the clique $B$, let $\Delta = \Delta(G^*)$ and recall that $S_i$ is the separator between the bag $B_i$ and $B$. We are now going to conclude the proof of Theorem 1 with a case analysis.

**Case 1:** There exists $i$ such that $|C_i| + b \leq \Delta + 1$.

**Case 1.1:** $k \geq 2$.

We may assume, without loss of generality, that $|C_1| \leq |C_2| \leq \ldots \leq |C_k|$. In particular, this implies that $|C_1| + b \leq \Delta + 1$. We will show that, in this case, all the vertices of $C_1$ are adjacent to all the vertices of $S_1 \cup \ldots \cup S_k$. This will lead

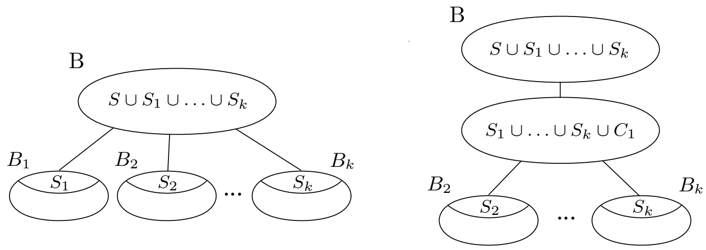



Fig. 2: To the left, the clique tree $\mathcal{T}$ and to the right, a clique tree of the updated graph $G^*$ that has less leaves than $\mathcal{T}$.

to a contradiction with the number of leaves of the clique tree of $G$. Suppose for a contradiction that there exists $v \in S_1 \cup \ldots \cup S_k$ that is not universal to $C_1$. This implies that $f_{G^*}(v, 1) < |C_1|$.

We will show that the graph $G^*$ can be modified in order to obtain another edge-extremal graph, also in $\mathcal{M}_{chordal}(d, \nu_1)$, in which $v$ is adjacent to every vertex of $S_1 \cup \ldots \cup S_k$.

First, note that it cannot be the case that $f_{G^*}(v, 1) > 0$, since by Lemma 5, if $0 < f_{G^*}(v, 1) < |C_1|$, then $v$ has maximum degree and has no neighbors outside $B \cup C_1$. However, this is a contradiction, since $|C_1| + b \leq \Delta + 1$. Thus, it holds that $f_{G^*}(v, 1) = 0$.

In what follows, we will modify the graph $G^*$ and the deletion of some edges might disrupt the membership in the class of chordal graphs. In these cases, we will use the following modification in order to restore it.

**Modification 3.** Let $H$ be any graph satisfying the conditions of Lemma 2. We do the folowing:

(i) Delete from $H$ all the edges $xy$ such that $x \in B$ and $y \in C_i$ for some $i$;
(ii) For each $v \in B$ and each $1 \leq i \leq k$, if $f_H(v, i) > 0$, add the edges between $v$ and the vertices $u_{i,1}, \ldots, u_{i,f_H(v,i)}$.

**Lemma 6 (♠).** *Modification 3 preserves membership in the class of chordal graphs and number of edges.*

We now modify $G^*$ as follows. Let $j$ be the largest index for which $f_{G^*}(v, j) > 0$. If $f_{G^*}(v, j) = |C_j|$, since $|C_1| \leq |C_j|$, we can delete $|C_1|$ edges between $v$ and $C_j$ and add all the edges between $v$ and $C_1$. We then apply Modification 3 to the obtained graph in order to obtain a graph that, by Lemma 6, is chordal. Note that the only vertices whose degree has increased are the ones in $C_1$. However, since $|C_1| + b \leq \Delta + 1$, we conclude that the maximum degree of $G^*$ did not increase.

If $f_{G^*}(v, j) < |C_j|$, then, by Lemma 5, $v$ has maximum degree and has no neighbors outside $B \cup C_1 \cup \ldots \cup C_j$. This implies that $\sum_{\ell=2}^{k} f_{G^*}(v, \ell) = \Delta - b + 1$. Since $|C_1| \leq \Delta - b + 1$ by assumption, we can delete $|C_1|$ edges between $v$ and vertices of $C_2 \cup \ldots \cup C_j$ and add all the edges between $v$ and $C_1$. We then

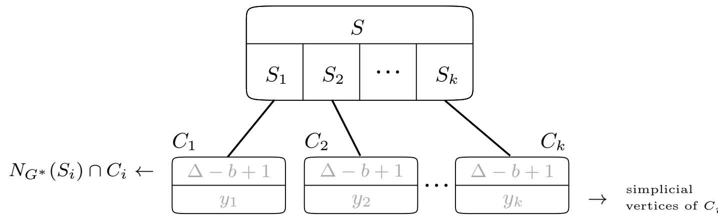



Fig. 3: Graph $G^*$ in case 2. Thick lines indicate all possible edges between the sets. Gray text indicates the cardinality of the vertex set.

apply Modification 3 to the obtained graph in order to obtain a graph that, by Lemma 6, is chordal. Again, the only vertices whose degree has increased in this process are the ones from $C_1$, thus we conclude the obtained graph still has degree at most $\Delta$.

Finally note that in both cases, the modifications do not change the number of edges of $G^*$, since $\sum_{\ell=1}^{k} f_{G^*}(v,\ell)$ remains the same. We perform this change for every $v \in S_1 \cup \ldots \cup S_k$ such that $f_{G^*}(v,1) > 0$ and obtain a new edge-extremal graph in $\mathcal{M}_{chordal}(d,\nu_1)$ such that all the vertices of $C_1$ are adjacent to all the vertices of $S_1 \cup \ldots \cup S_k$. Recall that among all the edge-extremal graphs in $\mathcal{M}_{chordal}(d,\nu_1)$ with $2\nu_1 - 1$ vertices, $G$ was the one that had a clique tree with minimum number of leaves. This new graph, however, has a clique tree that has less leaves than the clique tree $\mathcal{T}$ of $G$. This is because the clique $C_1 \cup S_1 \cup \ldots \cup S_k$ is contained in $B$ and contains the intersection between $B$ and each child of $B$ (see Figure 2). This contradicts the minimality of the number of leaves of $\mathcal{T}$.

**Case 1.2:** $k = 1$.

Since $B \cup C_1$ is not a clique by assumption, there exists $v \in S_1$ that is not universal to $C_1$ in $G^*$. By Lemma 5, $v$ has maximum degree and no neighbors outside $B \cup C_1$. Hence, $\deg_{G^*}(v) \leq b - 1 + |C_1| - 1$, which implies that $\Delta \leq b + |C_1| - 2$. This is a contradiction with the assumption of Case 1 that $|C_1| + b \leq \Delta + 1$.

**Case 2:** For every $i$, $|C_i| + b > \Delta + 1$.

Let $v \in S_1 \cup \ldots \cup S_k$. Let $a_v$ be the smallest index such that $f_{G^*}(v,a_v) > 0$. Note that $v$ cannot be universal to $C_{a_v}$ in $G^*$, since by assumption $|C_{a_v}| + b > \Delta + 1$. By Lemma 5, $\deg_{G^*}(v) = \Delta$ and $N_{G^*}(v) \subseteq B \cup C_{a_v}$. This implies that for every $v \in S_1 \cup \ldots \cup S_k$, there exists a unique index $a_v$ such that $f_{G^*}(v,a_v) > 0$. That is, for any $j \neq a_v$, $f_{G^*}(v,j) = 0$, and thus $S_i \cap S_j = \emptyset$ if $i \neq j$. Also, since $N_{G^*}(v) \subseteq B \cup C_{a_v}$ and $v$ has degree $\Delta$, we have that $f_{G^*}(v,a_v) = \Delta - b + 1$. That is, if $a_v = a_u$, then $u$ and $v$ are true twins in $G^*$. Moreover, for any $1 \leq i < j \leq k$, $|N_{G^*}(S_i) \cap C_i| = |N_{G^*}(S_j) \cap C_j|$. Let $S$ be the separator between $B$ and its parent in the clique tree $\mathcal{T}$. Since for every $v \in S_1 \cup \ldots \cup S_k$, $N_{G^*}(v) \subseteq B \cup C_{a_v}$, we know that $S \cap S_i = \emptyset$, for every $i$. Also, since the graph $G^*$ is connected, $S \neq \emptyset$. See Figure 3.

Let $u \in N_{G^*}(S_i) \cap C_i$. Suppose for a contradiction that $\deg_{G^*}(u) < \Delta$. Let $G_1$ be the graph obtained from $G^*$ by the deletion of one vertex of $S$ and addition of a new vertex $w$ in $S_i$, such that $N_{G_1}[w] = B \cup (N(S_i) \cap C_i)$.

*Claim 1 (♠).* $|E_{G_1}| \geq |E_{G^*}|$ and $\Delta(G_1) = \Delta(G^*)$.

If $G_1$ is disconnected or has more edges than $G^*$, we have a contradiction. We repeat the above modification until either the graph obtained is disconnected, that is, until $S = \emptyset$, or until for every $i$, the degree of the vertices in $N_{G_1}(S_i) \cap C_i$ is $\Delta$. Let $G_2$ be the graph obtained after exhaustive application of the above modification. If $G_2$ is disconnected, we have a contradiction with the maximality of the number of connected components of our initial edge-extremal graph. Otherwise, by Claim 1, $|E_{G_2}| \geq |E_{G^*}|$ and $\Delta(G_2) = \Delta(G^*)$. Therefore, we can now replace $G^*$ by $G_2$ in the edge-extremal graph $H$. Note that $G_2$ is such that:

1. For every $1 \leq i < j \leq k$, $S_i \cap S_j = \emptyset$;
2. For every $1 \leq i \leq k$, the vertices of $S_i$ and of $N_{G_2}(S_i) \cap C_i$ have degree $\Delta$ and $|N_{G_2}(S_i) \cap C_i| = \Delta - b + 1$.

**Case 2.1:** $k \geq 2$.

Let $y_i$ be the number of simplicial vertices in the clique $C_i$. Assume without loss of generality that $y_1 \geq y_2$. We perform the following modifications in the graph $G_2$: deletion of one simplicial vertex from $C_2$ and one vertex from $S_1$ and addition of one vertex to $S_2$ and one simplicial vertex to $C_1$. Note that, after this modification, the only vertices that had their degree changed are the simplicial vertices from $C_1$ and $C_2$. Since these simplicial vertices did not have maximum degree before, the degree of the obtained graph does not exceed the degree of $G_2$. Note that $y_2 - 1 + \Delta - b + 1 + \Delta$ edges were removed by the deletion of the two vertices and $y_1 + \Delta - b + 1 + \Delta$ were added by the addition of the other two vertices. However, since $y_1 \geq y_2$, we have that the obtained graph has strictly more edges than $G_2$, which is a contradiction.

**Case 2.2:** $k = 1$.

Since all vertices in $S_1$ and in $N_{G_2}(S_1) \cap C_1$ have maximum degree, we can perform the following modification in $G_2$: delete all vertices of $S_1$ and add $|S_1|$ vertices to $N_{G_2}(S_1) \cap C_1$. The graph obtained after this modification has the same number of edges as $G_2$, since $|S_1|$ vertices of degree $\Delta$ were removed and the same amount of vertices with the same degree was added. However, the obtained graph is disconnected, which is a contradiction with the maximality of the number of connected components of the edge-extremal graph $H$.

This concludes the proof of Theorem 1. □

By Theorem 1, we know that there is an edge-extremal graph in $\mathcal{M}_{chordal}(d, \nu)$ that is a disjoint union of cliques and stars. The next lemma gives a tight upper bound on the number of edges of such an edge-extremal graph when $d$ is even.

**Lemma 7 (♠).** *Let $G$ be a graph in $\mathcal{M}_{chordal}(d, \nu)$ that is a disjoint union of cliques and stars. If $d$ is even, then $|E_G| \leq (d-1)(\nu - 1)$.*

By Theorem 3, we already know the maximum number of edges that a graph that is a disjoint union of cliques and stars can have when $d$ is odd. From Theorem 1 and Lemma 7, we obtain our main result, Theorem 2 (see page 2), which establishes the upper bound on the number of edges that a chordal graph of $\mathcal{M}_{chordal}(d, \nu)$ can have and shows that the obtained bound is tight.

## 4 Final remarks and open problems

In this work, we determined the maximum number of edges that a chordal graph can have if its maximum degree and matching number are bounded. We also exhibt examples of graphs achieving this bound.

An interesting question that remains open comes from the fact that the graph $K_i'$ used in Theorem 3 has an induced $C_4$. *For each $d$ and $\nu$, what is the maximum number of edges of a graph in $\mathcal{M}_{C_4\text{-}free}(d, \nu)$?* We point out that the bound on the number of edges for chordal graphs does not hold for $C_4$-free graphs, as can be seen by the graph $P$, obtained from the famous Petersen graph by the subdivision of one edge (see Figure 4, ♠). We have that $\Delta(P) = 3$, $\nu(P) = 5$ and $|E_P| = 16$. In this case, the bound given by Theorem 1 when $d = 4$ and $\nu = 6$ is 15. This idea can be further generalized to create examples in the class of $\mathcal{H}$-free graphs, where $\mathcal{H}$ is any finite collection of cycles. Indeed, let $r$ be the size of a largest cycle of $\mathcal{H}$. A result due to Kochol [8] about snarks implies that for any $r \geq 5$ there exists an infinite family of 3-regular graphs of girth $r$ that have a perfect matching. Let $G$ be one such graph and let $H$ be the graph obtained from $G$ by the subdivision of one edge. The graph $H$ is clearly $\mathcal{H}$-free and is such that $\Delta(H) = 3$, $\nu(H) = \nu(G)$ and $|E_H| = 3\nu(H) + 1$, while the bound given by Theorem 1 when $d = 4$ and $\nu = \nu(H) + 1$ is $3\nu(H)$.

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# 5 Appendix

## 5.1 Proofs omitted in Section 3

*Observation 1 (restated).* For every $i$, the subgraph of $G$ induced by the edges $E_i = \{xy \mid x \in S_i \text{ and } y \in C_i\}$ is a chain graph.

*Proof.* Note that $(S_i, C_i)$ constitutes a partition of $V_{T_{B_i}}$, thus $G[E_i]$ is a bipartite graph. Suppose for a contradiction that there exists an induced $2K_2$ in $G[E_i]$ with vertex set $\{x_1, y_1, x_2, y_2\}$. Since $S_i$ and $C_i$ are cliques in $G$, the vertices $x_1$, $y_1$, $x_2$ and $y_2$ would form an induced $C_4$ in $G$, a contradiction with the fact that $G$ is chordal. Therefore $G[E_i]$ is indeed bipartite and $2K_2$-free, that is, a chain graph. ⌟

*Observation 2 (restated).* For every $i$, the subtree $T_{B_i}$ is a path.

*Proof.* Since $G[V_{T_{B_i}}]$ is a chordal graph, by Observation 1, the bipartite graph obtained from $G[V_{T_{B_i}}]$ by deleting the edges inside $S_i$ and $C_i$ is a chain graph. Because of the nested neighborhood property of chain graphs, $G[V_{T_{B_i}}]$ has a clique tree that is a path. Since $\mathcal{T}$ was chosen with minimum number of leaves, the subtree $T_{B_i}$ is a path, for every $i$. ⌟

**Lemma 2 (restated).** *Let $H$ be any graph and $B, C_1, \ldots, C_k$ be cliques of $H$ such that*

- $N_H(C_i) \subseteq B$*, for every* $1 \leq i \leq k$*;*
- $H[V_H \setminus (\cup_{i=1}^{k} C_i)]$ *is a chordal graph.*

*If the subgraph of $H$ induced by the edges $E_i = \{xy \mid x \in B \text{ and } y \in C_i\}$ is a chain graph for every $1 \leq i \leq k$, then $H$ is a chordal graph.*

*Proof.* We will show how to construct a perfect elimination ordering for the graph $H$. Note that for every $1 \leq i \leq k$, the vertex $u_{i,|C_i|}$ is simplicial in $H$. Moreover, for every $i \leq k$ and every $j \leq |C_i|$, the vertex $u_{i,j}$ is simplicial in $H[V_H \setminus \{u_{i,j+1}, \ldots, u_{i,|C_i|}\}]$. Finally, since $H[V_H \setminus (\cup_{i=1}^{k} C_i)]$ is a chordal graph, it has a perfect elimination ordering $\sigma'$ of its vertices. Let $\sigma_i = u_{i,|C_i|} \ldots u_{i,1}$. Then $\sigma_1 \sigma_2 \ldots \sigma_k \sigma'$ is a perfect elimination ordering for $H$, which concludes the proof that $H$ is chordal. ◊

**Lemma 4 (restated).** *Modification 2 preserves both membership in $\mathcal{M}_{chordal}(d, \nu)$ and number of edges.*

*Proof.* Let $G'$ be the graph obtained after the application of Modification 2. The only vertex that had its degree increased by this modification is $u_{i,f_G(v,i)+1}$. However, as in the proof of Lemma 3, since $vu_{i,f_G(v,i)} \in E_G$ and $vu_{i,f_G(v,i)+1} \notin E_G$, we have that $\deg_G(u_{i,f_G(v,i)+1}) < \deg_G(u_{i,f_G(v,i)})$. Thus, $\deg_{G'}(u_{i,f_G(v,i)+1}) \leq$

$\deg_{G'}(u_{i,f_G(v,i)}) = \deg_G(u_{i,f_G(v,i)}) < d$, implying that $\Delta(G') < d$. It is also easy to see that $|E_{G'}| = |E_G|$, since exactly one edge was deleted in step (i) and exactly one edge was added to the graph in step (ii). It remains to show that the obtained graph is still chordal. Indeed, note that both the deletion of the edge $vu_{j,f_G(v,j)}$ (resp. addition of the edge $vu_{i,f_G(v,i)+1}$) preserves the nested neighborhood property in the bipartite graph induced by the edges between $B$ and $C_j$ (resp. $C_i$). Thus, by Lemma 2, the graph $G'$ is a chordal graph. $\diamondsuit$

**Lemma 6 (restated).** *Modification 3 preserves membership in the class of chordal graphs and number of edges.*

*Proof.* Let $H'$ be the graph obtained from $H$ by Modification 3. We show that, for every $1 \leq i \leq k$, $H'[E_i]$ is a chain graph, where $E_i = \{xy \mid x \in B \text{ and } y \in C_i\}$. By Lemma 2, this shows that $H'$ is chordal. Suppose this is not the case, and let $v, w \in B$ and $u_{i,j}, u_{i,\ell} \in C_i$, with $j < \ell$, be such that $\{v, w, u_{i,j}, u_{i,\ell}\}$ induces a $2K_2$ in $H'[E_i]$ with edges $vu_{i,j}$ and $wu_{i,\ell}$. However, since $j < \ell$, the edge $wu_{i,j}$ was also added in step (ii), a contradiction. Finally, it is easy to see that $|E_{H'}| = |E_H|$, since the degrees of the vertices in $B$ remain unchanged. $\diamondsuit$

*Claim 1 (restated).* $|E_{G_1}| \geq |E_{G^*}|$ and $\Delta(G_1) = \Delta(G^*)$.

*Proof.* Since the vertices of $S_i$ have maximum degree, it holds that $\deg_{G_1}(w) = \Delta$ and hence the graph $G_1$ has at least as many edges as $G^*$. The only vertices whose degree has increased after this modification are those belonging to $N_{G_1}(S_i) \cap C_i$. However, note that if $x \in N_{G_1}(S_i) \cap C_i$, then $\deg_{G_1}(x) = \deg_{G^*}(x) + 1$. Since $\deg_{G^*}(x) < \Delta$ by assumption, we have that $\deg_{G_1}(x) \leq \Delta(G^*)$ and thus $\Delta(G_1) = \Delta(G)$. ⌟

**Lemma 7 (restated).** *Let $G$ be a graph in $\mathcal{M}_{chordal}(d, \nu)$ that is a disjoint union of cliques and stars. If $d$ is even, then $|E_G| \leq (d-1)(\nu-1)$.*

*Proof.* Let $G$ be a graph such that $\Delta(G) \leq d-1$ and $\nu(G) \leq \nu-1$ and that is a disjoint union of cliques and stars. We proceed by induction in $k$, the number of connected components of $G$. If $k = 1$ and $G$ is a star, then $\nu(G) = 1$ and since $\Delta(G) \leq d-1$, then $|E_G| \leq d-1$. If $G$ is a clique, then $|V_G| \leq d$. Since $d$ is even, $\nu(G) \leq \frac{d}{2}$ and therefore, $|E_G| \leq \frac{d(d-1)}{2} \leq \Delta(G)\nu(G) \leq (d-1)(\nu-1)$.

Let $G$ be a disjoint union of cliques and stars with $k > 1$ connected components. Let $H$ be a component of $G$ and $G'$ be the graph obtained from $G$ by the removal of the vertices of $H$. Then $\Delta(G') \leq \Delta(G)$ and by the induction hypothesis, $|E_{G'}| \leq \Delta(G')\nu(G')$. If $H$ is a star, then $\nu(G') = \nu(G) - 1$ and $|E_G| \leq |E_{G'}| + \Delta(G)$, which implies that $|E_G| \leq \Delta(G)\nu(G)$. If $H$ is a clique, then it is a clique of size at most $d$ and therefore $\nu(G') \leq \nu(G) - \frac{d}{2}$ and $|E_G| \leq |E_{G'}| + \binom{d}{2}$, which implies that $|E_G| \leq (d-1)(\nu(G) - \frac{d}{2}) + \binom{d}{2} \leq (d-1)(\nu-1)$. $\diamondsuit$

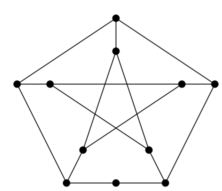

Fig. 4: A $C_4$-free graph with $\Delta = 3$, $\nu = 5$ and $|E| = 16$.