

On the Ordered List Subgraph Embedding Problems*

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Abstract

In the (parameterized) ORDERED LIST SUBGRAPH EMBEDDING problem (p-OLSE) we are given two graphs G and H , each with a linear order defined on its vertices, a function L that associates with every vertex in G a list of vertices in H , and a parameter k . The question is to decide if there is a subset $S \subseteq V(G)$ of k vertices and a mapping/embedding from S to $V(H)$ such that: (1) every vertex of S is mapped to a vertex from its associated list, (2) the linear orders inherited by S and its image under the mapping are respected, and (3) if there is an edge between two vertices in S then there is an edge between their images. If we change condition (3) to “there is an edge between two vertices in S if and only if there is an edge between their images”, we obtain the ORDERED LIST INDUCED SUBGRAPH EMBEDDING problem (p-OLISE). The p-OLSE and p-OLISE problems model various problems in Bioinformatics related to structural comparison/alignment of proteins.

We investigate the complexity of p-OLSE and p-OLISE with respect to the following structural parameters: the *width* Δ_L of the function L (size of the largest list), and the maximum degree Δ_H of H and Δ_G of G . In terms of the structural parameters under consideration, we draw a complete complexity landscape of p-OLSE and p-OLISE (and their optimization versions) with respect to the computational frameworks of classical complexity, parameterized complexity, and approximation.

1 Introduction

1.1 Problem Definition and Motivation

We consider the following problem that we refer to as the parameterized ORDERED LIST SUBGRAPH EMBEDDING problem, shortly (p-OLSE):

Given: Two graphs G and H with linear orders \prec_G and \prec_H defined on the vertices of G and H ; a function $L : V(G) \rightarrow 2^{V(H)}$; and $k \in \mathbb{N}$

Parameter: k

Question: Is there a subset $S \subseteq V(G)$ of k vertices and an injective map $\varphi : S \rightarrow V(H)$ such that: (1) $\varphi(u) \in L(u)$ for every $u \in S$; (2) for every $u, u' \in S$, if $u \prec_G u'$ then $\varphi(u) \prec_H \varphi(u')$; and (3) for every $u, u' \in S$, if $uu' \in E(G)$ then $\varphi(u)\varphi(u') \in E(H)$

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The parameterized ORDERED LIST INDUCED SUBGRAPH EMBEDDING (p-OLISE) problem is defined as follows:

Given: Two graphs G and H with linear orders \prec_G and \prec_H defined on the vertices of G and H ; a function $L : V(G) \rightarrow 2^{V(H)}$; and $k \in \mathbb{N}$

Parameter: k

Question: Is there a subset $S \subseteq V(G)$ of k vertices and an injective map $\varphi : S \rightarrow V(H)$ such that: (1) $\varphi(u) \in L(u)$ for every $u \in S$; (2) for every $u, u' \in S$, if $u \prec_G u'$ then $\varphi(u) \prec_H \varphi(u')$; and (3) for every $u, u' \in S$, $uu' \in E(G)$ if and only if $\varphi(u)\varphi(u') \in E(H)$

The optimization version of p-OLSE (resp. p-OLISE), denoted opt-OLSE (resp. opt-OLISE), asks for a subset $S \subseteq V(G)$ of maximum cardinality and an embedding $\varphi : S \rightarrow V(H)$ that satisfy conditions (1)-(3) in the definition of p-OLSE (resp. p-OLISE).

The p-OLSE and p-OLISE problems, and their optimization versions opt-OLSE and opt-OLISE, have applications in the area of Bioinformatics because they model numerous protein and DNA structural comparison problems (see [3, 8, 9, 24]).

In this paper we investigate the complexity of p-OLSE and p-OLISE with respect to the following structural parameters: the *width* Δ_L of the function L defined to be $\max\{|L(u)| : u \in V(G)\}$ and the maximum degree Δ_H of H and Δ_G of G . Restrictions on the structural parameters Δ_H , Δ_G and Δ_L are very natural in Bioinformatics. The parameters Δ_H and Δ_G model the maximum number of hydrophobic bonds that an amino acid in each protein can have; on the other hand, Δ_L is usually a parameter set by the Bioinformatics practitioners when computing the top few alignments of two proteins [24].

1.2 Previous Related Results

Goldman et al. [14] studied protein comparison problems using the notion of *contact maps*, which are undirected graphs whose vertices are linearly ordered. They formulated the protein comparison problem as a CONTACT MAP OVERLAP problem, in which we are given two contact maps and we need to identify a subset of vertices S in the first contact map, a subset of vertices S' in the second with $|S| = |S'|$, and an order-preserving bijection $f : S \rightarrow S'$, such that the number of edges in S that correspond to edges in S' is maximized. In [14], the authors proved that the CONTACT MAP OVERLAP problem is MAXSNP-complete, even when both contact maps have maximum degree one. The authors develop approximation algorithms by restricting their attention to special cases of the CONTACT MAP OVERLAP problem. In particular, they focus on cases where at least one of the graphs is a self-avoiding walk on the two dimensional grid. Such walks are formulated as a one-to-one mapping $g : \{1, 2, \dots, n\} \rightarrow \mathbb{Z} \times \mathbb{Z}$ such that $\|g(i) - g(i+1)\|_2 = 1$ for $i = 1, \dots, n-1$. Goldman et al. [14] observed that the geometry of proteins exhibit this structure, making self-avoiding walks a natural restriction for comparing protein contact maps. They create contact maps from walks by taking the set $\{1, 2, \dots, n\}$ as the set of vertices and including an edge for every pair ij such that $|i - j| > 1$ and $\|g(i) - g(j)\|_2 = 1$. While the problem remains \mathcal{NP} -complete even when comparing two contact maps that are self-avoiding walks, the authors give a polynomial-time $\frac{1}{4}$ -approximation algorithm when comparing two self-avoiding walks and a polynomial-time $\frac{1}{3}$ -approximation algorithm when comparing a self-avoiding walk and an arbitrary contact map. The main difference between the CONTACT MAP OVERLAP problem and the opt-OLSE and opt-OLISE problems under consideration is that in opt-OLSE and opt-OLISE the bijection f is restricted to mapping a vertex to one in its list, and the goal is to maximize the cardinality of the subset of vertices, not the number of edges, that can be embedded.

The p-OLISE problem generalizes the well-studied problem LONGEST ARC-PRESERVING COMMON SUBSEQUENCE (LAPCS) (see [1, 8, 9, 15, 16, 18, 20]). In LAPCS, we are given two sequences S_1 and S_2 over a fixed alphabet, where each sequence has arcs/edges between its characters, and the problem is to compute a longest common subsequence of S_1 and S_2 that respects the arcs. The LAPCS problem was introduced by [8, 9] where it was shown to be $W[1]$ -complete (parameterized by the length of the common subsequence sought) in the case when the arcs are *crossing*. Several works studied the complexity and approximation of LAPCS with respect to various restrictions on the types of the arcs (*e.g.*, *nested*, *crossing*, etc.) [1, 8, 9, 15, 18, 20]. The work in [1, 15] considered the problem in the case of nested arcs parameterized by the total number of characters that need to be deleted from S_1 and S_2 to obtain the arc-preserving common subsequence. They showed that the problem is \mathcal{FPT} with respect to this parameterization, and they also showed it to be \mathcal{FPT} when parameterized by the length of the common subsequence in the case when the alphabet consists of four characters. The p-OLISE problem generalizes LAPCS since no restriction is placed on the size of the alphabet, and a vertex can be mapped to any vertex from its list. Consequently, the “positive” results obtained in this paper about p-OLISE and opt-OLISE apply directly to their corresponding versions of LAPCS; on the other hand, we are able to borrow the $W[1]$ -hardness result from [9] to conclude the $W[1]$ -hardness results in Proposition 4.2 and Proposition 4.3.

A slight variation of p-OLSE was considered in [3, 24], where the linear orders imposed on G and H were replaced with partial orders; the problem was referred to as the GRAPH EMBEDDING problem in [3] and as the GENERALIZED SUBGRAPH ISOMORPHISM problem in [24]. The aforementioned problems were mainly studied in [3, 24] assuming no bound on Δ_H and Δ_G and, not surprisingly, only hardness results were derived. In [24], a parameterized algorithm with respect to the treewidth of G and the map width Δ_L combined was given. Most of the hardness results in [3, 24] were obtained by a direct reduction from the INDEPENDENT SET or CLIQUE problems. For example, it was shown in [3] that the problem of embedding the whole vertex-set $V(G)$ into H is \mathcal{NP} -hard, but is in \mathcal{P} if $\Delta_L = 2$. It was also shown that the problem of embedding a subset of $V(G)$ of size k into H is $W[1]$ -complete even when $\Delta_L = 1$, and cannot be approximated to a ratio $n^{1-\varepsilon}$ unless $\mathcal{P} = \mathcal{NP}$; we borrow these two hardness results as they also work for p-OLSE and p-OLISE.

Fagnot et al. [10] introduced Exact- (μ_G, μ_H) -Matching, a problem in which we are given two graphs, G and H , a mapping L as in p-OLISE and p-OLSE, and constants μ_G and μ_H where $\max\{|L(u)| : u \in G\} \leq \mu_G$ and $\max\{|L^{-1}(v)| : v \in H\} \leq \mu_H$. However, there is no ordering imposed on the vertices of G or H . The objective is to find an injective homomorphism from G to H . The authors proved that Exact- (μ_G, μ_H) -Matching is \mathcal{NP} -Complete when $\mu_G \geq 3$ and $\mu_H = 1$, even if G and H are bipartite graphs. Fertin et al. in [11, 12] studied Exact- (μ_G, μ_H) -Matching and an optimization version of it, called Max- (μ_G, μ_H) -Matching where the objective is to embed as many edges from G into H as possible, restricting their attention to graphs of bounded degree. While the authors found that for small values of Δ_G and Δ_H there are cases where Exact- (μ_G, μ_H) -Matching is in \mathcal{P} and cases where Max- (μ_G, μ_H) -Matching has a constant-factor approximation, generally the problem remains difficult. Fertin et al. [11, 12] also give an \mathcal{FPT} algorithm for solving Max- (μ_G, μ_H) -Matching by adding an arbitrary ordering to G and H which indicates that ordering G and H might improve the tractability of graph embedding problems. These results contrast with our results in Section 2 and Section 4 that show that p-OLSE and p-OLISE become easy when the ordering constraint is removed.

Finally, one can draw some similarities between p-OLISE and the SUBGRAPH ISOMORPHISM and GRAPH EMBEDDING problems. The main differences between p-OLISE and the aforementioned problems are: (1) in p-OLSE we have linear orders on G and H that need to be respected by the map sought, (2) we ask for an embedding of a subset of $V(G)$ rather than the whole vertex-set

$V(G)$, and (3) each vertex must be mapped to a vertex from its list. In particular, requirement (1) above precludes the application of well-known (logic) meta-theorems (see [13]) to the restrictions of p-OLISE that are under consideration in this paper.

1.3 Our Results and Techniques

We draw a complete complexity landscape of p-OLSE and p-OLISE with respect to the computational frameworks of classical complexity, parameterized complexity, and approximation, in terms of the structural parameters Δ_H , Δ_G and Δ_L . Table 1 outlines the obtained results about p-OLSE and p-OLISE and their optimization versions. Even though our hardness results are for specific values of the parameters Δ_H , Δ_G , and Δ_L , these results certainly hold true for restrictions of the problems to instances in which the corresponding parameters are upper bounded by (or equal to — by adding dummy vertices) any constant larger than these specific values. Observe also that the results we obtain *completely and tightly* characterize the complexity (with respect to all frameworks under consideration) of the problems with respect to Δ_H , Δ_G and Δ_L (unbounded vs. bounded, and when applicable, for different specific values).

Section 2 presents various classical complexity results. The \mathcal{NP} -hardness results are obtained by a reduction from the k -MULTI-COLORED INDEPENDENT SET problem. Section 3 presents various approximation results for the optimization versions of p-OLSE and p-OLISE. Section 4 presents parameterized complexity results for various restrictions of p-OLSE and p-OLISE. The $W[1]$ -hardness results are obtained by tweaking the $W[1]$ -hardness results given in the literature [3, 9], or by simple known reductions from the INDEPENDENT SET problem. The \mathcal{FPT} results in Theorem 4.10 for p-OLSE, when $\Delta_L = O(1)$, $\Delta_G = O(1)$, and $\Delta_H = \infty$, are derived using the random separation method [4]. This method is applied after transforming the problem — via reduction operations — to the INDEPENDENT SET problem on a graph composed of (1) a permutation graph and (2) a set of additional edges between the permutation graph vertices such that the number of additional edges incident to any vertex is at most a constant; Lemma 4.9 then shows that the INDEPENDENT SET problem on such graphs is \mathcal{FPT} . On the other hand, the \mathcal{FPT} results in Proposition 4.11, when $\Delta_H = 0$, $\Delta_G = O(1)$ (resp. $\Delta_G = 0$ and $\Delta_H = O(1)$ for p-OLISE by symmetry) and $\Delta_L = \infty$, are also derived using the random separation method [4], but the argument is simpler. We also point out in Propositions 2.3 and 4.4 that certain hard cases of p-OLSE and p-OLISE become easy if the order constraint (*i.e.*, the total order imposed on G and H) is no longer required.

To cope with the W -hardness of p-OLSE in certain cases, we consider a different parameterization of the problem in Section 5, namely the parameterization by the vertex cover number of G , and denote the associated problem by p-VC-OLSE. This parameterization is not interesting for p-OLISE since we prove that p-OLISE is \mathcal{NP} -complete in the case when $\Delta_G = 0$, $\Delta_H = 1$ and $\Delta_L = 1$, and hence the problem is *para- \mathcal{NP} -hard* [13] with respect to this parameterization. Proposition 5.1 shows that p-VC-OLSE is $W[1]$ -complete in the case when $\Delta_H = 1$, $\Delta_G = 1$ and Δ_L is unbounded (note that if either $\Delta_H = 0$ or $\Delta_G = 0$ then p-OLSE is \mathcal{FPT} when $\Delta_L = \infty$). So we restrict our attention to the case when $\Delta_L = O(1)$, and show in this case that the problem is \mathcal{FPT} even when both Δ_H and Δ_G are unbounded; the method relies on a bounded search tree approach, combined with the dynamic programming algorithm described in Proposition 2.1.

1.4 Background and Terminologies

Graphs. For a graph H we denote by $V(H)$ and $E(H)$ the set of vertices and edges of H , respectively; we write $|H|$ for $|V(H)|$. For a set of vertices $S \subseteq V(H)$, we denote by $H[S]$ the subgraph of H induced by the vertices in S . For a subset of edges $E' \subseteq E(H)$, we denote by $H - E'$

p-OLSE	Δ_H	Δ_G	Δ_L	Complexity
Classical	∞	0	∞	\mathcal{P}
	0	1	1	\mathcal{NP} -complete
Approximation	∞	$O(1)$	∞	\mathcal{APX} -complete
	0	∞	1	not approximable to $n^{1-\varepsilon}$
Parameterized	∞	$O(1)$	$O(1)$	\mathcal{FPT}
	1	1	∞	$W[1]$ -complete
	0	∞	1	$W[1]$ -complete
	0	$O(1)$	∞	\mathcal{FPT}
	∞	0	∞	\mathcal{FPT} (even in \mathcal{P})

p-OLISE	Δ_H	Δ_G	Δ_L	Complexity
Classical	0	0	∞	\mathcal{P}
	0	1	1	\mathcal{NP} -complete
	1	0	1	\mathcal{NP} -complete
Approximation	$O(1)$	$O(1)$	∞	\mathcal{APX} -complete
	0	∞	1	not approximable to $n^{1-\varepsilon}$
	∞	0	1	not approximable to $n^{1-\varepsilon}$
Parameterized	∞	0	1	$W[1]$ -complete
	0	∞	1	$W[1]$ -complete
	0	$O(1)$	∞	\mathcal{FPT}
	$O(1)$	0	∞	\mathcal{FPT}
	1	1	1	$W[1]$ -complete

Table 1: Classical, approximation, and parameterized complexity maps of p-OLSE and p-OLISE with respect to Δ_H , Δ_G and Δ_L . The inapproximability results are under the assumption that $\mathcal{P} \neq \mathcal{NP}$. The symbol ∞ stands for unbounded degree.

the graph $(V(H), E(H) \setminus E')$. For a vertex $v \in H$, $N(v)$ denotes the set of neighbors of v in H . The *degree* of a vertex v in H , denoted $deg_H(v)$, is $|N(v)|$. A vertex v is *isolated* in H if $deg_H(v) = 0$. The *degree* of H , denoted $\Delta(H)$, is $\Delta(H) = \max\{deg_H(v) : v \in H\}$. A *matching* in a graph is a set of edges M such that no two edges in M share an endpoint. An *independent set* of a graph H is a set of vertices I such that no two vertices in I are adjacent. A *maximum independent set* in H is an independent set of maximum cardinality. A *clique* is a subset of vertices $Q \subseteq V(H)$ such that for each pair of vertices $u, u' \in Q$, $uu' \in E(H)$. A *vertex cover* of H is a set of vertices such that each edge in H is incident to at least one vertex in this set; we denote by $\tau(H)$ the cardinality of a minimum vertex cover of H . Let L and L' be two parallel lines in the plane. A *permutation graph* P is the intersection graph of a set of line segments such that one endpoint of each of those segments lies on L and the other endpoint lies on L' . For more information about graphs, we refer the reader to [25].

Optimization Problems. An \mathcal{NP} *optimization problem* Q is a 4-tuple (I_Q, S_Q, f_Q, g_Q) . I_Q is the set of input instances, which is recognizable in polynomial time. For each instance $x \in I_Q$, $S_Q(x)$ is the set of feasible solutions for x , which is defined by a polynomial p and a polynomial-time computable predicate π (p and π depend only on Q) as $S_Q(x) = \{y : |y| \leq p(|x|) \wedge \pi(x, y)\}$. The function $f_Q(x, y)$ is the objective function mapping a pair $x \in I_Q$ and $y \in S_Q(x)$ to a non-negative integer. The function f_Q is computable in polynomial time. The function g_Q is the *goal* function, which is one of the two functions $\{\max, \min\}$, and Q is called a *maximization problem* if $g_Q = \max$, or a *minimization problem* if $g_Q = \min$. We will denote by $opt_Q(x)$ the value $g_Q\{f_Q(x, z) \mid z \in S_Q(x)\}$, and if there is no confusion about the underlying problem Q , we will write $opt(x)$ to denote $opt_Q(x)$.

Polynomial Time Approximation Algorithms. A polynomial time algorithm A is an *approximation algorithm* for a (maximization) problem Q if for each input instance $x \in I_Q$ the algorithm A returns a feasible solution $y_A(x) \in S_Q(x)$. The solution $y_A(x)$ has an *approximation factor* $r(|x|)$ if it satisfies the following condition:

$$\text{opt}_Q(x)/f_Q(x, y_A(x)) \leq r(|x|).$$

The approximation algorithm A has an *approximation factor* $r(|x|)$ if for every instance x in I_Q the solution $y_A(x)$ constructed by the algorithm A has an approximation factor bounded by $r(|x|)$.

An optimization problem Q has a *constant-factor approximation algorithm* if it has an approximation algorithm whose approximation factor is a constant.

Given two maximization problems $Q = (I_Q, S_Q, f_Q, g_Q)$ and $Q' = (I_{Q'}, S_{Q'}, f_{Q'}, g_{Q'})$, we say that Q *L-reduces* to Q' [23] if there are two polynomial-time computable functions/algorithms h, h' and constants α and β such that:

- (i) For all $x \in I_Q$, h produces an instance $x' = h(x)$ of $I_{Q'}$, such that $\text{opt}_{Q'}(x') \leq \alpha \cdot \text{opt}_Q(x)$, and
- (ii) for any solution $y' \in S_{Q'}(x')$, h' produces a solution $y \in S_Q(x)$ such that $|f_Q(y) - \text{opt}_Q(x)| \leq \beta \cdot |f_{Q'}(y') - \text{opt}_{Q'}(x')|$.

Parameterized Complexity. A *parameterized problem* is a set of instances of the form (x, k) , where $x \in \Sigma^*$ for a finite alphabet set Σ , and k is a non-negative integer called the *parameter*. A parameterized problem Q is *fixed parameter tractable (FPT)*, if there exists an algorithm that on input (x, k) decides if (x, k) is a yes-instance of Q in time $f(k)|x|^{O(1)}$, where f is a computable function independent of $|x|$; we will denote by *fpt-time* a running time of the form $f(k)|x|^{O(1)}$. A hierarchy of fixed-parameter intractability, *the W-hierarchy* $\bigcup_{t \geq 0} W[t]$, was introduced based on the notion of *fpt-reduction*, in which the 0-th level $W[0]$ is the class FPT . It is commonly believed that $W[1] \neq \text{FPT}$. The asymptotic notation $O^*(\cdot)$ suppresses a polynomial factor in the input length. For more information about parameterized complexity, we refer the reader to [7, 13, 22].

Randomized Algorithms. A *randomized algorithm* is an algorithm that relies on random data as part of its input. Such algorithms can be *de-randomized* using techniques to enumerate values of the randomized input, yielding a deterministic algorithm. In this paper we design randomized algorithms using the color coding technique introduced by Alon et al. [2] and the random separation technique introduced by Cai et al. [4]. Both techniques are applicable to problems where the objective is to find a subset of size k satisfying certain properties in a universe of size n .

Given a problem where we seek a subset of size k satisfying certain properties in a universe of n elements, color coding works by randomly assigning k colors to the set of n elements and then looking for a subset of size k that is *colorful*, that is, where each element of the subset has a distinct color. The algorithm can be de-randomized using families of k -perfect hash functions, or colorings, constructed as shown in [2] in time $\mathcal{O}(2^{\mathcal{O}(k)} \log(n))$. Thus, color coding algorithms run in time $\mathcal{O}(2^{\mathcal{O}(k)} \log(n) f(n, k))$ where $f(n, k)$ is the running time to verify that a colorful subset of size k satisfying the desired properties exists for a given coloring. If f is fpt-time (k is the parameter and n is the input size) then the de-randomized algorithm runs in fpt-time.

Random separation [4] works by randomly partitioning the vertices of a graph G into red and black vertices and testing whether a solution $S \subseteq V(G)$ exists such that S is entirely contained in the black partition and $N(S)$ is entirely contained in the red partition. The likelihood that a solution S is colored black and $N(S)$ is colored red is $2^{-(k+|N(S)|)}$. To derandomize the algorithm we use (n, t) -universal sets where $t = k + |N(S)|$. Naor et al. [21] give a construction of (n, t) -universal sets

that runs in time $\mathcal{O}(2^t t^{\mathcal{O}(\log(t))} \log(n))$. Thus, the running time for the de-randomized algorithm is $\mathcal{O}(2^t t^{\mathcal{O}(\log(t))} \log(n) f(n, k))$, where $f(n, k)$ is the time it takes to verify that a solution exists in the black partition and its neighbors are in the red partition. If f is fpt-time and t is bounded by a function of k , then the algorithm runs in fpt-time.

We will denote an instance of p-OLSE or p-OLISE by the tuple $(G, H, \prec_G, \prec_H, L, k)$. We shall call an injective map φ satisfying conditions (1)-(3) in the definition of p-OLSE and p-OLISE (given in Section 1) for some subset $S \subseteq V(G)$, a *valid list embedding*, or simply a *valid embedding*. Constraint (3) will be referred to as the *embedding constraint* (note that constraint (3) is different in the two problems). We define the *width* of L , denoted Δ_L , as $\max\{|L(u)| : u \in G\}$. It is often more convenient to view/represent the map L as a set of edges joining every vertex $u \in G$ to the vertices of H that are in $L(u)$.

2 Classical Complexity Results

In this section we consider p-OLSE and p-OLISE, and their optimization versions, with respect to classical complexity. Consider the restrictions of the opt-OLSE and opt-OLISE problems to instances in which $\Delta_G = \Delta_H = 0$ (for p-OLSE we can even assume that $\Delta_H = \infty$ as the edges in H do not play any role when $\Delta_G = 0$). This version of the problem can be easily shown to be solvable in polynomial time by dynamic programming. The algorithm is very similar to that of the LONGEST COMMON SUBSEQUENCE problem; so we only give the recursive definition of the solution necessary for the dynamic programming approach.

Proposition 2.1. *The opt-OLSE problem restricted to instances in which $\Delta_G = 0$ and $\Delta_H = \infty$, and the opt-OLISE problem restricted to instances in which $\Delta_G = \Delta_H = 0$ are solvable in $O(|V(G)| \cdot |V(H)|)$ time (and hence are in \mathcal{P}).*

Proof. Assume that the vertices of G are ordered as u_1, \dots, u_n with respect to \prec_G , and that the vertices of H are ordered as v_1, \dots, v_N with respect to \prec_H . We maintain a two-dimensional table T , in which $T[i, j]$ is the maximum cardinality of a subset of $\{u_1, \dots, u_i\}$ from which there exists a valid list embedding into $\{v_1, \dots, v_j\}$. The entry $T[i, j]$ can be computed recursively as follows: $T[i, j] = 1 + T[i-1, j-1]$ if $v_j \in L(u_i)$, and $T[i, j] = \max\{T[i, j-1], T[i-1, j]\}$ otherwise. We can now use a standard dynamic programming approach to compute T and to construct the subset of vertices S and the embedding φ . The running time of the algorithm is $O(|V(G)| \cdot |V(H)|)$. (Note that deciding if a vertex $v_j \in H$ is in $L(u_i)$ for a vertex $u_i \in G$ can be done in $O(1)$ time after preprocessing the instance in $O(|V(G)| \cdot |V(H)|)$ time: We can compute a two-dimensional Boolean matrix M , one dimension of which corresponds to the vertex-set of G and the other dimension to that of H , such that $M[i, j] = 1$ if and only if $v_j \in L(u_i)$.) \square

Proposition 2.1 will be useful for Proposition 3.1 and Theorem 5.2.

As we show next, if $\Delta_G > 0$, the p-OLSE and p-OLISE problems become \mathcal{NP} -complete, even in the simplest case when $\Delta_G = 1$, $\Delta_H = 0$ and $\Delta_L = 1$. The same proof shows the \mathcal{NP} -completeness of p-OLISE when $\Delta_H = 1$, $\Delta_G = 0$ and $\Delta_L = 1$ by symmetry.

Theorem 2.2. *The p-OLSE and p-OLISE problems restricted to instances in which $\Delta_H = 0$, $\Delta_G = 1$, and $\Delta_L = 1$ are \mathcal{NP} -complete.*

Proof. We first present the proof for p-OLSE.

It is easy to see that p-OLSE $\in \mathcal{NP}$, so it suffices to show that p-OLSE is \mathcal{NP} -hard. We do so by providing a polynomial time reduction from the k -MULTI-COLORED INDEPENDENT SET (k -MCIS) problem: decide whether for a given graph $M = (V(M), E(M))$ and a proper k -coloring of

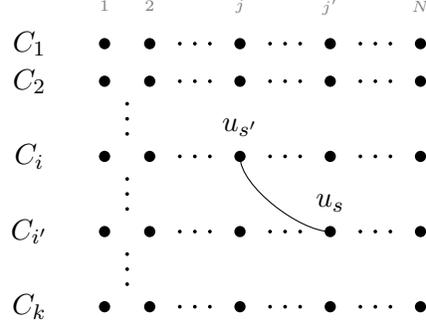


Figure 1: An instance of k -MCIS consists of kN vertices partitioned into k color classes C_1, \dots, C_k each of size N . All edges are between vertices in different color classes. The edge $(u_{s'}, u_s)$, for example, connects vertices $u_{s'}$ and u_s where $s' = (i-1)N + j$ and $s = (i'-1)N + j'$.

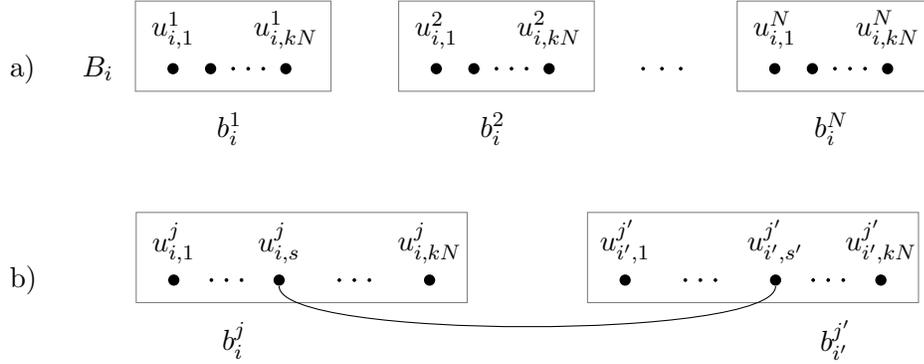


Figure 2: a) Associated with every color class C_i is a block B_i of vertex sequences b_i^1, \dots, b_i^N such that each b_i^j is a sequence of kN vertices $u_{i,1}^j, \dots, u_{i,kN}^j$. The vertex sequences b_i^1, \dots, b_i^N correspond to the N vertices of C_i . b) $(u_{i,s}^j, u_{i',s'}^{j'})$ is an edge in $E(G)$ if and only if $s = (i'-1)N + j'$, $s' = (i-1)N + j$, and $(u_s, u_{s'}) \in E(M)$.

the vertices $f : V(M) \rightarrow C$, where $C = \{1, 2, \dots, k\}$ and each color class has the same cardinality, there exists an independent set $I \subseteq V(M)$ of size k such that, $\forall u, v \in I$, $f(u) \neq f(v)$ [17].

Let (M, f) be an instance of k -MCIS. We assume that the set C_i , $i = 1, \dots, k$, of vertices that are mapped to color $i \in C$ is of size N , and we label the vertices of M u_1, \dots, u_{kN} such that $C_i = \{u_{(i-1)N+1}, \dots, u_{iN}\}$ (see Figure 1). We describe how we construct the corresponding instance $(G, H, \prec_G, \prec_H, L, k)$ of p-OLSE.

To construct $G = (V(G), E(G))$, we associate with every color class C_i a block B_i of vertex sequences b_i^1, \dots, b_i^N such that each b_i^j , $j = 1, \dots, N$, is a sequence of kN vertices $u_{i,1}^j, \dots, u_{i,kN}^j$. $V(G)$ is then the set of all vertices $u_{i,s}^j$ for $i = 1, \dots, k$, $j = 1, \dots, N$, and $s = 1, \dots, kN$. The ordering \prec_G of vertices in G is defined as follows: for all $u_{i,s}^j, u_{i',s'}^{j'} \in G$

$$u_{i,s}^j \prec_G u_{i',s'}^{j'} \iff i < i' \text{ or } (i = i' \text{ and } j < j'), \text{ or } (i = i' \text{ and } j = j' \text{ and } s < s').$$

(See Figure 2-(a).) Lastly we define the set $E(G)$ as follows: $(u_{i,s}^j, u_{i',s'}^{j'})$ is an edge in $E(G)$ if and only if $i < i'$, $s = (i'-1)N + j'$, $s' = (i-1)N + j$, and $(u_s, u_{s'}) \in E(M)$. (See Figure 2-(b).)

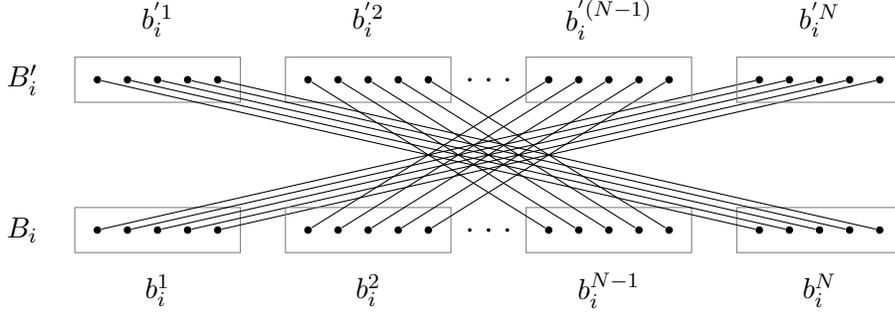


Figure 3: The mapping L maps vertices of B_i to vertices of B'_i . More precisely, it maps vertices of b_i^j to vertices of b'_i^{N-j+1} in a way that forbids an embedding that would map a vertex in b_i^j and a vertex in $b_i^{j'}$ simultaneously, for any $i = 1, \dots, k$, $j, j' = 1, \dots, N$, $j \neq j'$.

To construct $H = (V(H), E(H))$, we again associate with every color class C_i a block B'_i of vertex sequences b'_i^1, \dots, b'_i^N such that each b'_i^j is a sequence of kN vertices $v_{i,1}^j, \dots, v_{i,kN}^j$. $V(H)$ is the set of all vertices $v_{i,r}^j$ for $i = 1, \dots, k$, $j = 1, \dots, N$, and $r = 1, \dots, kN$. The ordering \prec_H of vertices in H is defined just as it was in G . The edge set $E(H)$ is simply \emptyset : in this way, for any valid list embedding $\varphi : S \rightarrow V(H)$, S must be an independent set of G .

We complete the construction of the instance of p-OLSE by defining the mapping $L : G \rightarrow 2^{V(H)}$. L is a bijection that maps the vertices in each block $B_i \in G$ to the block $B'_i \in H$ as follows:

$$\forall u_{i,s}^j \in b_i^j, L(u_{i,s}^j) = \{v_{i,s}^{(N-j+1)}\}.$$

This mapping is illustrated in Figure 3.

This completes the construction of the instance of p-OLSE. Observe that in the constructed instance we have $\Delta_H = 0$, $\Delta_G = 1$, and $\Delta_L = 1$. We show next that M has an independent set of size k if and only if the constructed instance of p-OLSE has a subset $S \subseteq V(G)$ of size k^2N , and a valid embedding $\varphi : S \rightarrow V(H)$.

Let I be solution to the instance (M, f) of k -MCIS. Each vertex $u_s \in I$ corresponds to a sequence $b_i^j \in B_i$ where $s = (i-1)N + j$. For any two sequences $b_i^j \in B_i, b_{i'}^{j'} \in B_{i'}$, there is an edge between a vertex of b_i^j and a vertex of $b_{i'}^{j'}$ if and only if $(u_s, u_{s'}) \in E(M)$ for $s' = (i'-1)N + j'$ and $s = (i-1)N + j$ ($i < i'$). Since for every pair $u_s, u_{s'} \in I, (u_s, u_{s'}) \notin E(M)$ and since $f(u_s) \neq f(u_{s'})$, there are k sequences b_i^j with the property that for any two vertices $u_{i,s}^j \in b_i^j$ and $u_{i',s'}^{j'} \in b_{i'}^{j'}$, $(u_{i,s}^j, u_{i',s'}^{j'}) \notin E(G)$. By construction of G , the vertices of any two sequences $b_i^j, b_{i'}^{j'}$ ($i \neq i'$) that do not have edges between them are simultaneously embeddable in H . Therefore, we can define a valid embedding $\varphi : S \rightarrow V(H)$, where $|S| = k \cdot (kN) = k^2N$, and S consists of all the vertices in each b_i^j that corresponds to a vertex in I .

Conversely, let $\varphi : S \rightarrow V(H)$, $|S| = k^2N$, be a valid embedding from a subset $S \subseteq V(G)$ into H . By construction of L , φ can embed at most kN vertices from any block B_i into H and, furthermore, all vertices in B_i embedded by φ must belong to the same sequence b_i^j of B_i . Therefore, exactly k sequences, one from each of the k blocks, are fully embedded by φ . Moreover, because $\Delta_H = 0$, φ embeds all kN vertices of the k sequences b_i^j in S if and only if for any two vertices $u_{i,s}^j, u_{i',s'}^{j'} \in S, (u_{i,s}^j, u_{i',s'}^{j'}) \notin E(G)$. By construction of G , each sequence b_i^j corresponds to the j^{th} vertex of the color class C_i . Therefore, we can find a subset $I \subseteq M, |I| = k$ such that for any two vertices $u_s, u_{s'} \in I, f(u_s) \neq f(u_{s'})$ and $(u_s, u_{s'}) \notin E(M)$; that is, the set I is a solution to the instance (M, f) of k -MCIS.

For p-OLISE, since $\Delta_H = 0$, the same reduction given above can be used to show its \mathcal{NP} -hardness when $\Delta_H = 0$, $\Delta_G = 1$ and $\Delta_L = 1$. To show that p-OLISE is \mathcal{NP} -hard when $\Delta_H = 1$, $\Delta_G = 0$ and $\Delta_L = 1$ we need only tweak the construction of the p-OLISE instance as follows. We construct H as we constructed G in the previous case (with edges), and we construct G as we constructed H in the previous case (without edges). The \mathcal{NP} -hardness when $\Delta_H = 1$, $\Delta_G = 0$ and $\Delta_L = 1$ follows by symmetry. \square

We contrast this hardness result with a result for a version of the problem where the injective map need not preserve the ordering of the vertices. First, we define the parameterized LIST SUBGRAPH EMBEDDING problem (p-LSE):

Given: Two graphs G and H ; a function $L : V(G) \rightarrow 2^{V(H)}$; and $k \in \mathbb{N}$

Parameter: k

Question: Is there a subset $S \subseteq V(G)$ of k vertices and an injective map $\varphi : S \rightarrow V(H)$ such that: (1) $\varphi(u) \in L(u)$ for every $u \in S$ and (2) for every $u, u' \in S$, if $uu' \in E(G)$ then $\varphi(u)\varphi(u') \in E(H)$

Proposition 2.3. *The p-LSE problem restricted to instances where $\Delta_G = 1$, $\Delta_H = 0$, and $\Delta_L = 1$ is in \mathcal{P} .*

Proof. It suffices to prove that the optimization version opt-LSE of p-LSE is in \mathcal{P} .

For each vertex $v \in H$ define $L^{-1}(v) = \{u \in G : L(u) = v\}$. Note that for each $v \in H$, $L^{-1}(v)$ defines a subset of vertices in $V(G)$ from which only one vertex can be included in the solution. Let $\mathcal{S} = \{S_i : i = 1, \dots, |V(H)|\}$ be the set of subsets of $V(G)$ where $S_i = L^{-1}(v_i), v_i \in V(H)$. Since $\Delta_L = 1$, the elements of \mathcal{S} are pairwise disjoint. We say that two subsets S_i and S_j , $i \neq j$, are *adjacent* if there exists an edge $uu' \in E(G)$ where $u \in S_i$ and $u' \in S_j$. We define a *cycle* of subsets to be a sequence of distinct subsets $C = \langle S_1, \dots, S_l \rangle$, $l > 2$, such that S_i and S_{i+1} are adjacent, for $i = 1, \dots, l-1$, and S_l and S_1 are adjacent; or a sequence of two subsets $C = \langle S_i, S_j \rangle$ such that there are two distinct vertices u_i, u'_i in S_i and two distinct vertices u_j, u'_j in S_j such that u_i is adjacent to u_j and u'_i is adjacent to u'_j . Using these constructs, we can define the following rules to simplify the problem:

- (i) For each subset $S_i \in \mathcal{S}$ with a vertex u such that $|N_G(u)| = 0$, or u has a neighbor in S_i , we add u to the solution S and we define $\varphi(u) = v_i$. We remove the vertices in S_i from G and vertex v_i from H .
- (ii) For each cycle of subsets $C = \langle S_1, \dots, S_l \rangle$, if $l > 2$, choose two vertices u_i, u'_i in each S_i , $i = 1, \dots, l$ such that u'_i is adjacent to u_{i+1} , for $i = 1, \dots, l-1$, and u'_l is adjacent to u_1 . Add u_i to the solution S , define $\varphi(u_i) = v_i$, and remove the vertices in S_i from G and v_i from H , for $i = 1, \dots, l$. If $l = 2$, let $u_1, u'_1 \in S_1$ and $u_2, u'_2 \in S_2$ such that u_1 is adjacent to u_2 and u'_1 is adjacent to u'_2 . Add u_1, u'_2 to the solution S , define $\varphi(u_1) = v_1$, $\varphi(u'_2) = v_2$, and remove the vertices in $S_1 \cup S_2$ from G and v_1, v_2 from H .

Clearly, each of the above rules is safe in the sense that it is possible to extend the solution obtained after each application of the rule to an optimal solution. After the above rules are no longer applicable, we are left with a graph G (we call the resulting graph G for simplicity) where the remaining subsets in \mathcal{S} (considered as nodes), together with their adjacency relation, form a tree-like structure. We say that a subset $S_i \in \mathcal{S}$ is a *leaf-subset* (in \mathcal{S}) if it is adjacent to exactly one other subset in \mathcal{S} ; observe that every subset S_i must be adjacent to at least one other subset in \mathcal{S} by rule (i). Since each leaf-subset is adjacent to one subset in \mathcal{S} , each leaf-subset must consist

of exactly one vertex for the following reasons. First, by rule (i), each vertex in a leaf-subset S_i must have a neighbor in the (single) subset S_j that is adjacent to S_i . By rule (ii), S_i must consist of exactly one vertex; otherwise, rule (ii) would apply to S_i and S_j . To complete the construction of our solution, we iteratively apply a new rule, rule (iii), that includes the vertex $u_i \in S_i$ in our solution S , for each leaf-subset S_i , defines $\varphi(u_i) = v_i$, and removes the vertices in S_i from G and v_i from H . Rule (iii) is safe because any optimal solution must contain exactly one of u_i and its only neighbor w in the subset adjacent to S_i , and in the case when the solution contains w , w can be exchanged with u_i , and the map φ can be updated by defining $\varphi(u_i) = v_i$. When rule (iii) is no longer applicable, the graph G is empty, and the algorithm has computed an optimal solution. Clearly, the running time is polynomial because each application of a rule removes at least one vertex from G (and from H). \square

3 Approximation Results

In this section we consider opt-OLSE and opt-OLISE with respect to the framework of approximation theory. We begin with the following proposition:

Proposition 3.1. *The opt-OLSE problem restricted to instances in which $\Delta_G = O(1)$ has an approximation algorithm of approximation factor $(\Delta_G + 1)$, and the opt-OLISE problem restricted to instances in which $\Delta_G = O(1)$ and $\Delta_H = O(1)$ has an approximation algorithm of factor $(\Delta_H + 1) \cdot (\Delta_G + 1)$.*

Proof. Let $(G, H, \prec_G, \prec_H, L)$ be an instance of opt-OLSE, and consider the following algorithm. Apply the dynamic programming algorithm in Proposition 2.1 to $(G, H, \prec_G, \prec_H, L)$ after removing the edges of G and the edges of H , and let S and φ be the set of vertices and map obtained, respectively. Apply the following trivial approximation algorithm to compute an independent set I of S : pick a vertex u in S , include u in I , remove u and $N(u)$ from S , and repeat until S is empty. Return the set I , and the restriction of φ to I , φ_I . Clearly, we have $|I| \geq |S|/(\Delta_G + 1)$.

Since φ is a valid list embedding of S after the edges of G and H have been removed, and since I is an independent set of $G[S]$, it is clear that φ_I is a valid list embedding of I into H . Therefore, the algorithm is an approximation algorithm. Now let S_{opt} be an optimal solution of the instance. $|S|$ is an upper bound on the size of an optimal solution S_{opt} since S_{opt} is a solution to opt-OLSE that respects the embedding constraint. Therefore, $|S_{opt}|/|I| \leq |S|/|I| \leq (\Delta_G + 1)$.

For opt-OLISE, we apply the algorithm above to obtain I , however, there may be vertices $u, v \in I$ where $\varphi_I(u)\varphi_I(v) \in E(H)$, violating the embedding constraint of opt-OLISE. We use a similar method to that used to obtain the approximation algorithm described above to overcome this hurdle. This time we pick a vertex v in the image $\varphi(I)$ of the independent set I , include v in the independent set $I' \subseteq V(H)$, remove v and $N(v)$ from $\varphi(I)$, and repeat until $\varphi(I)$ is empty. As was the case for I , it follows that $|I'| \geq |\varphi(I)|/(\Delta_H + 1)$. Let $I'' \subseteq V(G)$ be the inverse image $\varphi_I^{-1}(I')$. Since $|I| \geq |S|/(\Delta_G + 1)$ and $|I'| \geq |\varphi(I)|/(\Delta_H + 1)$, it follows that $|I''| \geq |S|/((\Delta_G + 1)(\Delta_H + 1))$. $|S|$ remains an upper bound on the size of an optimal solution S_{opt} . Therefore, $|S_{opt}|/|I''| \leq |S|/|I''| \leq (\Delta_G + 1)(\Delta_H + 1)$. \square

The inapproximability results outlined in Table 1 follow by simple reductions from the MAXIMUM INDEPENDENT SET problem. We introduce the following reduction modeled after a reduction in [3] for a variation of opt-OLSE.

Lemma 3.2. *Let G be a graph. Construct an instance $I_G = (G, H, \prec_G, \prec_H, L)$ of opt-OLSE from G as follows. Let \prec_G be an arbitrary ordering on the vertices of G , u_1, \dots, u_n where $u_1 \prec_G u_2 \cdots \prec_G$*

u_n . Let H be the graph consisting of the isolated vertices v_1, \dots, v_n , where $v_1 \prec_H v_2 \cdots \prec_H v_n$. Define the mapping $L : V(G) \rightarrow 2^{V(H)}$ such that for each vertex $u_i \in V(G)$, $L(u_i) = \{v_i\}$. We have:

- (i) G has an independent set of size k if and only if I_G has a solution of size k .
- (ii) There is an fpt-reduction from INDEPENDENT SET to p -OLSE where $\Delta_G = \infty, \Delta_H = 0$ and $\Delta_L = 1$, and to p -OLISE where $\Delta_G = \infty, \Delta_H = 0$ (resp. $\Delta_G = 0, \Delta_H = \infty$ by symmetry) and $\Delta_L = 1$.
- (iii) There is an L -reduction from MAXIMUM INDEPENDENT SET to opt -OLSE where $\Delta_G = \infty, \Delta_H = 0$ and $\Delta_L = 1$, and to opt -OLISE where $\Delta_G = \infty, \Delta_H = 0$ (resp. $\Delta_G = 0, \Delta_H = \infty$ by symmetry) and $\Delta_L = 1$.

Proof. (i) Let $I \subseteq V(G)$ be an independent set in G where $|I| = k$. Since $\Delta_H = 0$, for any subset $S \subseteq V(G)$ and a valid embedding $\varphi : S \rightarrow V(H)$, no two vertices $u, u' \in S$ are adjacent. Given that I is an independent set and that for each vertex $u_i \in V(G)$, $L(u_i) = \{v_i\}$, letting $S = I$, we can define the embedding $\varphi : S \rightarrow V(H)$, where $\varphi(u_i) = v_i$ for $u_i \in S$, satisfying that for any two vertices $u_i, u_j \in S$ if $u_i \prec_G u_j$ then $\varphi(u_i) \prec_H \varphi(u_j)$. Thus, there is a valid embedding from S to $V(H)$ where $|S| = |I| = k$.

Conversely, let $\varphi : S \rightarrow V(H)$ be an embedding such that $|S| = k$. As observed earlier, S must be an independent set since $\Delta_H = 0$. Thus, G has an independent set I where $|I| = |S| = k$.

(ii) Let (G, k) be an instance of INDEPENDENT SET. We construct the instance $I_G = (G, H, \prec_G, \prec_H, L, k)$ by following the same procedure outlined in the statement of the lemma. By part (i), G contains an independent set I where $|I| = k$ if and only if there exists a valid embedding $\varphi : S \rightarrow V(H)$ where $S = I$. Since the same parameter is shared between the instance of INDEPENDENT SET and the instance I_G of p -OLSE, and since the construction of I_G can be clearly done in polynomial time, we have an fpt-reduction from INDEPENDENT SET to p -OLSE. The reduction extends immediately to the case of p -OLISE where $\Delta_G = \infty, \Delta_H = 0$ and $\Delta_L = 1$ because $\Delta_H = 0$. By symmetry, it also can be used for the case of p -OLISE where $\Delta_G = 0, \Delta_H = \infty$ and $\Delta_L = 1$.

(iii) The L -reduction is given by the function h that maps the instance G of MAXIMUM INDEPENDENT SET into the instance I_G of opt -OLSE, as described in the statement of the lemma, and the function h' that, for each solution S, φ of I_G corresponds the solution $I = S$ of G , and the constants $\alpha = \beta = 1$. By part (i), any solution I of G corresponds to a solution S, φ of I_G such that $|I| = |S|$, and vice versa. Therefore, $opt(G) = opt(I_G)$, and $|I| - opt(G) = |S| - opt(I_G)$. This shows that MAXIMUM INDEPENDENT SET L -reduces to opt -OLSE.

The L -reduction extends immediately to the case of opt -OLISE where $\Delta_G = \infty, \Delta_H = 0$ and $\Delta_L = 1$, and by symmetry it extends to the case of opt -OLISE where $\Delta_G = 0, \Delta_H = \infty$ and $\Delta_L = 1$. \square

We can now demonstrate that opt -OLSE is APX-complete when $\Delta_G = O(1)$ and opt -OLISE is APX-complete when $\Delta_G = O(1)$ and $\Delta_H = O(1)$.

Proposition 3.3. *The opt -OLSE problem restricted to instances in which $\Delta_G = O(1)$ and the opt -OLISE problem restricted to instances in which $\Delta_G = O(1)$ and $\Delta_H = O(1)$ are APX-complete.*

Proof. Proposition 3.1 showed that opt -OLSE is in APX when $\Delta_G = O(1)$. We demonstrate that opt -OLSE is APX-complete when $\Delta_G = O(1), \Delta_H = 0$ and $\Delta_L = 1$ to yield the result. Since the case of opt -OLSE where $\Delta_G = O(1), \Delta_H = 0$ and $\Delta_L = 1$ is a subset of opt -OLSE where $\Delta_G = \infty$,

$\Delta_H = 0$ and $\Delta_L = 1$ we can apply Lemma 3.2 part (iii) to L -reduce MAXIMUM INDEPENDENT SET to the cases of opt-OLSE where G has bounded degree. Since MAXIMUM INDEPENDENT SET on bounded degree graphs is APX-complete, it follows that opt-OLSE is APX-complete when $\Delta_G = O(1)$ [23]. Similar arguments can be applied to opt-OLISE to show that it is APX-complete when restricted to instances in which $\Delta_G = O(1)$ and $\Delta_H = O(1)$. \square

The inapproximability results for opt-OLSE and opt-OLISE when $\Delta_G = \infty$, $\Delta_L = 1$ and $\Delta_H = 0$ (and the symmetric case for opt-OLISE when $\Delta_G = 0$, $\Delta_L = 1$ and $\Delta_H = \infty$) also follow from Lemma 3.2.

Proposition 3.4. *The opt-OLSE problem restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ and $\Delta_L = 1$ and opt-OLISE restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ and $\Delta_L = 1$ or $\Delta_G = 0$, $\Delta_H = \infty$ and $\Delta_L = 1$ cannot be approximated within a factor of $n^{1-\varepsilon}$ unless $\mathcal{P} = \mathcal{NP}$, where n is the number of vertices in G .*

Proof. Zuckerman showed in [26] that MAXIMUM INDEPENDENT SET is not approximable within a factor of $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $\mathcal{P} = \mathcal{NP}$. The proposition follows immediately from the L -reduction in part (iii) of Lemma 3.2 for the given cases of opt-OLSE and opt-OLISE. \square

4 Parameterized Complexity Results

In this section we consider p-OLSE and p-OLISE with respect to the framework of parameterized complexity. First, it is easy to establish the membership of p-OLSE and p-OLISE in $W[1]$ via an fpt-reduction to INDEPENDENT SET, which is known to be in $W[1]$ (e.g., see [6]). The fpt-reduction associates with an instance $(G, H, \prec_G, \prec_H, L, k)$ of p-OLSE (resp. p-OLISE) an instance (Z, k) of INDEPENDENT SET constructed as follows. For each vertex $u \in G$ and $v \in L(u)$, associate a vertex $z \in Z$; choosing z to be in the independent set corresponds to mapping u to v . For two distinct vertices $z, z' \in V(Z)$, where z represents vertices $u \in G$ and $v \in L(u)$, and z' represents $u' \in G$ and $v' \in L(u')$, z is adjacent to z' in Z if and only if $u = u'$ or $v = v'$, or simultaneously mapping u to v and u' to v' violates either the ordering constraint or the embedding constraint (or both) in the instance $(G, H, \prec_G, \prec_H, L, k)$. Clearly, the above reduction is an fpt-reduction from p-OLSE (resp. p-OLISE) to INDEPENDENT SET.

We start by applying part (ii) of Lemma 3.2 to p-OLSE and p-OLISE to obtain the following result:

Proposition 4.1. *The p-OLSE problem restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ and $\Delta_L = 1$ is $W[1]$ -complete, and the p-OLISE problem restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ (resp. $\Delta_G = 0$ and $\Delta_H = \infty$ by symmetry) and $\Delta_L = 1$ is $W[1]$ -complete.*

Proof. Downey and Fellows showed in [6] that INDEPENDENT SET is $W[1]$ -complete. By part (ii) of Lemma 3.2 there is an fpt-reduction from INDEPENDENT SET to p-OLSE restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ and $\Delta_L = 1$, and to p-OLISE restricted to instances in which $\Delta_G = \infty$, $\Delta_H = 0$ (resp. $\Delta_H = \infty$ and $\Delta_G = 0$ by symmetry) and $\Delta_L = 1$. The statement of the proposition follows. \square

Next, we consider the case of p-OLSE and p-OLISE when $\Delta_G = 1$, $\Delta_H = 1$ and $\Delta_L = \infty$. Evans [9] reduces the $W[1]$ -complete problem CLIQUE [7] to the LONGEST ARC-PRESERVING COMMON SUBSEQUENCE (LAPCS) to show that LAPCS is $W[1]$ -hard when the length of the common subsequence, l , is the parameter, and when no two arcs in the instance share an endpoint. The reduction encodes a graph H as an arc-annotated sequence (S_1, P_1) where each vertex in $V(H)$

is a substring of S_1 of the form (ba^nb) and where the edges in H are encoded as arcs between the substrings of S_1 with endpoints corresponding to the endpoints of the edge in H . A clique of size k is encoded as an arc-annotated sequence (S_2, P_2) where each of the k vertices in the clique is encoded as a substring of S_2 of the form (ba^kb) with similarly constructed arcs between substrings. A clique of size k exists in H if (S_2, P_2) is an arc-preserving subsequence of (S_1, P_1) with length $l = k(k + 2)$. Since the length of S_2, l , is a function of k , CLIQUE is fpt-reducible to LAPCS.

We show that the $W[1]$ -hard restriction of LAPCS to instances in which no two arcs share an endpoint, is a special case of p-OLISE where $\Delta_G = 1, \Delta_H = 1$ and $\Delta_L = \infty$. Taking each character in S_1 and S_2 to be a vertex in $V(H)$ and $V(G)$, respectively, and preserving the arcs in P_1 and P_2 as edges in $E(H)$ and $E(G)$, respectively, (S_1, P_1) and (S_2, P_2) are reduced to the graphs H and G , respectively, and $|V(G)| = k(k + 2)$. Each vertex $u \in V(G)$ has a list in $V(H)$ that consists of the vertices $v \in V(H)$ such that the character for $u \in S_2$ is the same as the character for $v \in S_1$. The size of the parameter l remains the same, completing the reduction from LAPCS in which no two arcs share an endpoint to p-OLISE in which $\Delta_G = 1$ and $\Delta_H = 1$. Note that a clique of size k is encoded by the property that the embedding from G to H preserves the arcs in G , which is a property of solutions for both p-OLSE and p-OLISE. Thus, by the same arguments, LAPCS reduces to p-OLSE as well. Finally, note that this reduction results in instances of p-OLSE and p-OLISE where the question is whether we can embed the whole graph G into H , yielding the following proposition:

Proposition 4.2. *The p-OLSE and p-OLISE problems restricted to instances in which $\Delta_H = 1$ and $\Delta_G = 1$ are $W[1]$ -complete. Moreover, the problems remain $W[1]$ -complete when the parameter k is the number of vertices in $V(G)$.*

Proposition 4.3. *The p-OLISE problem restricted to instances where $\Delta_H = 1, \Delta_G = 1$, and $\Delta_L = 1$ is $W[1]$ -complete under Turing fpt-reductions.*

Proof. Let \mathcal{I} be the set of instances of p-OLISE where $\Delta_H = 1, \Delta_G = 1$, and $\Delta_L = 1$. For an instance $I = (G, H, \prec_G, \prec_H, L, k) \in \mathcal{I}$ define $I^{-1} = (H, G, \prec_H, \prec_G, L^{-1}, k)$ where, for each $v \in V(H), L^{-1}(v) = \{u : L(u) = \{v\}\}$. As a consequence of $\Delta_L = 1$, for every instance I^{-1} , and for every vertex $u \in G$, we have $|\{v \in V(H) : L^{-1}(v) \supseteq \{u\}\}| = 1$. Let \mathcal{I}^{-1} be the set of all I^{-1} . It is easy to see that an instance of p-OLISE I is a yes-instance if and only if I^{-1} is a yes-instance. Therefore, to prove the proposition we can equivalently prove that p-OLISE restricted to instances in \mathcal{I}^{-1} is $W[1]$ -complete. Denote by p-OLISE-simple p-OLISE restricted to instances in which $\Delta_H = 1, \Delta_G = 1, \Delta_L = \infty$, and $|V(G)| = k$. Proposition 4.2 showed that p-OLISE-simple is $W[1]$ -complete. We give a Turing fpt-reduction from p-OLISE-simple to \mathcal{I}^{-1} to yield the result.

Let $I' = (G, H, \prec_G, \prec_H, L)$ be an instance of p-OLISE-simple where $V(G) = u_1, \dots, u_k$. Using color-coding, there is a family \mathcal{F} of fpt-many k -colorings for the vertices in $V(H)$ using the colors u_1, \dots, u_k — corresponding to the labels of the vertices of G — such that if I' is a yes-instance, then there is a coloring $c \in \mathcal{F}$, and a valid embedding $\varphi_c : V(G) \rightarrow V(H)$, such that if $\varphi_c(u_i) = v_i$ then $c(v_i) = u_i$. To reduce I' , for every coloring $c \in \mathcal{F}$ we create the instance I^c of p-OLISE where $I^c = (G, H, \prec_G, \prec_H, L^c, k)$, where for each $u \in V(G), L^c(u) = \{v \in H : c(v) = u\}$. Observe that by the definition of L^c , for each vertex $v \in V(H), |\{u : L^c(u) \supseteq \{v\}\}| = 1$. Thus, $I^c \in \mathcal{I}^{-1}$.

Hence there is a Turing fpt-reduction from p-OLISE-simple to \mathcal{I}^{-1} , and if there is an algorithm that runs in fpt-time that solves \mathcal{I}^{-1} , then we can find a solution for p-OLISE-simple in fpt-time by enumerating the colorings in \mathcal{F} and applying the algorithm for \mathcal{I}^{-1} . Given that p-OLISE-simple is $W[1]$ -complete, it follows that \mathcal{I}^{-1} is $W[1]$ -complete under Turing fpt-reductions. It follows that \mathcal{I} is $W[1]$ -complete as well under Turing fpt-reductions. \square

Again, relaxing the ordering constraint significantly reduces the difficulty of the problem. For example, when we relax the ordering constraint in p-OLISE we have the parameterized LIST INDUCED SUBGRAPH EMBEDDING problem, shortly (p-LISE):

Given: Two graphs G and H ; a function $L : V(G) \rightarrow 2^{V(H)}$; and $k \in \mathbb{N}$

Parameter: k

Question: Is there a subset $S \subseteq V(G)$ of k vertices and an injective map $\varphi : S \rightarrow V(H)$ such that: (1) $\varphi(u) \in L(u)$ for every $u \in S$ and (2) for every $u, u' \in S$, $uu' \in E(G)$ if and only if $\varphi(u)\varphi(u') \in E(H)$

Proposition 4.4. *The p-LISE problem restricted to instances in which $\Delta_G = 1$ and $\Delta_H = 1$ is in \mathcal{P} .*

Proof. It suffices to show that the optimization (maximization) version opt-LISE of p-LISE is in \mathcal{P} . We proceed by reducing opt-LISE in the case where $\Delta_G = 1$ and $\Delta_H = 1$ to the well-known polynomial-time computable problem WEIGHTED MAXIMUM MATCHING ON BIPARTITE GRAPHS (WMMBG): Given a graph $B(V, E)$, a bipartition $V(B) = (X, Y)$, and weight function $w : E(B) \rightarrow \mathbb{R}$, find a matching $M \subseteq E(B)$ with maximum weight, where the weight of a matching $w(M) = \sum_{e \in M} w(e)$.

Let (G, H, L, k) be an instance of opt-LISE. To create B , we first create the vertices of X from the edges and vertices of G . For each edge $e \in E(G)$ where $e = uu'$, we add a vertex to X . For each vertex $u \in G$ where $|N_G(u)| = 0$, we add a vertex to X . We follow the same procedure to create the vertices of Y from H . To complete B we add edges as follows:

- (i) For any two edges $e_1 = uu' \in E(G)$ and $e_2 = vv' \in E(H)$ such that $v \in L(u)$ and $v' \in L(u')$, we add an edge of weight 2 between the vertex in X corresponding to e_1 and the vertex in Y corresponding e_2 .
- (ii) For each vertex $u \in G$ such that $|N_G(u)| = 0$ and for each vertex $v \in L(u)$ such that $|N_H(v)| = 0$, we add an edge of weight 1 between the vertex corresponding to u in X and the vertex corresponding to v in Y .
- (iii) For each vertex $u \in G$ such that $|N_G(u)| = 0$, and for each vertex $v \in L(u)$ with neighbor $v' \in H$, we add an edge of weight 1 between the vertex corresponding to u in X and the vertex corresponding to the edge vv' in Y . If both v and v' appear in $L(u)$ we add *only* one edge between the vertex corresponding to u in X and that corresponding to vv' in Y so that no multi-edges are created in B .
- (iv) Finally, for each vertex $u \in G$ with neighbor u' in G , and for each vertex $v \in L(u)$ where $|N_H(v)| = 0$, we add an edge of weight 1 between the vertex in X corresponding to uu' and the vertex in Y corresponding to v . If v appears in both $L(u)$ and $L(u')$, we add *only* one edge between the vertex corresponding to uu' in X and that corresponding to v in Y so that no multi-edges are created in B .

This completes the construction of B . We show next that B has a matching M where $w(M) = |S|$ if and only if there is a subset $S \subseteq V(G)$ where $|S| = w(M)$ and there is a valid embedding $\varphi : S \rightarrow V(H)$. We begin by showing that for every matching $M \subseteq E(B)$ we can construct a subset $S \subseteq V(G)$ with a valid embedding $\varphi : S \rightarrow V(H)$ such $|S| = w(M)$. Given a matching M , for each edge $e \in M$ we add vertices to S and define $\varphi(u)$ for each vertex u added using the following rules:

- (i) Let $e = xy \in M$ be an edge of weight 2, where $x \in X$ corresponds to an edge $uu' \in G$ and $y \in Y$ corresponds to an edge $vv' \in H$. By the construction of B , one of the two vertices v, v' , say v , must be in $L(u)$ and the other $v' \in L(u')$. Add u and u' to S , define $\varphi(u) = v$, and define $\varphi(u') = v'$.
- (ii) Let $e = xy \in M$ be an edge of weight 1 where $x \in X$ corresponds to a vertex $u \in V(G)$, and $y \in Y$ corresponds to a vertex $v \in V(H)$. By the construction of B , $v \in L(u)$, and $|N_G(u)| = |N_H(v)| = 0$. Add u to S and define $\varphi(u) = v$.
- (iii) Let $e = xy \in M$ be an edge of weight 1, where $x \in X$ corresponds to a vertex $u \in V(G)$ and $y \in Y$ corresponds to an edge $vv' \in H$. By the construction of B , $v \in L(u)$ (without loss of generality). Add u to S and define $\varphi(u) = v$.
- (iv) Let $e = xy \in M$ be an edge of weight 1, where $x \in X$ corresponds to an edge $uu' \in G$, and $y \in Y$ corresponds to a vertex $v \in V(H)$. By the construction of B , $v \in L(u)$ (without loss of generality). Add u to S and define $\varphi(u) = v$.

The resulting subset S satisfies $|S| = w(M)$ because for each edge $e \in M$ where $w(e) = 2$, we add two vertices to S , and for each edge $e \in M$ where $w(e) = 1$ we add one vertex to S . Moreover, the resulting map $\varphi : S \rightarrow V(H)$ is a valid embedding. In particular, according to the above rules, for each pair of vertices $u, u' \in V(G)$ added to S , $uu' \in E(G)$ if and only if $\varphi(u)\varphi(u') \in E(H)$.

It remains to show that for a subset $S \subseteq V(G)$ with a valid embedding $\varphi : S \rightarrow V(H)$ there is a matching $M \subseteq E(B)$ such that $w(M) = |S|$. Similarly to how a $S \subseteq V(G)$ and embedding $\varphi : S \rightarrow V(H)$ were defined from a matching $M \subseteq E(B)$, there are a few rules by which we can construct a matching $M \subseteq E(B)$ from a subset $S \subseteq V(G)$ and a valid embedding $\varphi : S \rightarrow V(H)$:

- (i) For an edge $e = uu' \in S$, $\varphi(u)\varphi(u')$ must be an edge in H . By the construction of B , there is a corresponding vertex $x \in X$ for uu' , a corresponding vertex $y \in Y$ for the edge $\varphi(u)\varphi(u') \in H$, and an edge $e = xy \in E(B)$ of weight 2. Add e to M .
- (ii) For a vertex $u \in S$ where $|N_G(u)| = 0$ and $|N_H(\varphi(u))| = 0$, by the construction of B , there is a corresponding vertex $x \in X$ for u , a corresponding vertex $y \in Y$ for $\varphi(u)$, and an edge $e = xy \in E(B)$ of weight 1. Add e to M .
- (iii) For an edge $uu' \in G$ where $u \in S$, $u' \notin S$, and $|N_H(\varphi(u))| = 0$, by the construction of B , there is a corresponding vertex $x \in X$ for uu' , a corresponding vertex $y \in Y$ for $\varphi(u)$, and an edge $e = xy \in E(B)$ of weight 1. Add e to M .
- (iv) For a vertex $u \in S$ where $|N_G(u)| = 0$ and an edge $vv' \in E(H)$ where $v = \varphi(u)$, by the construction of B , there is a vertex $x \in X$ that corresponds to u , a vertex $y \in Y$ that corresponds to vv' , and an edge $e = xy \in E(B)$ of weight 1. Add e to M .

Since φ is a valid embedding that embeds S into H , it is not difficult to verify that the set of constructed edges M is a matching, and that $w(M) = |S|$. This completes the proof. \square

We turn our attention next to discussing the fixed-parameter tractability results for p-OLSE when both Δ_G and Δ_L are $O(1)$ (Δ_H may be unbounded). Let $(G, H, \prec_G, \prec_H, L, k)$ be an instance of p-OLSE in which both Δ_G and Δ_L are upper bounded by a fixed constant. Consider the graph \mathcal{G} whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $E(G) \cup E(H) \cup E_L$, where $E_L = \{uv \mid u \in G, v \in H, v \in L(u)\}$; that is, \mathcal{G} is the union of G and H plus the edges that represent the mapping L . We perform the following *splitting* operation on the vertices of \mathcal{G} (see Figure 4 for illustration):

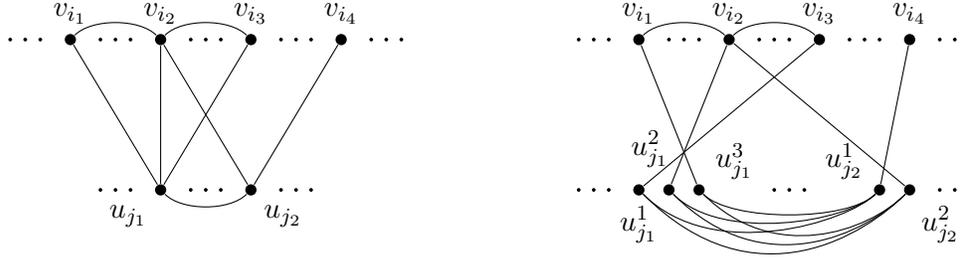


Figure 4: Illustration of the splitting operation when applied to vertices u_{j_1} and u_{j_2} .

Definition 4.5. Let u be a vertex in \mathcal{G} and assume that $u \in G$ (the operation is similar when $u \in H$). Suppose that the vertices of G are ordered as $\langle u_1, \dots, u_n \rangle$ with respect to \prec_G , and suppose that $u = u_i$, for some $i \in \{1, \dots, n\}$. Let $e_1 = uv_{i_1}, \dots, e_r = uv_{i_r}$ be the edges incident to u in E_L , and assume that $v_{i_1} \prec_H v_{i_2} \prec_H \dots \prec_H v_{i_r}$. By splitting vertex u we mean: (1) replacing u in \mathcal{G} with vertices u_i^1, \dots, u_i^r such that the resulting ordering of the vertices in G with respect to \prec_G is $\langle u_1, \dots, u_{i-1}, u_i^1, \dots, u_i^r, u_{i+1}, \dots, u_n \rangle$; (2) removing all the edges e_1, \dots, e_r from \mathcal{G} and replacing them with the edges $u_i^1 v_{i_r}, u_i^2 v_{i_{r-1}}, \dots, u_i^r v_{i_1}$; and (3) replacing every edge uu_j in G with the edges $u_i^s u_j$, for $s = 1, \dots, r$.

Let \mathcal{G}_{split} be the graph resulting from \mathcal{G} by splitting every vertex in G and every vertex in H (in an arbitrary order), where G_{split} is the graph resulting from splitting the vertices in G and H_{split} that resulting from splitting the vertices of H . Let E_{split} be the set of edges having one endpoint in G_{split} and the other in H_{split} , $L_{split} : G_{split} \rightarrow 2^{V(H_{split})}$ defined by $L_{split}(u) = \{v \mid uv \in E_{split}\}$ for $u \in V(G_{split})$, and let $\prec_{G_{split}}$ and $\prec_{H_{split}}$ be the orders on G_{split} and H_{split} , respectively, resulting from \prec_G, \prec_H after the splitting operation.

Lemma 4.6. The graph \mathcal{G}_{split} satisfies the properties: (i) for every $u \in V(G_{split})$ we have $\deg_{G_{split}}(u) \leq \Delta_L \cdot \Delta_G$,¹ (ii) in the graph $(V(G_{split}) \cup V(H_{split}), E_{split})$ every vertex has degree exactly 1 (in particular $|L_{split}(u)| = 1$ for every $u \in V(G_{split})$), and (iii) the instance $(G, H, \prec_G, \prec_H, L, k)$ is a yes-instance of p -OLSE if and only if $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$ is.

Proof. (i) Consider a vertex u_i^r that is split from vertex u . u_i^r is adjacent in G_{split} to every vertex that was split from a neighbor of u . Since u has at most Δ_G many neighbors in G_{split} , and since every neighbor of u was split into at most Δ_L vertices, it follows that u_i^r has at most $\Delta_L \cdot \Delta_G$ many neighbors in G_{split} .

(ii) After splitting a vertex $u \in G$, all resulting vertices are of degree 1 in $(V(G_{split}) \cup V(H_{split}), E_L)$. Moreover, splitting the neighbors of u does not change the degree of u in $(V(G_{split}) \cup V(H_{split}), E_L)$. Therefore, all vertices have degree 1 in $(V(G_{split}) \cup V(H_{split}), E_L)$ after splitting.

(iii) First, we show that if the instance $(G, H, \prec_G, \prec_H, L, k)$ is a yes-instance of p -OLSE then $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$ is also a yes-instance. Let $S \subseteq V(G)$ where $|S| = k$ and let $\varphi : S \rightarrow V(H)$ be a valid embedding. Let $u_i \in S$ and $v_j \in H$ where $\varphi(u_i) = v_j$, without loss of generality, suppose that u_i was split before v_j . After splitting u_i and before splitting v_j there is exactly one vertex u_i^p resulting from splitting u_i such that there is an edge between u_i^p and v_j . After splitting v_j there exists exactly one vertex v_j^q resulting from splitting v_j such that there is an edge $u_i^p v_j^q \in E_{L_{split}}$. Add u_i^p to the constructed solution S_{split} in G_{split} and define (the constructed embedding) $\varphi_{split}(u_i^p) = v_j^q$. Given that for any two vertices $u_i, u_j \in S$ and vertices $u_i^r, u_j^s \in S_{split}$ split from u_i and u_j (and similarly for vertices in H and H_{split}), $u_i \prec_G u_j$ if and only if $u_i^r \prec_{G_{split}} u_j^s$,

¹The degree of a vertex in H_{split} may be unbounded.

and $u_i u_j \in E(G)$ if and only if $u_i^r u_j^s \in E(G_{split})$, it follows that φ_{split} is a valid embedding from S_{split} into $V(H_{split})$.

Conversely, we show that if $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$ is a yes-instance of p-OLSE then so is $(G, H, \prec_G, \prec_H, L, k)$. Let S_{split} be a subset of vertices in G_{split} where $|S_{split}| = k$ and let $\varphi_{split} : S_{split} \rightarrow V(H_{split})$ be a valid embedding. For each pair of vertices $u_i^p \in S_{split}$ and $v_j^q \in H_{split}$ where $\varphi_{split}(u_i^p) = v_j^q$ there are vertices $u_i \in G$ and $v_j \in H$ such that u_i^p and v_j^q were split from u_i and v_j , respectively, and $u_i v_j \in E_L$. Furthermore, for each pair of vertices $u_i^p, u_{i'}^q \in S_{split}$, u_i^p and $u_{i'}^q$ are split from vertices $u_i, u_{i'}$, respectively, where $i \neq i'$. Note that $i \neq i'$ follows from item (2) in the description of the splitting operation (Definition 4.5), which implies that no two vertices that result from splitting the same vertex can be embedded simultaneously. For the same reason, no two distinct vertices v_j^q and $v_{j'}^q$ in H_{split} that are the images under φ_{split} of two vertices in S_{split} can be split from the same vertex in H . Therefore, we can define a subset $S \subseteq V(G)$ where $u_i \in S$ if any vertex u_i^p split from u_i is in S_{split} and a map $\varphi : S \rightarrow V(H)$ where $\varphi(u_i) = v_j$ if $\varphi_{split}(u_i^p) = v_j^q$. Given that for any two vertices $u_i, u_j \in S$ and vertices $u_i^r, u_j^s \in S_{split}$ split from u_i and u_j (and similarly for vertices in H and H_{split}), $u_i \prec_G u_j$ if and only if $u_i^r \prec_{G_{split}} u_j^s$, and $u_i u_j \in E(G)$ if and only if $u_i^r u_j^s \in E(G_{split})$, it follows that φ is a valid embedding that embeds S into H . \square

Next, we perform the following operation, denoted **Simplify**, to \mathcal{G}_{split} . Observe that every vertex in $(V(G_{split}) \cup V(H_{split}), E_{split})$ has degree 1 by part (ii) of Lemma 4.6. Let $u, u' \in G_{split}$ and let $\{v\} = L_{split}(u)$ and $\{v'\} = L_{split}(u')$. If either (1) $uu' \notin E(G_{split})$ but $vv' \in E(H_{split})$, or (2) both $uu' \in E(G_{split})$ and $vv' \in E(H_{split})$, then we can remove edge vv' from $E(H_{split})$ in case (1) and we can remove both edges uu' and vv' in case (2) without affecting any embedding constraint. Without loss of generality, we will still denote by $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$ the resulting instance after the removal of the edges satisfying cases (1) and (2) above. Note that $E(H_{split}) = \emptyset$ at this point, and hence if $uu' \in E(G_{split})$ then no valid list embedding can be defined on a subset of vertices that includes both u and u' . Note also that $\mathcal{G}_{split} - E(G_{split})$ is a realization of a permutation graph P in which the vertices of G_{split} can be arranged on one line according to the order induced by $\prec_{G_{split}}$, and the vertices of H_{split} can be arranged on a parallel line according to the order induced by $\prec_{H_{split}}$. The vertex-set of P corresponds to the edges in E_{split} , and two vertices in P are adjacent if and only if their two corresponding edges cross. Note that two vertices in P correspond to two edges of the form $e = uv$ and $e' = u'v'$, where $u, u' \in G_{split}$ and $v, v' \in H_{split}$. Let \mathcal{I} be the graph whose vertex-set is $V(P)$ and whose edge set is $E(P) \cup E_c$, where $E_c = \{ee' \mid e, e' \in V(P), e = uv, e' = u'v', uu' \in E(G_{split})\}$ is the set of *conflict edges*; that is, \mathcal{I} consists of the permutation graph P plus the set of conflict edges E_c , where each edge in E_c joins two vertices in P whose corresponding endpoints in G_{split} cannot both be part of a valid solution.

Lemma 4.7. *For every vertex $e \in \mathcal{I}$, the number of conflict edges incident to e in \mathcal{I} , denoted $deg_c(e)$, is at most $\Delta_L \cdot \Delta_G$.*

Proof. By part (i) of Lemma 4.6, the degree of a vertex $u \in G_{split}$ is at most $\Delta_L \cdot \Delta_G$. Note that every conflict edge incident to $e \in \mathcal{I}$ corresponds to an edge in \mathcal{G} incident to some $u \in G_{split}$. Thus for each vertex $e \in \mathcal{I}$ the vertex $u \in G$ incident to e has at most $\Delta_L \cdot \Delta_G$ neighbors. The result follows. \square

Lemma 4.8. *The instance $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$, and hence $(G, H, \prec_G, \prec_H, L, k)$, before **Simplify** is applied is a yes-instance of p-OLSE if and only if \mathcal{I} has an independent set of size k .*

Proof. A size- k independent set I in \mathcal{I} corresponds to a set of k edges $u_{i_1} v_{j_1}, \dots, u_{i_k} v_{j_k}$ in \mathcal{G}_{split} such that $u_{i_1} \prec_{G_{split}} \dots \prec_{G_{split}} u_{i_k}, v_{j_1} \prec_{H_{split}} \dots \prec_{H_{split}} v_{j_k}$, and $S = \{u_{i_1}, \dots, u_{i_k}\}$ is an independent

set in G_{split} . Clearly, the embedding $\varphi(u_{i_s}) = \{v_{j_s}\}$, $s = 1, \dots, k$, is a valid embedding that embeds S into H_{split} because it respects both $\prec_{G_{split}}$, $\prec_{H_{split}}$, and because it respects the embedding constraints. To see why the latter statement is true, note that, for any two vertices u_{i_s} and u_{i_r} ($r \neq s$) in S , either there was no edge between u_{i_s} and u_{i_r} before the application of the operation **Simplify**, or there was an edge and got removed by **Simplify**, and in this case there must be also an edge between v_{j_s} and v_{j_r} in H_{split} ; in either case, φ respects the embedding constraints.

Conversely, let φ be a valid embedding that embeds a subset S of size k where $S = \{u_{i_1}, \dots, u_{i_k}\}$, and $\varphi(u_{i_s}) = v_{j_s}$, for $s = 1, \dots, k$. We claim that the set of vertices $I = \{e_1 = u_{i_1}v_{j_1}, \dots, e_k = u_{i_k}v_{j_k}\}$ is an independent set in \mathcal{I} . Since φ is a valid list embedding, no edge in P exists between any two vertices in I . Let $e_r = u_{i_r}v_{j_r}$ and $e_s = u_{i_s}v_{j_s}$ be two vertices in I , where $r \neq s$. If there is no edge between u_{i_r} and u_{i_s} in $E(G_{split})$, then no edge exists between e_s and e_r in \mathcal{I} . On the other hand, if there is an edge between u_{i_r} and u_{i_s} in $E(G_{split})$, then because φ is a valid embedding, there must be an edge as well between v_{i_r} and v_{i_s} . After applying **Simplify**, the edge between u_{i_r} and u_{i_s} will be removed, and hence no edge exists between e_s and e_r in \mathcal{I} . It follows that I is a size- k independent set in \mathcal{I} . \square

Lemma 4.9. *Let \mathcal{C} be a hereditary class² of graphs on which the INDEPENDENT SET problem is solvable in polynomial time, and let $\Delta \geq 0$ be a fixed integer constant. Let $\mathcal{C}' = \{\mathcal{I} = (V(P), E(P) \cup E_c) \mid P \in \mathcal{C}, E_c \subseteq V(P) \times V(P)\}$, where at most Δ edges in E_c are incident to any vertex in \mathcal{I} . Assuming that a graph in \mathcal{C}' is given as $(V(P), E(P) \cup E_c)$ (E_c is given), the INDEPENDENT SET problem can be solved in $\mathcal{O}^*(2^{(\Delta+1)k}(\Delta(k+1))^{\mathcal{O}(\log(\Delta+1)k)})$ time on graphs in the class \mathcal{C}' .*

Proof. Let $(\mathcal{I} = (V(P), E(P) \cup E_c), k)$ be an instance of INDEPENDENT SET, where $\mathcal{I} \in \mathcal{C}'$. We use the random separation method introduced by Cai et al. [4]; this method can be de-randomized in fpt-time using the notion of universal sets and perfect hash functions as explained in Subsection 1.4.

We apply the random separation method to the subgraph $(V(P), E_c)$, and color the vertices in $V(P)$ with two colors, “black” or “red”, randomly and independently. If I is an independent set of size k in \mathcal{I} , since there are at most Δ edges of E_c that are incident to any vertex in \mathcal{I} , the probability that all vertices in I are colored black and all their neighbors along the edges in E_c are colored red, is at least $2^{-k+\Delta k} = 2^{-(\Delta+1)k}$. Using universal sets and perfect hash functions, in time $\mathcal{O}^*(2^{(\Delta+1)k}(\Delta k + k)^{\mathcal{O}(\log(\Delta+1)k)})$ (see the discussion about random separation in Subsection 1.4) we can enumerate 2-colorings such that if a size- k independent set exists, then one of the enumerated 2-colorings will result in the independent set vertices colored black, and all their neighbors along edges in E_c colored red. Therefore, it suffices to determine, given a 2-colored graph \mathcal{I} , whether there is an independent set of size k consisting of black vertices whose neighbors along the edges in E_c are red vertices. We explain how to do so next.

Suppose that the vertices in \mathcal{I} are colored black or red, and let \mathcal{I}_g be the subgraph of \mathcal{I} induced by the black vertices, and \mathcal{I}_r that induced by the red vertices. Notice that if there is an independent set I consisting of k black vertices whose neighbors along the edges in E_c are red, then for each vertex u in I , u is an isolated vertex in the graph $(V(\mathcal{I}_g), E_c)$. Moreover, since I is an independent set, then no edge in $E(P)$ exists between any two vertices in I . Therefore, if we form the subgraph $G_0 = (V_0, E_0)$, where V_0 is the set of vertices in \mathcal{I}_g that are isolated with respect to the set of edges E_c , and E_0 is the set of edges in $E(P)$ whose both endpoints are in V_0 , then I is an independent set in G_0 . On the other hand, any independent set of G_0 is also an independent set of \mathcal{I} . Since G_0 is a subgraph of $P \in \mathcal{C}$ and \mathcal{C} is hereditary, it follows that $G_0 \in \mathcal{C}$ and we can compute a maximum independent set I_{max} in G_0 in polynomial time. If $|I_{max}| \geq k$, then we accept the instance; otherwise, we try the next 2-coloring. If no 2-coloring results in an independent set of

²The class \mathcal{C} is closed under taking subgraphs; that is, every subgraph of a graph in \mathcal{C} is also in \mathcal{C} .

size at least k , we reject. From the above, given a 2-coloring, clearly we can decide if there is an independent set of size k consisting of black vertices whose neighbors along the edges in E_c are red vertices. The result follows. \square

Theorem 4.10. *The p -OLSE problem restricted to instances in which $\Delta_G = O(1)$ and $\Delta_L = O(1)$ is FPT.*

Proof. Let $(G, H, \prec_G, \prec_H, L, k)$ be an instance of p -OLSE in which both Δ_G and Δ_L are upper bounded by a fixed constant. We form the graph \mathcal{G} and perform the splitting operation described in Definition 4.5 to obtain the instance $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$. By Lemma 4.6, $(G, H, \prec_G, \prec_H, L, k)$ is a yes-instance of p -OLSE if and only if $(G_{split}, H_{split}, \prec_{G_{split}}, \prec_{H_{split}}, L_{split}, k)$ is. We apply the operation **Simplify** to the instance and construct the graph $\mathcal{I} = (V(P), E(P) \cup E_c)$ as described above, where P is a permutation graph. Note that the set of edges E_c is known to us. By Lemma 4.8, $(G, H, \prec_G, \prec_H, L, k)$ is a yes-instance of p -OLSE if and only if \mathcal{I} has an independent set of size k . Observe that we can perform the splitting and the **Simplify** operations, and construct \mathcal{I} in polynomial time. Since the INDEPENDENT SET problem is solvable in polynomial time on the class of permutation graphs (e.g., see [19]), the class of permutation graphs is hereditary, and every vertex in \mathcal{I} has at most $\Delta_L \cdot \Delta_G$ edges in E_c incident to it by Lemma 4.7, it follows from Lemma 4.9 that we can decide if \mathcal{I} has an independent set of size k in fpt-time, and hence decide the instance $(G, H, \prec_G, \prec_H, L, k)$ in fpt-time. \square

Unfortunately, the above result does not hold true for p -OLISE, even when $\Delta_G = O(1)$, $\Delta_H = O(1)$, and $\Delta_L = O(1)$, because the number of vertices in G that have the same vertex $v \in H$ in their list may be unbounded.

From Proposition 4.2, we know that in the case when Δ_L is unbounded, and both $\Delta_G = 1$ and $\Delta_H = 1$, p -OLSE is $W[1]$ -hard. The following proposition says that the condition $\Delta_H = 1$ is essential for this W -hardness result:

Proposition 4.11. *The p -OLSE and p -OLISE problems restricted to instances in which $\Delta_H = 0$, $\Delta_G = O(1)$ (resp. $\Delta_G = 0$ and $\Delta_H = O(1)$ for p -OLISE by symmetry) and $\Delta_L = \infty$ are FPT.*

Proof. We prove the result for p -OLSE. The proof is exactly the same for p -OLISE. The proof uses the random separation method, but is simpler than the proof of Lemma 4.9. Let $(G, H, \prec_G, \prec_H, L, k)$ be an instance of p -OLSE. Observe that if S is the solution that we are looking for then S must be an independent set since $\Delta_H = 0$. Use the random separation method to color G with black or red. Since $\Delta_G = O(1)$, if a solution S exists, then in (deterministic) fpt-time we can enumerate 2-colorings such that one of the enumerated 2-colorings will result in all vertices in S colored black and their neighbors in G colored red. So we can work under this assumption (all vertices in S are colored black and their neighbors in G are colored red). Let G_g be the subgraph of G induced by the black vertices, and G_r that induced by the red vertices. Observe that any black vertex in G_g that is not isolated in G_g can be discarded by our assumption (since all neighbors of a vertex in S must be in G_r). Therefore, we can assume that G_g is an independent set. We can now compute a maximum cardinality subset of vertices of G_g that can be (validly) embedded into H using the dynamic programming algorithm in Proposition 2.1; if the subset has size at least k we accept; otherwise, we try another 2-coloring of G . If no 2-coloring of G results in a solution of size at least k , we reject. \square

Remark 4.12. *By examining the running time of the algorithm described in the proof of Theorem 4.10, we observe that p -OLSE is FPT when parameterised jointly by k , Δ_G , and Δ_L , as none of these three parameters appears in the exponent of the polynomial in the input size in the running*

time of this algorithm. Similarly, examining the running time of the algorithm described in the proof of Proposition 4.11 shows that each of p -OLSE and p -OLISE is \mathcal{FPT} when parameterised jointly by k and Δ_G (resp. k and Δ_H) in the case when $\Delta_H = 0$ (resp. $\Delta_G = 0$).

5 Parameterization by the Vertex Cover Number

In this section we study the parameterized complexity of p -OLSE parameterized by the size of a vertex cover ν in the graph G , shortly (p -VC-OLSE), defined formally as follows:

Given: Two graphs G and H with linear orders \prec_G and \prec_H defined on the vertices of G and H ; a function $L : V(G) \rightarrow 2^{V(H)}$; and $k \in \mathbb{N}$

Parameter: ν where $\nu = \tau(G)$ is the size of a minimum vertex cover of G

Question: Is there a subset $S \subseteq V(G)$ of k vertices and an injective map $\varphi : S \rightarrow V(H)$ such that: (1) $\varphi(u) \in L(u)$ for every $u \in S$; (2) for every $u, u' \in S$, if $u \prec_G u'$ then $\varphi(u) \prec_H \varphi(u')$; and (3) for every $u, u' \in S$, $uu' \in E(G)$ if and only if $\varphi(u)\varphi(u') \in E(H)$

As noted earlier, the reduction used in Proposition 4.2 to prove the $W[1]$ -hardness of p -OLSE when restricted to instances in which $\Delta_H = 1$, $\Delta_G = 1$ and $\Delta_L = \infty$ results in an instance in which the number of vertices in G , is upper bounded by a function of the parameter. Since $\nu < |V(G)|$ it follows immediately that p -VC-OLSE is $W[1]$ -hard in this case. Therefore:

Proposition 5.1. *p -VC-OLSE restricted to instances in which $\Delta_H = 1$ and $\Delta_G = 1$ is $W[1]$ -complete.*

Therefore, we can focus our attention on studying the complexity of p -VC-OLSE restricted to instances in which $\Delta_L \in O(1)$.

Theorem 5.2. *p -VC-OLSE restricted to instances in which $\Delta_L = O(1)$ can be solved in time $O^*((2\Delta_L)^\nu)$ and hence is \mathcal{FPT} .*

Proof. Let $\Delta \geq 0$ be any fixed integer, and suppose that $\Delta_L \leq \Delta$. Let $(G, H, \prec_G, \prec_H, L, k, \nu)$ be an instance of p -VC-OLSE, where k is the desired solution size and ν is the size of a minimum vertex cover in G . In fpt -time (in $O^*(1.274^\nu)$ time [5]) we can compute a minimum vertex cover C of G of size ν (if no such vertex cover exists we reject the instance). Let $I = V(G) \setminus C$, and note that I is an independent set of G . Suppose that the solution we are seeking (if it exists) is S , and the valid mapping of S is φ . Let $S_C = S \cap C$, where S_C is possibly empty, and let φ_C be the restriction of φ to S_C . We enumerate each subset of C as S_C in time $O^*(2^\nu)$, enumerate each possible mapping from S_C to $L(S_C)$ as φ_C in time $O^*(\Delta_L^\nu)$, and check the validity of φ_C in polynomial time (if no valid φ_C exists, we reject the enumeration); since $\Delta_L \leq \Delta$, the enumeration can be carried out in time $O^*((2\Delta)^\nu)$. Therefore, we will work under the assumption that the desired solution intersects C at a known subset S_C , and that the restriction of φ to S_C is a known map φ_C , and reject the instance if this assumption is proved to be wrong.

We remove all vertices in $C \setminus S_C$ from G (together with their incident edges) and update L accordingly; without loss of generality, we will still use G to refer to the resulting graph whose vertex-set at this point is $I \cup S_C$. Let u_{i_1}, \dots, u_{i_r} be the vertices in S_C , where $u_{i_1} \prec_G \dots \prec_G u_{i_r}$. Since φ_C is valid, we have $\varphi(u_{i_1}) \prec_H \dots \prec_H \varphi(u_{i_r})$. We now perform the following operation. For each vertex u in I and each vertex $v \in L(u)$, if setting $\varphi(u) = v$ violates the embedding constraint in the sense that either (1) there is a vertex $u_{i_j} \in S_C$ such that $uu_{i_j} \in E(G)$ but $v\varphi_C(u_{i_j}) \notin E(H)$ or (2) there is a vertex $u_{i_j} \in S_C$ such that $u \prec u_{i_j}$ (resp. $u_{i_j} \prec u$) but $\varphi_C(u_{i_j}) \prec v$ (resp.

$v \prec \varphi_C(u_{i_j})$), then remove v from $L(u)$. Afterwards, partition the vertices in G into at most $r + 1$ intervals, I_0, \dots, I_r , where I_0 consists of the vertices preceding u_{i_1} (with respect to \preceq_G), I_r consists of those vertices following u_{i_r} , and I_j consists of those vertices that fall strictly between vertices $u_{i_{j-1}}$ and u_{i_j} , for $j = 1, \dots, r$. Similarly, partition H into $r + 1$ intervals, I'_0, \dots, I'_r , where I'_0 consists of the vertices preceding $\varphi_C(u_{i_1})$ (with respect to \prec_H), I'_r those vertices following $\varphi_C(u_{i_r})$, and I'_j those vertices that fall strictly between vertices $\varphi_C(u_{i_{j-1}})$ and $\varphi_C(u_{i_j})$, for $j = 1, \dots, r$. Clearly, any valid mapping φ that respects \prec_G and \prec_H must map vertices in the solution that belong to I_j to vertices in I'_j , for $j = 0, \dots, r$, in a way that respects the restrictions of \prec_G and \prec_H on I_j and I'_j , respectively. On the other hand, since after the above operation every vertex u in I can be validly mapped to any vertex $v \in L(u)$, any injective mapping φ_j that maps a subset of vertices in I_j to a subset in I'_j in a way that respects the restrictions of \prec_G and \prec_H to I_j and I'_j , respectively, can be extended to a valid embedding whose restriction to S_C is φ_C . Therefore, our problem reduces to determining whether there exist injective maps φ_j , $j = 0, \dots, r$, mapping vertices in I_j to vertices in I'_j , such that the total number of vertices mapped in the I_j 's is $k - r$. Consider the subgraphs $G_j = G[I_j]$ and $H_j = H[I'_j]$, for $j = 0, \dots, r$. Since G_j is an independent set, the presence of edges in H_j does not affect the existence of a valid list mapping from vertices in G_j to H_j , and hence those edges can be removed. Therefore, we can solve the opt-OLSE problem using Proposition 2.1 on the two graphs G_j and H_j to compute a maximum cardinality subset of vertices S_j in G_j that can be validly embedded into H_j via an embedding φ_j . If the union of the S_j 's with S_C has cardinality at least k , we accept. If after all enumerations of S_C and φ_C we do not accept, we reject the instance. This algorithm runs in time $O^*((2\Delta_L)^\nu)$. \square

6 Conclusion

We drew a complete complexity landscape of p-OLSE and p-OLISE, and their optimization versions, with respect to the computational frameworks of classical complexity, parameterized complexity, and approximation, in terms of the structural parameters Δ_H , Δ_G and Δ_L . We also showed that relaxing the order constraint of p-OLSE and p-OLISE makes the problems significantly easier (in \mathcal{P}). There are few interesting open questions that result from our research:

1. What is the complexity of the weighted versions of the problems presented in this paper (vertex-weights or edge-weights), with respect to the frameworks of classical complexity, approximation, and parameterized complexity?
2. Are there meta-theorems, possibly similar to those for the SUBGRAPH ISOMORPHISM and GRAPH EMBEDDING problems (see [13]), in terms of the structures of G and/or H , that would unify some of the complexity results about p-OLSE and p-OLISE presented in the paper?
3. Finally, graph representations of proteins in practice often fall within the bounds of the parameters assumed for the fixed-parameter algorithms (and approximation algorithms) presented in the paper. How do these algorithms perform in practice?

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