COVERING VECTORS BY SPACES: REGULAR MATROIDS*

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Abstract. Seymour's decomposition theorem for regular matroids is a fundamental result with a number of combinatorial and algorithmic applications. In this work we demonstrate how this theorem can be used in the design of parameterized algorithms on regular matroids. We consider the problem of covering a set of vectors of a given finite dimensional linear space (vector space) by a subspace generated by a set of vectors of minimum size. Specifically, in the Space Cover problem, we are given a matrix M and a subset of its columns T; the task is to find a minimum set F of columns of M disjoint with T such that the linear span of F contains all vectors of T. For graphic matroids this problem is essentially Steiner Forest and for cographic matroids this is a generalization of Multiway Cut.

Our main result is the algorithm with running time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ solving Space Cover in the case when M is a totally unimodular matrix over rationals, where k is the size of F. In other words, we show that on regular matroids the problem is fixed-parameter tractable parameterized by the rank of the covering subspace.

Key words. Regular matroids, spanning set, parameterized complexity

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1. Introduction.

We consider the problem of covering a subspace of a given finite dimensional linear space (vector space) by a set of vectors of minimum size. The input of the problem is a matrix M given together with a function w assigning a nonnegative weight to each column of M and a set T of terminal column-vectors T of M. The task is to find a minimum set of column-vectors F of M (if such a set exists) which is disjoint with T and generates a subspace containing the linear space generated by T. In other words, $T \subseteq \text{span}(F)$, where $\text{span}(F)$ is the linear span of F. We refer to this problem as the Space Cover problem.

The Space Cover problem encompasses various problems arising in different domains. The Minimum Distance problem in coding theory asks for a minimum dependent set of columns in a matrix over GF(2). This problem can be reduced to Space Cover by finding for each column t in matrix M a minimum set of columns in the remaining part of the matrix that cover $T = \{t\}$. The complexity of this problem was asked by Berlekamp et al. [2] and remained open for almost 30 years. It was resolved only in 1997, when Vardy showed it to be NP-complete [43]. The parameterized version of the Minimum Distance problem, namely Even Set, asks whether there is a dependent set $F \subseteq X$ of size at most k. The parameterized complexity of Even Set is a long-standing open question in the area, see e.g. [10]. In the language of matroid theory, the problem of finding a minimum dependent set is known as Matroid Girth, i.e. the problem of finding a circuit in matroid of minimum length [44]. In machine learning this problem is known as the Subspace

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RECOVERY problem [22]. This problem also generalizes the problem of computing the rank of a tensor.

For our purposes, it is convenient to rephrase the definition of the SPACE COVER problem in the language of matroids. Given a matrix N, let $M = (E, \mathcal{I})$ denote the matroid where the ground set E corresponds to the columns of N and $\mathcal{I}$ denote the family of subsets of linearly independent columns. This matroid is called the vector matroid corresponding to matrix N. Then for matroids, finding a subspace covering T corresponds to finding $F \subseteq E \setminus T$, $F \in \mathcal{I}$, such that $|F| \leq k$ and T is spanned by F. Let us remind that in a matroid set F spans T, denoted by $T \subseteq \text{span}(F)$, if $r(F) = r(T \cup F)$. Here $r \colon 2^E \to \mathbb{N}_0$ is the rank function of M. (We use $\mathbb{N}_0$ to denote the set of nonnegative integers.)

Then SPACE COVER is defined as follows.

> SPACE COVER **Parameter:** k
> **Input:** A binary matroid $M = (E, \mathcal{I})$ given together with its matrix representation over GF(2), a weight function $w \colon E \to \mathbb{N}_0$, a set of *terminals* $T \subseteq E$, and a nonnegative integer k.
> **Question:** Is there a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that $T \subseteq \text{span}(F)$?

Since a representation of a binary matroid is given as a part of the input, we always assume that the *size* of M is $||M|| = |E|$. For regular matroids, testing matroid regularity can be done efficiently, see e.g. [42], and when the input binary matroid is regular, the requirement that the matroid is given together with its representation can be omitted.

It is known (see, e.g., [28]) that SPACE COVER on special classes of binary matroids, namely graphic and cographic matroids, generalizes two well-studied optimization problems on graphs, namely STEINER TREE and MULTIWAY CUT. Both problems play fundamental roles in parameterized algorithms.

Recall that in the STEINER FOREST problem we are given a (multi) graph G, a weight function $w \colon E \to \mathbb{N}$, a collection of pairs of distinct vertices $\{x_1, y_1\}, \ldots, \{x_r, y_r\}$ of G, and a nonnegative integer k. The task is to decide whether there is a set $F \subseteq E(G)$ with $w(F) \leq k$ such that for each $i \in \{1, \ldots, r\}$, graph $G[F]$ contains an (x_i, y_i)-path. To see that STEINER FOREST is a special case of SPACE COVER, for instance $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ of STEINER FOREST, we construct the following graph. For each $i \in \{1, \ldots, r\}$, we add a new edge $x_i y_i$ to G and assign an arbitrary weight to it; notice that we can create multiple edges this way. Denote by G' the obtained multigraph and let T be the set of added edges and let $M(G')$ be the graphic matroid associated with G'. Then a set of edges $F \subseteq E(G)$ forms a graph containing all (x_i, y_i)-paths if an only if $T \subseteq \text{span}(F)$ in $M(G')$.

The special case of STEINER FOREST when $x_1 = x_2 = \cdots = x_r$, i.e. when set F should form a connected subgraph spanning all demand vertices, is the STEINER TREE problem, the fundamental problem in network optimization. By the classical result of Dreyfus and Wagner [12], STEINER TREE is fixed-parameter tractable (FPT) parameterized by the number of terminals. The study of parameterized algorithms for STEINER TREE has led to the design of important techniques, such as Fast Subset Convolution [3] and the use of branching walks [33]. Research on the parameterized complexity of STEINER TREE is still on-going, with recent significant advances for the planar version of the problem [37]. Algorithms for STEINER TREE are frequently used as a subroutine in FPT algorithms for other problems; examples include vertex cover problems [21], near-perfect phylogenetic tree reconstruction [4], and connectivity augmentation problems [1].

The dual of SPACE COVER, i.e., the variant of SPACE COVER asking whether there is a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that $T \subseteq \text{span}(F)$ in the dual matroid M^*, is equivalent to the RESTRICTED SUBSET FEEDBACK SET problem. In this problem the task is for a given matroid M, a weight function $w\colon E \to \mathbb{N}_0$, a set $T \subseteq E$ and a nonnegative integer k, to decide whether there is a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that matroid M' obtained from M by deleting the elements of F has no circuit containing an element of T. Hence, SPACE COVER for cographic matroids is equivalent to RESTRICTED SUBSET FEEDBACK SET for graphic matroids. RESTRICTED SUBSET FEEDBACK SET for graphs was introduced by Xiao and Nagamochi [45], who showed that this problem is FPT parameterized by $|F|$. Let us note that in order to obtain an algorithm for SPACE COVER with a single-exponential dependence in k, we also need to design a new algorithm for SPACE COVER on cographic matroids which improves significantly the running time achieved by Xiao and Nagamochi [45].

MULTIWAY CUT, another fundamental graph problem, is the special case of RESTRICTED SUBSET FEEDBACK SET, and therefore of SPACE COVER. In the MULTIWAY CUT problem we are given a (multi) graph G, a weight function $w\colon E \to \mathbb{N}$, a set $S \subseteq V(G)$, and a nonnegative integer k. The task is to decide whether there is a set $F \subseteq E(G)$ with $w(F) \leq k$ such that the vertices of S are in distinct connected components of the graph obtained from G by deleting edges of F. Indeed, let (G, w, S, k) be an instance of MULTIWAY CUT. We construct graph G' by adding a new vertex u and connecting it to the vertices of S. Denote by T the set of added edges and assign weights to them arbitrarily. Then (G, w, S, k) is equivalent to the instance $(M(G'), w, T, k)$ of RESTRICTED SUBSET FEEDBACK SET. If $|S| = 2$, MULTIWAY CUT is exactly the classical min-cut problem which is solvable in polynomial time. However, as it was proved by Dahlhaus et al. [6] already for three terminals the problem becomes NP-hard. Marx, in his celebrated work on important separators [31], has shown that MULTIWAY CUT is FPT when parameterized by the size of the cut $|F|$.

While STEINER TREE is FPT parameterized by the number of terminal vertices, the hardness results for MULTIWAY CUT with three terminals yields that SPACE COVER parameterized by the size of the terminal set T is Para-NP-complete even if restricted to cographic matroids. This explains why a meaningful parameterization of SPACE COVER is by the rank of the span and not the size of the terminal set.

It follows from the result of Downey et al. [11] on the hardness of the MAXIMUM-LIKELIHOOD DECODING problem, that SPACE COVER is W[1]-hard for binary matroids when parameterized by k even if restricted to the inputs with one terminal and unit-weight elements. However, it is still possible to establish the tractability of the problem on a large class of binary matroids. Sandwiched between graphic and cographic (where the problem is FPT) and binary matroids (where the problem is intractable) is the class of regular matroids.

Our results. Our main theorem establishes the tractability of SPACE COVER on regular matroids.

THEOREM 1.1. SPACE COVER *on regular matroids is solvable in time* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$.

We believe that due to the generality of SPACE COVER, Theorem 1.1 will be useful in the study of various optimization problems on regular matroids. As an example, we consider the RANK h-REDUCTION problem, see e.g. [26]. Here we are given a binary matroid M and positive integers h and k, the task is to decide whether it is possible to decrease the rank of M by at least h by deleting k elements. For graphic matroids, this is the h-WAY CUT problem, which is for a connected graph G and positive integers h and k, to decide whether it is possible to separate G into at least h connected components by deleting at most k edges. By the celebrated result of Kawarabayashi and Thorup [27], h-WAY CUT is FPT parameterized by k even if h is a part of the input. The result of Kawarabayashi and Thorup cannot be extended to cographic matroids; we show that for cographic matroids the problem is W[1]-hard when parameterized by $h + k$. On the other hand, by making use of Theorem 1.1, we solve RANK h-REDUCTION in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(h)}$ on regular matroids (Theorem 8.3).

Let us also remark that the running time of our algorithm is asymptotically optimal: unless Exponential Time Hypothesis fails, there is no algorithm of running time $2^{o(k)} \cdot ||M||^{\mathcal{O}(1)}$ solving SPACE COVER on graphic (STEINER TREE) or cographic (MULTIWAY CUT) matroids, see e.g. [5].

Related work. The main building block of our algorithm is the fundamental theorem of Seymour [38] on a decomposition of regular matroids. Roughly speaking (we define it properly in Section 4), Seymour's decomposition provides a way to decompose a regular matroid into much simpler *base* matroids that are graphic, cographic or of constant size. Then all "communication" between base matroids is limited to "cuts" of small rank (we refer to the monograph of Truemper [42] and the survey of Seymour [40] for the introduction to matroid decompositions). This theorem has a number of important combinatorial and algorithmic applications. Among the classic algorithmic applications of Seymour's decomposition are the polynomial time algorithms of Truemper [41] (see also [42]) for finding maximum flows, and shortest routes and the polynomial algorithm of Golynski and Horton [20] for constructing a minimum cycle basis. More recent applications of Seymour's decomposition can be found in approximation, on-line and parameterized algorithms. Goldberg and Jerrum [19] used Seymour's decomposition theorem for obtaining a fully polynomial randomized approximation scheme (FPRAS) for the partition function of the ferromagnetic Ising model on regular matroids. Dinitz and Kortsarz in [8] applied the decomposition theorem for the MATROID SECRETARY problem. In [14], Gavenciak, Král and Oum initiated the study of the MINIMUM SPANNING CIRCUIT problem for matroids that generalizes the classical CYCLE THROUGH ELEMENTS problem for graphs. The problem asks for a matroid M, a set $T \subseteq E$ and a nonnegative integer ℓ, whether there is a circuit C of M with $T \subseteq C$ of size at most ℓ. Gavenciak, Král and Oum [14] proved that the problem is FPT when parameterized by ℓ if $|T| \leq 2$. Recently, in [13], we extended this result by showing that MINIMUM SPANNING CIRCUIT is FPT parameterized by $k = \ell - |T|$.

On a very superficial level, all the algorithmic approaches based on Seymour's decomposition theorem utilize the same idea: solve the problem on base matroids and then "glue" solutions into a global solution. Of course, such a view is a significant oversimplification. First of all, the original decomposition of Seymour in [38] was not meant for algorithmic purposes and almost every time to use it algorithmically one has to apply nontrivial adjustments to the original decomposition. For example, in order to solve MATROID SECRETARY on regular matroids, Dinitz and Kortsarz in [8] have to give a refined decomposition theorem suitable for their algorithmic needs. Similarly, in order to use the decomposition theorem for approximation algorithms, Goldberg and Jerrum in [19] have to add several new ingredients to original Seymour's construction. We face exactly the same nature of difficulties in using Seymour's decomposition theorem. Our starting point is the variant of Seymour's decomposition theorem proved by Dinitz and Kortsarz in [8]. However, even the decomposition of Dinitz and Korsatz cannot be used as a black box for our purposes. Our algorithm, while recursively constructing a solution has to transform the decomposition "dynamically". This occurs when the algorithm processes cographic matroids "glued" with other matroids and for that part of the algorithm the transformation of the decomposition is essential.

2. Organization of the paper and outline of the algorithm.

In this section we explain the structure of the paper and give a high-level overview of our algorithm.

2.1. Organization of the paper.

The remaining part of the paper is organizied as follows. In Section 3 we give basic definitions and prove some simple auxiliary results. In Section 4 we define decompositions of regular matroids. In Section 5 we provide a number of reduction rules for Space Cover which will be used in the algoritm. In Section 6 we provide algorithms for basic matroids: graphic and cographic. The algorithm for the general case, which is the most technical part of the paper, is described in Section 7. In Section 8 we discuss the application of our main result to the Rank h-Reduction problem. We conclude with some open questions in Section 9.

2.2. Outline of the algorithm.

One of the crucial components of our algorithm is the classical theorem of Seymour [38] on a decomposition of regular matroids and in Section 4 we briefly introduce these structural results. Roughly speaking, the theorem of Seymour says that every regular matroid can be decomposed via "small sums" into basic matroids which are graphic, cographic and very special matroid of constant size called R_{10}. Our general strategy is: First solve Space Cover on basic matroids, second move through matroid decomposition and combine solutions from basic matroids. However when it comes to the implementation of this approach, many difficulties arise. In what follows we give an overview of our algorithm.

To describe the decomposition of matroids, we need the notion of "ℓ-sums" of matroids; we refer to [36, 42] for a formal introduction to matroid sums. However, for our purpose, it is sufficient that we restrict ourselves to binary matroids and up to 3-sums [38].

Definition 2.1 ($\oplus$-Sums of matroids). *For two binary matroids M_1 and M_2, the* sum *of M_1 and M_2, denoted by $M_1 \oplus M_2$, is the matroid M with the ground set $E(M_1) \triangle E(M_2)$ whose cycles are all subsets $C \subseteq E(M_1) \triangle E(M_2)$ of the form $C = C_1 \triangle C_2$, where C_1 is a cycle of M_1 and C_2 is a cycle of M_2. We will be using only the following sums.*

($\oplus_1$) *If $E(M_1) \cap E(M_2) = \emptyset$ and $E(M_1), E(M_2) \neq \emptyset$, then M is the* 1-sum *of M_1 and M_2 and we write $M = M_1 \oplus_1 M_2$.*

($\oplus_2$) *If $|E(M_1) \cap E(M_2)| = 1$, the unique $e \in E(M_1) \cap E(M_2)$ is not a loop or coloop of M_1 or M_2, and $|E(M_1)|, |E(M_2)| \geq 3$, then M is the* 2-sum *of M_1 and M_2 and we write $M = M_1 \oplus_2 M_2$.*

($\oplus_3$) *If $|E(M_1) \cap E(M_2)| = 3$, the 3-element set $Z = E(M_1) \cap E(M_2)$ is a circuit of M_1 and M_2, Z does not contain a cocircuit of M_1 or M_2, and $|E(M_1)|, |E(M_2)| \geq 7$, then M is the* 3-sum *of M_1 and M_2 and we write $M = M_1 \oplus_3 M_2$.*

An $\{1, 2, 3\}$-*decomposition* of a matroid M is a collection of matroids $\mathcal{M}$, called the *basic matroids* and a rooted binary tree T in which M is the root and the elements of $\mathcal{M}$ are the leaves such that any internal node is 1, 2 or 3-sum of its children.

By the celebrated result of Seymour [38], every regular matroid M has an $\{1, 2, 3\}$-decomposition in which every basic matroid is either graphic, cographic, or isomorphic to R_{10}. Moreover, such a decomposition (together with the graphs whose cycle and bond matroids are isomorphic to the corresponding basic graphic and cographic matroids) can be found in time polynomial in $|E(M)|$. The matroid R_{10} is a binary matroid represented over GF(2) by the 5×10-matrix whose columns are formed by vectors that have exactly three non-zero entries (or rather three ones) and no two columns are identical.

In this paper we use a variant of Seymour's decomposition suggested by Dinitz and Kortsarz in [8]. With a regular matroid one can associate a *conflict graph*, which is an intersection graph of the basic matroids. In other words, the nodes of the conflict graph are the basic matroids and two nodes are adjacent if and only if the intersection of the corresponding matroids is nonempty. It was shown by Dinitz and Kortsarz in [8] that every regular matroid M can be decomposed into basic matroids such that the corresponding conflict graph is a forest. Thus every node of this forest is one of the basic matroids that are either graphic, or cographic, or isomorphic to R_{10} (we can relax this condition and allow variations of R_{10} obtained by adding parallel elements to participate in a decomposition). Two nodes are adjacent if the corresponding matroids have some elements in common, the edge connecting these nodes corresponds to 2-, or 3-sum. We complement this forest into a *conflict tree* $\mathcal{T}$ by edges which correspond to 1-sums. As it was shown by Dinitz and Kortsarz, then regular matroid M can be obtained from $\mathcal{T}$ by taking the sums between adjacent matroids in any order.

In matroid language, it is much more convenient to speak in terms of minimal dependent sets, i.e. circuits. In this language, a set $F \subseteq E(M) \setminus T$ spans $T \subseteq E(M)$ in matroid M if and only if for every $t \in T$, there is a circuit C of M such that $t \in C \subseteq F \cup \{t\}$. In what follows, we often will use an equivalent reformulation of Space Cover, namely the problem of finding a minimum-sized set F, such that for every terminal element t, the set $F \cup \{t\}$ contains a circuit with t.

We start our algorithm with solving Space Cover on basic matroids in Section 6. The problem is trivial for R_{10}. If M is a graphic matroid, then there is a graph G such that M is isomorphic to the cycle matroid $M(G)$ of G. That is, the circuits of $M(G)$ are exactly the cycles of G. Hence, $F \subseteq E(G)$ spans $t = uv \in E(G)$ if and only if F contains an (u, v)-path. By this observation, we can reduce an instance of Space Cover to an instance of Steiner Forest. The solution to Steiner Forest is very similar to the classical algorithm for Steiner Tree [12].

Recall that Space Cover on cographic matroids is equivalent to Restricted Edge-Subset Feedback Edge Set. Xiao and Nagamochi proved in [45] that this problem can be solved in time $(12k)^{6k}2^k \cdot n^{\mathcal{O}(1)}$ on n-vertex graphs. To get a single-exponential in k algorithm for regular matroids, we improve this result and construct a single-exponential algorithm for Space Cover on cographic matroids. We consider a graph G such that M is isomorphic to the bond matroid $M^*(G)$ of G. The set of circuits of M is the set of inclusion-minimal edge cut-sets of G, and we can restate Space Cover as a cut problem in G: for a given set $T \subseteq E(G)$, we need to find a set $F \subseteq E(G) \setminus T$ such that the edges of T are bridges of $G - F$. To resolve this problem, we use a powerful technique of Marx [31] based on *important separators* or *cuts*. Unfortunately, for our purposes this technique cannot be applied directly and web have to introduce special important edge-cuts tailored for Space Cover. We call such edge-cuts *semi-important* and obtain structural results for semi-important cuts. Then a branching algorithm based on the enumeration of semi-important cuts solves the problem in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

The algorithm for the general case is described in Section 7. Suppose that we have an instance of Space Cover for a regular matroid M. First, we apply some reduction rules described in Section 5 to simplify the instance. In particular, for technical reasons we allow zero weights of elements, but a nonterminal element of zero weight can always be taken into a solution. Hence, we can contract such elements. Also, if the set of terminals T contains a circuit C, then the deletion from M of any $e \in C$ leads to an equivalent instance of the problem. This way, we can bound the number of terminals in the parameter k.

In the next step, we construct a conflict tree $\mathcal{T}$. If $\mathcal{T}$ has one node, then M is graphic, cographic or a copy of R_{10}, and we solve the problem directly. Otherwise, we select arbitrarily a root node r of $\mathcal{T}$, and its selection defines the parent-child relation on $\mathcal{T}$. We say that u is a *sub-leaf* if its children are leaves of $\mathcal{T}$. Clearly, such a node exists and can be found in polynomial time. Let a basic matroid M_s be a sub-leaf of $\mathcal{T}$. We say that a child of M_s is a 1, 2 or 3-*leaf* respectively if the edge between M_s and the leaf corresponds to 1, 2 or 3-sum respectively. We either reduce a leaf M_ℓ that is a child of M_s by deleting M_ℓ from the decomposition and modifying M_s, or we branch on M_ℓ or M_s. For each branch, we delete M_ℓ or/and modify M_s in such a way that the parameter k decreases.

The case when there is an 1-leaf M_ℓ is trivial, because we can solve the problem for M_ℓ independently. For the cases of 2 and 3-leaves, we recall that a solution F together with T is a union of circuits and analyze the possible structure of these circuits.

If M_ℓ is a 2-leaf, we have two cases: either M_ℓ contains a terminal or not. If M_ℓ contains no terminal, we are able to delete M_ℓ from the decomposition and assign to the unique element $e \in E(M_s) \cap E(M_\ell)$ the minimum weight of $F_\ell \subseteq E(M_\ell) \setminus \{e\}$ that spans e in M_ℓ. If $T_\ell = E(M_\ell) \cap T \neq \emptyset$, then we have three possible cases for $F_\ell = E(M_\ell) \cap F$, where F is a (potential) solution:

i) F_ℓ spans T_ℓ and e in M_ℓ, then we can use the elements of F_ℓ that together with e form a circuit of M_ℓ to span $t \in T \setminus T_\ell$,
ii) the symmetric case, where $F_\ell \cup \{e\}$ spans T_ℓ and we need the elements of $F \setminus F_\ell$ that together with e form a circuit to span the elements of T_ℓ, and
iii) F_ℓ spans T_ℓ in M_ℓ and no element of F_ℓ is needed to span the remaining terminals.

Respectively, we branch according to these cases. It can be noticed that in ii), we have a degenerate possibility that e spans T_ℓ. Then the branching does not decrease the parameter. To avoid this situation, we observe that if there is $t \in T_\ell$ that is parallel to e in M_ℓ, then we modify the decomposition by deleting t from M_ℓ and by adding a new element t to M_ℓ that is parallel to e.

The analysis of the cases when we have only 3-leaves is done in similar way but is more complicated. If we have a 3-leaf M_ℓ that contains terminals, then we branch. Here we have 6 types of branches, and the total number of branches is 15. Moreover, for some of branches, we have to solve a special variant of the problem called Restricted Space Cover for the leaf to break the symmetry. If there is no a 3-leaf with terminals, then our strategy depends on the type of M_s that can be graphic or cographic.

If M_s is a graphic matroid, then we consider a graph G such that the cycle matroid $M(G)$ is isomorphic to M_s and assume that $M(G) = M_s$. If M_ℓ is a 3-leaf, then the elements of $E(M_s) \cap E(M_\ell)$ form a cycle Z of size 3 in G. We delete M_ℓ from the decomposition and modify G as follows: construct a new vertex u and join u with the vertices of Z be edges. Then we assign the weights to the edges of Z and the edges incident to u to emulate all possible selections of elements of M_ℓ for a solution.

As with the basic matroids, the case of cographic matroids proved to be most difficult. If M_s is cographic, then there is a graph G such that the bond matroid $M^*(G)$ is isomorphic to M_s. Recall that the circuits of $M^*(G)$ are exactly the minimal edge cut-sets of G. In particular, the intersections of the sets of elements of the 3-leafs with $E(M_s)$ are mapped by an isomorphism of M_s and $M^*(G)$ to minimal cut-sets of G. We analyze the structure of these cuts. It is well-known that *minimum* cut-sets of odd size form a tree-like structure (see [7]). In our case, we can assume that G has no bridges, but still G is not necessarily 3 connected. We show that we always can find an isomorphism α of M_s to $M^*(G)$ and a 3-leaf M_ℓ such that a minimal cut-set $Z = \alpha(E(M_s) \cap E(M_\ell))$ separates G into two components in such a way with the following condition: There is a component H such that H has no bridges, moreover, no element of a basic matroid $M' \neq M_s$ is mapped by α to an edge of H. In the case of a graphic sub-leaf, we are able to get rid of a leaf by making a simple local adjustment of the corresponding graph. For the cographic case, this approach does not work as we are working with cuts. Still, if H contains no terminal, then we make a replacement but we are replacing the leaf M_ℓ and H in G simultaneously by a gadget. If H has terminals, we branch on H: we decompose further $M^*(G)$ into a sum of two cographic matroids and obtain a new leaf of the considered sub-leaf from H. Then we either reduce the new leaf if it is an 1-leaf or branch on it if it is a 2 or 3-leaf.

3. Preliminaries. Parameterized Complexity. Parameterized complexity is a two dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a parameter k. It is said that a problem is *fixed parameter tractable* (or FPT), if it can be solved in time $f(k) \cdot n^{O(1)}$ for some function f. We refer to the recent books of Cygan et al. [5] and Downey and Fellows [10] for the introduction to parameterized complexity.

It is standard for a parameterized algorithm to use *(data) reduction rules*, i.e., polynomial or FPT algorithms that either solve an instance or reduce it to another one that typically has a smaller input size and/or a lesser value of the parameter. A reduction rule is *safe* if it either correctly solves the problem or outputs an equivalent instance.

Our algorithm for SPACE COVER uses the bounded search tree technique or *branching*. It means that the algorithm includes steps, called *branching rules*, on which we either solve the problem directly or recursively call the algorithm on several instances (*branches*) for lesser values of the parameter. We say that a branching rule is *exhaustive* if it either correctly solves the problem or the considered instance is a yes-instance if and only if there is a branch with a yes-instance.

Graphs. We consider finite undirected (multi) graphs that can have loops or multiple edges. We use n and m to denote the number of vertices and edges of the considered graphs respectively if it does not create confusion. For a graph G and a subset $U \subseteq V(G)$ of vertices, we write $G[U]$ to denote the subgraph of G induced by U. We write $G - U$ to denote the subgraph of G induced by $V(G) \setminus U$, and $G - u$ if $U = \{u\}$. Respectively, for $S \subseteq E(G)$, $G[S]$ denotes the graph induced by S, i.e., the graph with the edges S whose vertices are the vertices of G incident to the edges of S. We denote by $G - S$ the graph obtained from G by the deletion of the edges of G; for a single element set, we write $G - e$ instead of $G - \{e\}$. For $e \in E(G)$, we denote by G/e the graph obtained by the contraction of e. Since we consider multigraphs, it is assumed that if $e = uv$, then to construct G/e, we delete u and v, construct a new vertex w, and then for each $ux \in E(G)$ and each $vx \in E(G)$, where

$x \in V(G) \setminus \{u, v\}$, we construct new edge wx (and possibly obtain multiple edges), and for each $e' = uv \neq e$, we add a new loop ww. A set $S \subseteq E(G)$ is an *(edge) cut-set* if the deletion of S increases the number of components. A cut-set S is *(inclusion) minimal* if any proper subset of S is not a cut-set. A *bridge* is a cut-set of size one.

Matroids. We refer to the book of Oxley [36] for the detailed introduction to the matroid theory. Recall that a matroid M is a pair $(E, \mathcal{I})$, where E is a finite *ground* set of M and $\mathcal{I} \subseteq 2^E$ is a collection of *independent* sets that satisfy the following three axioms:

I1. $\emptyset \in \mathcal{I}$,
I2. if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$,
I3. if $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there is $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$.

We denote the ground set of M by $E(M)$ and the set of independent set by $\mathcal{I}(M)$ or simply by E and $\mathcal{I}$ if it does not create confusion. If a set $X \subseteq E$ is not independent, then X is *dependent*. Inclusion maximal independent sets are called *bases* of M. We denote the set of bases by $\mathcal{B}(M)$ (or simply by $\mathcal{B}$). The matroid M^* with the ground set $E(M)$ such that $\mathcal{B}(M^*) = \mathcal{B}^*(M) = \{E \setminus B \mid B \in \mathcal{B}(M)\}$ is *dual* to M. The bases of M^* are *cobases* of M.

A function $r\colon 2^E \to \mathbb{Z}_0$ such that for any $Y \subseteq E$, $r(Y) = \max\{|X| \mid X \subseteq Y \text{ and } X \in \mathcal{I}\}$ is called the *rank* function of M. Clearly, $X \subseteq E$ is independent if and only if $r(X) = |X|$. The *rank* of M is $r(M) = r(E)$. Repectively, the *corank* $r^*(M) = r(M^*)$.

Recall that a set $X \subseteq E$ *spans* $e \in E$ if $r(X \cup \{e\}) = r(X)$, and $\text{span}(X) = \{e \in E \mid X \text{ spans } e\}$. Respectively, X *spans a set* $T \subseteq E$ if $T \subseteq \text{span}(X)$. Let $T \subseteq E$. Notice that if $F \subseteq T$ spans every element of T, then an independent set of maximum size $F' \subseteq F$ spans T as well by the definition. Hence, we can observe the following.

Observation 3.1. *Let $T \subseteq E$ for a matroid M, and let $F \subseteq E \setminus T$ be an inclusion minimal set such that F spans T. Then F is independent.*

An (inclusion) minimal dependent set is called a *circuit* of M. We denote the set of all circuits of M by $\mathcal{C}(M)$ or simply $\mathcal{C}$ if it does not create a confusion. The circuits satisfy the following conditions (*circuit axioms*):

C1. $\emptyset \notin \mathcal{C}$,
C2. if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$,
C3. if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

An one-element circuit is called *loop*, and if $\{e_1, e_2\}$ is a two-element circuit, then it is said that e_1 and e_2 are *parallel*. An element e is *coloop* if e is a loop of M^* or, equivalently, $e \in B$ for every $B \in \mathcal{B}$. A *circuit* of M^* is called *cocircuit* of M. A set $X \subseteq E$ is a *cycle* of M if X either empty or X is a disjoint union of circuits. By $\mathcal{S}(M)$ (or $\mathcal{S}$) we denote the set of all cycles of M. We often use the property that the sets of circuits and cycles completely define matroid. Indeed, a set is independent if and only if it does not contain a circuits, and the circuits are exactly inclusion minimal nonempty cycles.

We can observe the following.

Observation 3.2. *Let $\{e_1, e_2\} \in \mathcal{C}$ for distinct $e_1, e_2 \in E$ and let $C \in \mathcal{C}$ for a matroid M. If $e_1 \in C$ and $e_2 \notin C$, then $C' = (C \setminus \{e_1\}) \cup \{e_2\}$ is a circuit.*

Proof. By axiom C3, $(\{e_1, e_2\} \cup C) \setminus \{e_1\} = (C \setminus \{e_1\}) \cup \{e_2\} = C'$ contains a circuit C''. Suppose that $C'' \neq C'$. Notice that because $C \setminus \{e_1\}$ contains no circuit, we have that $e_2 \in C''$. As $e_1 \notin C''$, we obtain that $(\{e_1, e_2\} \cup C'') \setminus \{e_2\}$ contains a circuit, but $(\{e_1, e_2\} \cup C'') \setminus \{e_2\}$ is a proper subset of C, which is a contradiction. Hence, $C'' = C'$ and thus C' is a circuit. ◻

Often it is convenient to express the property that a set X spans an element e in terms of circuits or, equivalently, cycles.

OBSERVATION 3.3. *Let $e \in E$ and $X \subseteq E \setminus \{e\}$ for a matroid M. Then $e \in \operatorname{span}(X)$ if and only if there is a circuit (cycle) C such that $e \in C \subseteq X \cup \{e\}$.*

Proof. Denote by r the rank function of M. Let $e \in \operatorname{span}(X)$. Then $r(X \cup \{e\}) = r(X)$. Let Y be an independent set such that $Y \subseteq X$ and $r(X) = r(Y)$. We have that $r(Y \cup \{e\}) \leq r(X \cup \{e\}) = r(X) = r(Y)$. Hence, $Y \cup \{e\}$ is not independent. Therefore, there is a circuit (cycle) C such that $C \subseteq Y \cup \{e\} \subseteq X \cup \{e\}$. Because Y is independent, we have that $C \not\subseteq Y$ and $e \in C$. Hence $e \in C \subseteq X \cup \{e\}$.

Suppose that there is a circuit C such that $e \in C \subseteq X \cup \{e\}$. Let $Y = C \cap X$. Since $e \in C$ and $e \notin X$, we have that Y is a proper subset of C, i.e., Y is independent. Denote by Z an (inclusion) maximal independent set such that $Y \subseteq Z \subseteq X$ and let Z' be a maximal independent set such that $Z' \subseteq X \cup \{e\}$. If $|Z'| > |Z|$, then by axiom I3, there is $e' \in Z' \setminus Z$ such that $Z \cup \{e'\}$ is independent. Because Z is a maximal independent set such that $Y \subseteq Z \subseteq X$, it follows that $e' \notin X$. Hence, $e' = e$, but then $C = Y \cup \{e\} \subseteq Z \cup \{e\}$ contradicting the independence of $Z \cup \{e\}$. It means that $|Z| = |Z'|$. Therefore, $r(X) \leq r(X \cup \{e\}) = |Z'| = |Z| \leq r(X)$. Hence, $e \in \operatorname{span}(X)$.

Finally, if there is a cycle C such that $e \in C \subseteq X \cup \{e\}$, then there is a circuit $C' \subseteq C$ such that $e \in C' \subseteq X \cup \{e\}$ and, therefore, $e \in \operatorname{span}(X)$ by the previous case. ◻

By Observation 3.3, we can reformulate SPACE COVER in the following equivalent form.

SPACE COVER (reformulation)	**Parameter:** k
Input: A binary matroid $M = (E, \mathcal{I})$ given together with its matrix representation over GF(2), a weight function $w\colon E \to \mathbb{N}_0$, a set of *terminals* $T \subseteq E$, and a nonnegative integer k.	
Question: Is there a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that for any $e \in T$, there is a circuit (or cycle) C such that $e \in C \subseteq F \cup \{e\}$?	

We use this equivalent definition in the majority of the proofs without referring to Observation 3.3.

Let M be a matroid and $e \in E(M)$ is not a loop. We say that M' is obtained from M by *adding of a parallel to e element* if $E(M') = E(M) \cup \{e'\}$, where e' is a new element, and $\mathcal{I}(M') = \mathcal{I}(M) \cup \{(X \setminus \{e\}) \cup \{e'\} \mid X \in \mathcal{I}(M) \text{ and } e \in X\}$. It is straightforward to verify that $\mathcal{I}(M')$ satisfies the axioms I.1-3, i.e., M' is a matroid with the ground set $E(M) \cup \{e'\}$. It is also easy to see that $\{e, e'\}$ is a circuit, that is, e and e' are parallel elements of M'.

Deletions and contractions. Let M be a matroid, $e \in E(M)$. The matroid $M' = M - e$ is obtained by *deleting* e if $E(M') = E(M) \setminus \{e\}$ and $I(M') = \{X \in \mathcal{I}(M) \mid e \notin X\}$. It is said that $M' = M/e$ is obtained by *contracting* e if $M' = (M - e)^*$. In particular, if e is not a loop, then $I(M') = \{X \setminus \{e\} \mid e \in X \in \mathcal{I}(M)\}$. Notice that deleting an element in M is equivalent to contracting it in M^* and vice versa. Let $X \subseteq E(G)$. Then $M - X$ denotes the matroid obtained from M by the deletion of the elements of X and M/X is the matroid obtained by the consecutive contractions of the elements of X. The *restriction* of M to X, denoted by $M|X$, is the matroid obtained by the deletion of the elements of $E(G) \setminus X$.

Matroids associated with graphs. Let G be a graph. The *cycle* matroid $M(G)$ has the ground set $E(G)$ and a set $X \subseteq E(G)$ is independent if $X = \emptyset$ or $G[X]$ has no cycles. Notice that C is a circuit of $M(G)$ if and only if C induces a cycle of G. The *bond* matroid $M^*(G)$ with the ground set $E(G)$ is dual to $M(G)$, and X is a circuit of $M^*(G)$ if and only if X is a minimal cut-set of G. It is said that M is a *graphic* matroid if M is isomorphic to $M(G)$ for some graph G. Respectively, M is *cographic* if there is graph G such that M is isomorphic to $M^*(G)$. Notice that $e \in E$ is a loop of a cycle matroid $M(G)$ if and only if e is a loop of G, and e is a loop of $M^*(G)$ if and only if e is a bridge of G. Notice also that by the addition of an element parallel to $e \in E$ for $M(G)$ we obtain $M(G')$ for the graph G' obtained by adding a new edge with the same end vertices as e. Respectively, by adding of an element parallel to $e \in E$ for $M^*(G)$ we obtain $M^*(G')$ for the graph G' obtained by subdividing e.

Matroid representations. Let M be a matroid and let F be a field. An $n \times m$-matrix A over F is a *representation of M over F* if there is one-to-one correspondence f between E and the set of columns of A such that for any $X \subseteq E$, $X \in \mathcal{I}$ if and only if the columns $f(X)$ are linearly independent (as vectors of F^n); if M has such a representation, then it is said that M has a *representation over F*. In other words, A is a representation of M if M is isomorphic to the *column matroid* of A, i.e., the matroid whose ground set is the set of columns of A and a set of columns is independent if and only if these columns are linearly independent. A matroid is *binary* if it can be represented over GF(2). A matroid is *regular* if it can be represented over any field. In particular, graphic and cographic matroids are regular. Notice that any matroid obtained from a regular matroid by deleting and contracting its elements is regular. Observe also that the matroid obtained from a regular matroid by adding a parallel element is regular as well.

We stated in the introduction that we assume that we are given a representation over GF(2) of the input matroid of an instance of Space Cover. Then it can be checked in polynomial time whether a subset of the ground set is independent by checking the linear independence of the corresponding columns.

We use the following observation about cycles of binary matroids.

Observation 3.4 (see [36]). *Let C_1 and C_2 be circuits (cycles) of a binary matroid M. Then $C_1 \triangle C_2$ is a cycle of M.*

The dual of Space Cover. We recall the definition of Restricted Subset Feedback Set.

Restricted Subset Feedback Set
Input: A binary matroid M, a weight function $w\colon E \to \mathbb{N}_0$, $T \subseteq E$, and a nonnegative integer k.
Question: Is there a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that matroid $M' = M - F$ has no circuit containing an element of T.

This problem is dual to Space Cover.

Proposition 3.1. Restricted Subset Feedback Set *on matroid M is equivalent to* Space Cover *on the dual of M.*

Proof. Let M be a binary matroid and $T \subseteq E$. By Observation 3.3, it is sufficient to show that for every $F \subseteq E \setminus T$, $M - F$ has no circuit containing an element of T if and only if for each $t \in T$ there is a cocircuit C of M such that $t \in C \subseteq F \cup \{t\}$.

Suppose that for each $t \in T$, there is a cocircuit C of M such that $t \in C \subseteq F \cup \{t\}$. We show that $M - F$ has no circuit containing an element of T. To obtain a contradiction, assume that there is $t \in T$ and a circuit C' of M such that $t \in C'$ and $C' \cap F = \emptyset$. Let C be a cocircuit of M such that $t \in C \subseteq F \cup \{t\}$. Then $C \cap C' = \{t\}$, but it contradicts the well-known property (see [36]) that for every circuit and every cocircuit of a matroid, their intersection is either empty of contains at least two elements.

Suppose now that $M - F$ has no circuit containing an element of T. In particular, it means that T is independent in M, and hence in $M - F$. Then there is a basis B of $M - F$ such that $T \subseteq B$. Clearly, B is an independent set of M. Hence, there is a basis B' of M such that $B \subseteq B'$. Consider cobasis$B^* = E \setminus B'$. Let $t \in T$. The set $B^* \cup \{t\}$ contains a unique cocircuit C and $t \in C$. We claim that $C \subseteq F \cup \{t\}$. To obtain a contradiction, assume that there is $e \in C \setminus (F \cup \{t\})$. Since $C \cap B' = \{t\}$, $e \notin B$ and, therefore, $e \notin B'$. The set $B \cup \{e\}$ contains a unique circuit C' of $M - F$ such that $e \in C'$. Notice that C' is a circuit of M as well. Observe that $e \in C \cap C' \subseteq \{e, t\}$. Since $C \cap C' \neq \emptyset$, $|C \cap C'| \geq 2$. Hence, $t \in C'$. We obtain that C' is a circuit of M containing t but $C' \cap F = \emptyset$; a contradiction. ∎

The variant of RESTRICTED SUBSET FEEDBACK SET for graphs, i.e., RESTRICTED SUBSET FEEDBACK SET for graphic matroids, was introduced by Xiao and Nagamochi in [45]. They proved that this problem can be solved in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$ for n-vertex graphs. In fact, they considered the problem without weights, but their result can be generalized for weighted graphs. They also considered the unweighted variant of the problem without the restriction $F \subseteq E \setminus S$. They proved that this problem can be solved in polynomial time. We observe that this results holds for binary matroids. More formally, we consider the following problem.

SUBSET FEEDBACK SET **Input:** A binary matroid M, $T \subseteq E$ and a nonnegative integer k. **Question:** Is there a set $F \subseteq E \setminus T$ with $|F| \leq k$ such that the matroid M' obtained from M by the deletion of the elements of F has no circuits containing elements of T.

PROPOSITION 3.2. SUBSET FEEDBACK SET *is solvable in polynomial time.*

Proof. To see that SUBSET FEEDBACK SET is solvable in polynomial time, it is sufficient to notice that it is dual to the similar variant of SPACE COVER without weights and without the condition $F \subseteq E \setminus T$. The proof of this claim is almost the same as the proof of Proposition 3.1; the only difference is that $F \subseteq E$ spans T in M if and only if for every $t \in T \setminus F$, there is a circuit C such that $t \in C \subseteq F \cup \{t\}$. This variant of SPACE COVER is solvable in polynomial time because the set of minimum size that spans T can be chosen to be a maximal independent subset of T. ∎

Notice also that if we allow weights but do not restrict $F \subseteq E \setminus T$, then this variant of SPACE COVER is at least as hard as the original variant of the problem, because by assigning the weight $k + 1$ to the elements of T we can forbid their usage in the solution.

Restricted Space Cover problem. For technical reasons, in the algorithm we have to solve the following restricted variant of SPACE COVER on graphic and cographic matroids.

RESTRICTED SPACE COVER	**Parameter:** k
Input: Matroid M with a ground set E, a weight function $w\colon E \to \mathbb{N}_0$, a set of terminals $T \subseteq E$, a nonnegative integer k, and $e^* \in E$ with $w(e^*) = 0$ and $t^* \in T$.	
Question: Is there a set $F \subseteq E \setminus T$ with $w(F) \leq k$ such that $T \subseteq \text{span}(F)$ and $t^* \in \text{span}(F \setminus \{e^*\})$?	

In fact, we have to solve this problem only in one special case (see Branching Rule 7.2) when we deal with 3-sums in our branching algorithm and have to break symmetry between summands to be able to recurse. Nevertheless, we cannot avoid solving this variant of the problem separately for graphic and cographic matroids.

We conclude the section by some hardness observations.

PROPOSITION 3.3. SPACE COVER *is* W[1]*-hard for binary matroids when parameterized by* k *even if restricted to the inputs with one terminal and unit-weight elements.*

Proof. Downey et al. proved in [11] that the following parameterized problem is W[1]-hard:

MAXIMUM-LIKELIHOOD DECODING	**Parameter:** k
Input: A binary $n \times m$ matrix A, a target binary n-element vector s, and a positive integer k.	
Question: Is there a set of at most k columns of A that sum to s?	

The W[1]-hardness is proved in [11] for nonzero s; in particular, it holds if s is the vector of ones.

Let (A, s, k) be an instance of MAXIMUM-LIKELIHOOD DECODING for nonzero s. We define the matrix A' by adding the column s to A. Let M be the column matroid of A' and $T = \{s\}$. For every $e \in E(M)$, we set $w(e) = 1$.

Suppose that there are at most k columns of A that sum to s. Then there are at most k linearly independent columns that sum to s. Clearly, these columns span s in M. If there is a set $F \subseteq E(M) \setminus \{s\}$ of size at most k that spans s, then there is a circuit C of M such that $s \in C \subseteq F \cup \{s\}$. It immediately implies that the sum of columns of C is the zero vector and, therefore, the columns of $C \setminus \{s\}$ sum to s. □

We noticed that Steiner Tree is a special case of SPACE COVER for the cycle matroid of an input graph. This reduction together with the result of Dom, Lokshtanov and Saurabh [9] that Steiner Tree has no polynomial kernel (we refer to [5] for the formal definitions of kernels) unless P ⊆ coNP/poly immediately implies the following statement.

PROPOSITION 3.4. SPACE COVER *has no polynomial kernel unless* P ⊆ coNP/poly *even if restricted to graphic matroids and the inputs with unit-weight elements.*

Finally, it was proved by Dahlhaus et al. [6] that MULTIWAY CUT is NP-complete even if $|S| = 3$. It implies as the following proposition.

PROPOSITION 3.5. *The version of* SPACE COVER*, where the parameter is* $|T|$*, is* Para-NP*-complete even if restricted to cographic matroids and the inputs with unit-weight elements.*

4. Regular matroid decompositions.

In this section we describe matroid decomposition theorems that are pivotal for algorithm for SPACE COVER. In particular we start by giving the structural decomposition for regular matroids given by Seymour [38]. Recall that, for two sets X and Y, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$ denotes the *symmetric difference* of X and Y. To describe the decomposition of matroids we need the notion of "ℓ-sums" of matroids for $\ell = 1, 2, 3$. We already defined these sums in Section 2, Definition 2.1 (see also [36, 42]). If $M = M_1 \oplus_\ell M_2$ for some $\ell \in \{1, 2, 3\}$, then we write $M = M_1 \oplus M_2$.

DEFINITION 4.1. *A* $\{1, 2, 3\}$-decomposition *of a matroid* M *is a collection of matroids* $\mathcal{M}$*, called the* basic matroids *and a rooted binary tree* T *in which* M *is the root and the elements of* $\mathcal{M}$ *are the leaves such that any internal node is either* 1-*,* 2- *or* 3-*sum of its children.*

We also need the special binary matroid R_{10} to be able to define the decomposition theorem for regular matroids. It is represented over GF(2) by the 5×10-matrix whose columns are formed by vectors that have exactly three non-zero entries (or rather three ones) and no two columns are identical. Now we are ready to give the decomposition theorem for regular matroids due to Seymour [38].

THEOREM 4.2 ([38]). *Every regular matroid* M *has an* $\{1, 2, 3\}$*-decomposition in which every basic matroid is either graphic, cographic, or isomorphic to* R_{10}*. Moreover, such a decomposition (together with the graphs whose cycle and bond matroids are isomorphic to the corresponding basic graphic and cographic matroids) can be found in time polynomial in* $|E(M)|$*.*

4.1. Modified Decomposition. For our algorithmic purposes we will not use the Theorem 4.2 but rather a modification proved by Dinitz and Kortsarz in [8]. Dinitz and Kortsarz in [8] first observed that some restrictions in the definitions of 2- and 3-sums are not important for the algorithmic purposes. In particular, in the definition of the 2-sum, the unique $e \in E(M_1) \cap E(M_2)$ is not a loop or coloop of M_1 or M_2, and $|E(M_1)|, |E(M_2)| \geq 3$ could be dropped. Similarly, in the definition of 3-sum the conditions that $Z = E(M_1) \cap E(M_2)$ does not contain a cocircuit of M_1 or M_2, and $|E(M_1)|, |E(M_2)| \geq 7$ could be dropped. We define *extended* 1-, 2- and 3-sums by omitting these restrictions. Clearly, Theorem 4.2 holds if we replace sums by extended sums in the definition of the $\{1, 2, 3\}$-decomposition. To simplify notation, we use $\oplus_1, \oplus_2, \oplus_3$ and $\oplus$ to denote these extended sums. Finally, we also need the notion of a conflict graph associated with a $\{1, 2, 3\}$-*decomposition* of a matroid M given by Dinitz and Kortsarz in [8].

DEFINITION 4.3 ([8]). *Let* $(T, \mathcal{M})$ *be a* $\{1, 2, 3\}$*-decomposition of a matroid* M*. The* intersection *(or* conflict*) graph of* $(T, \mathcal{M})$ *is the graph* G_T *with the vertex set* $\mathcal{M}$ *such that distinct* $M_1, M_2 \in \mathcal{M}$ *are adjacent in* G_T *if and only if* $E(M_1) \cap E(M_2) \neq \emptyset$*.*

Dinitz and Kortsarz in [8] showed how to modify a given decomposition in order to make the conflict graph a forest. In fact they proved a slightly stronger condition that for any 3-sum (which by definition is summed along a circuit of size 3), the circuit in the intersection is contained entirely in two of the lowest-level matroids. In other words, while the process of summing matroids might create new circuits that contain elements that started out in different matroids, any circuit that is used as the intersection of a sum existed from the very beginning.

Let $(T, \mathcal{M})$ be a $\{1, 2, 3\}$-decomposition of a matroid M. A node of $V(T)$ with degree at least 2 is called an *internal* node of T. Note that with each internal node t of T one can associate a matroid M_t, but also the set of elements that is the intersection of the ground sets of its children (and is thus not in the ground set of M_t). This set is either the empty set (if M_t is the 1-sum of its children), a single element (if it is the 2-sum), or three elements that form a circuit in both of its children (if it is the 3-

sum). For an internal node t, let Z_{M_t} denote this set. Essentially, corresponding to an internal node of $t \in V(T)$ with children t_1 and t_2, denote by $Z_{M_t} = E(M_{t_1}) \cap E(M_{t_2})$ its *sum-set*.

Let t be an internal node of T and t_1 and t_2 be its children. Using the terminology of Dinitz and Kortsarz in [8], we say that Z_{M_t} is *good* if all the elements of Z_{M_t} belong to the same basic matroid that is a descendant of M_{t_1} in T and they belong to the same basic matroid that is a descendant of M_{t_2} in T. We say that a $\{1,2,3\}$-decomposition of M is *good* if all the sum-sets are good. We state the result of [8] in the following form that is convenient for us.

THEOREM 4.4 ([8]). *Every regular matroid M has a good $\{1,2,3\}$-decomposition in which every basic matroid is either graphic, cographic, or isomorphic to a matroid obtained from R_{10} by (possibly) adding parallel elements. Moreover, such a decomposition (together with the graphs whose cycle and bond matroids are isomorphic to the corresponding basic graphic and cographic matroids) can be found in time polynomial in $||M||$.*

Using this theorem, for a given regular matroid, we can obtain in polynomial time a good $\{1,2,3\}$-decomposition with a collection $\mathcal{M}$ of basic matroids, where every basic matroid is either graphic, or cographic, or is isomorphic to a matroid obtained from R_{10} by deleting some elements and adding parallel elements and deleting. Then we obtain a conflict forest G_T, whose nodes are basic matroids and the edges correspond to extended 2- or 3-sums such that their sum-sets are the elements of the basic matroids that are the endpoints of the corresponding edge. By adding bridges between components of G_T corresponding to 1-sums, we obtain a *conflict tree* $\mathcal{T}$ for a good $\{1,2,3\}$-decomposition, whose edges correspond to extended 1, 2 or 3-sums between adjacent matroids. Hence we obtain the following corollary.

COROLLARY 4.5. *For a given regular matroid M, there is a (conflict) tree $\mathcal{T}$ whose set of nodes is a set of matroids $\mathcal{M}$, where each element of $\mathcal{M}$ is a graphic or cographic matroid, or a matroid obtained from R_{10} by adding (possibly) parallel elements, that has the following properties:*

i) if two distinct matroids $M_1, M_2 \in \mathcal{M}$ have nonempty intersection, then M_1 and M_2 are adjacent in $\mathcal{T}$,

ii) for any distinct $M_1, M_2 \in \mathcal{M}$, $|E(M_1) \cap E(M_2)| = 0,\ 1$ or 3,

iii) M is obtained by the consecutive performing extended 1, 2 or 3-sums for adjacent matroids in any order.

Moreover, $\mathcal{T}$ can be constructed in a polynomial time.

If $\mathcal{T}$ is a conflict tree for a matroid M, we say that M is defined by $\mathcal{T}$.

5. Elementary reductions for SPACE COVER.

In this section we give some elementary reduction rules that we apply on the instances of SPACE COVER and RESTRICTED SPACE COVER to make it more structured and thus easier to design an FPT algorithm. Throughout this section we will assume that the input matroid $M = (E, \mathcal{I})$ is regular.

5.1. Reduction rules for SPACE COVER.

Let (M, w, T, k) be an instance of SPACE COVER, where M is a regular matroid. For technical reasons, we permit the weight function w to assign 0 to the elements of E. However, observe that if M has a nonterminal element e with $w(e) = 0$, then we can always include it in a (potential) solution. This simple observation is formulated in the following reduction rule.

REDUCTION RULE 5.1 (**Zero-element reduction rule**). *If there is an element*

$e \in E \setminus T$ with $w(e) = 0$, then contract e.

The next rule is used to remove irrelevant terminals.

Reduction Rule 5.2 (**Terminal circuit reduction rule).** *If there is a circuit $C \subseteq T$, then delete an arbitrary element $e \in C$ from M.*

Lemma 5.1. *Reduction Rule 5.2 is safe.*

Proof. We first prove the forward direction. Suppose that there is a circuit $C \subseteq T$ and $e \in C$. Clearly, if $F \subseteq E \setminus T$ spans T, then F spans $T \setminus \{e\}$ as well. For the reverse direction, assume that $F \subseteq E \setminus T$ spans $T \setminus \{e\}$. Let $C = \{e, e_1, \ldots, e_r\}$. Since $F \subseteq E \setminus T$ spans $T \setminus \{e\}$, there are circuits $C_1, \ldots, C_r$ such that $e_i \in C_i \subseteq F \cup \{e_i\}$. By Observation 3.4, $\tilde{C} = (C_1 \triangle \ldots \triangle C_r) \triangle C$ is a cycle. However, observe that $\tilde{C}$ only contains elements from $F \cup \{e\}$. In fact, since $e \notin C_i$ for $i \in \{1, \ldots, r\}$, $e \in \tilde{C}$ and thus there is a circuit C' such that $e \in C' \subseteq \tilde{C}$. This implies that $e \in C' \subseteq F \cup \{e\}$ and thus F spans e. This completes the proof. ∎

Now we remove irrelevant nonterminals. It is clearly safe to delete loops as there always exists a solution F such that $F \in \mathcal{I}$.

Reduction Rule 5.3 (**Loop reduction rule).** *If $e \in E \setminus T$ is a loop, then delete e.*

We remark that it is safe to apply Reduction Rule 5.3 even for Restricted Space Cover. Our next rule removes parallel elements.

Reduction Rule 5.4 (**Parallel reduction rule).** *If there are two elements $e_1, e_2 \in E \setminus T$ such that e_1 and e_2 are parallel and $w(e_1) \leq w(e_2)$, then delete e_2.*

Lemma 5.2. *Reduction Rule 5.4 is safe.*

Proof. Let $e_1, e_2 \in E \setminus T$ be parallel elements such that $w(e_1) \leq w(e_2)$. Then, by Observations 3.2, for any $F \subseteq E \setminus T$ that spans T such that $e_2 \in F$, $F' = (F \setminus \{e_2\}) \cup \{e_1\}$ also spans T. Hence, it is safe to delete e_2. ∎

To sort out the trivial yes-instance or no-instance after the exhaustive applications of the above rules, we apply the next rule.

Reduction Rule 5.5 (**Stopping rule).** *If $T = \emptyset$, then return yes and stop. Else, if $E \setminus T = \emptyset$ or $|T| > k$, then return no and stop.*

Lemma 5.3. *Reduction Rule 5.5 is safe.*

Proof. Indeed if $T = \emptyset$, then we have a yes-instance of the problem, and if $T \neq \emptyset$ and $E \setminus T = \emptyset$, then the considered instance is a no-instance. If we cannot apply Reduction Rule 5.2 (**Terminal circuit reduction rule**), then T is an independent set of M. Hence, if $F \subseteq E \setminus T$ spans T, $|F| \geq |T|$. Since we have no element with zero weight after the exhaustive application of Reduction Rule 5.1 (**Zero-element reduction rule**), if $k < |T|$, then the input instance is a no-instance. ∎

5.2. Reduction rules for Restricted Space Cover. For Restricted Space Cover, we use the following modifications of Reduction Rules 5.1-5.5, where applicable. Proofs of these rules are analogous to its counter-part for Space Cover and thus omitted.

Reduction Rule 5.6 (**Zero-element reduction rule*).** *If there is an element $e \in E \setminus (T \cup \{e^*\})$ with $w(e) = 0$, then contract e.*

Reduction Rule 5.7 (**Terminal circuit reduction rule*).** *If there is a circuit $C \subseteq T$, then delete an arbitrary element $e \in C$ such that $e \neq t^*$ from M. If t^* is* *a loop, then delete* t^*.

REDUCTION RULE 5.8 (**Parallel reduction rule***). *If there are two elements* $e_1, e_2 \in E \setminus T$ *such that* e_1 *and* e_2 *are parallel,* $e_1 \neq e^*$ *and* $w(e_1) \leq w(e_2)$*, then delete* e_2.

Since $w(e^*) = 0$, we obtain the following variant of Reduction Rule 5.5.

REDUCTION RULE 5.9 (**Stopping rule***). *If* $T = \emptyset$*, then return yes and stop. Else, if* $E \setminus T = \emptyset$ *or* $|T| > k + 1$*, then return no and stop.*

5.3. Final lemma. If we have an independence oracle for $M = (E, \mathcal{I})$ or if we can check in polynomial time using a given representation of M whether a given subset of E belongs to $\mathcal{I}$, then we get the following lemma.

LEMMA 5.4. *Reduction Rules 5.1-5.9 can be applied in time polynomial in* $||M||$.

6. Solving Space Cover for basic matroids. In this section we solve (RESTRICTED) SPACE COVER on basic matroids that are building blocks of regular matroid. In particular, we solve SPACE COVER for R_{10} and (RESTRICTED) SPACE COVER for graphic and cographic matroids. We first give an algorithm on R_{10}, followed by algorithms on graphic matroids based on algorithms for STEINER TREE and its generalization. Finally, we give algorithms on cographic matroids based on ideas inspired by important separators.

6.1. SPACE COVER on R_{10}. In this section we give an algorithm for SPACE COVER about matroids that could be obtained from R_{10} by either adding parallel elements, or by deleting elements or by contracting elements. Observe that an instance of (RESTRICTED) SPACE COVER for such a matroid is reduced to an instance with a matroid that has at most 20 elements by the exhaustive application of **Terminal circuit reduction rule** and **Parallel reduction rule**. Indeed, in the worst case we obtain the matroid from R_{10} by adding exactly one parallel element for each element of R_{10}. Since the matroid, $M = (E, \mathcal{I})$, of the reduced instance has at most 20 elements we can solve SPACE COVER by examining all subsets of E of size at most k. This brings us to the following.

LEMMA 6.1. SPACE COVER *can be solved in polynomial time for matroids that can be obtained from* R_{10} *by adding parallel elements, element deletions and contractions.*

6.2. SPACE COVER for graphic matroids. Recall that STEINER FOREST restated below can be seen as a special case of SPACE COVER on graphic matroids by a simple reduction.

> STEINER FOREST **Parameter:** k
> **Input:** A (multi) graph G, a weight function $w\colon E \to \mathbb{N}$, a collection of pairs of distinct vertices (*demands*) $\{x_1, y_1\}, \ldots, \{x_r, y_r\}$ of G, and a nonnegative integer k
> **Question:** Is there a set $F \subseteq E(G)$ with $w(F) \leq k$ such that for any $i \in \{1, \ldots, r\}$, $G[F]$ contains an (x_i, y_i)-path?

In this section, we "reverse this reduction" in a sense and use this reversed reduction to solve (RESTRICTED) SPACE COVER. In particular we utilize an algorithm for STEINER FOREST to give an FPT algorithm for (RESTRICTED) SPACE COVER on graphic matroids. It seems a folklore knowledge that STEINER FOREST is FPT when parameterized by the number of edges in a solution. We provide this algorithm here for completeness.

6.2.1. A single-exponential algorithm for Steiner Forest**.** Our algorithm is based on the FPT algorithm for the following well-known parameterization of Steiner Tree. Let us remind that in Steiner Tree, we are given a (multi) graph G, a weight function $w\colon E \to \mathbb{N}$, a set of vertices $S \subseteq V(G)$ called *terminals*, and a nonnegative integer k. The task is to decide whether there is a set $F \subseteq E(G)$ with $w(F) \leq k$ such that the subgraph of G induced by F is a tree that contains the vertices of S.

It was already shown by Dreyfus and Wagner [12] in 1971, that Steiner Tree can be solved in time $3^p \cdot n^{\mathcal{O}(1)}$, where p is the number of terminals. The current best FPT-algorithms for Steiner Tree are given by Björklund et al. [3] and Nederlof [33] (the first algorithm demands exponential in p space and the latter uses polynomial space) and runs in time $2^p \cdot n^{\mathcal{O}(1)}$. Finally, we are ready to describe the algorithm for Steiner Forest.

Lemma 6.2. Steiner Forest *is solvable in time* $4^k \cdot n^{\mathcal{O}(1)}$.

Proof. Let $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ be an instance of Steiner Forest. Consider the auxiliary graph H with $V(H) = \cup_{i=1}^{r}\{x_i, y_i\}$ and $E(H) = \{x_1, y_1\}, \ldots, \{x_r, y_r\}$. Let $S_1, \ldots, S_t$ denote the sets of vertices of the connected components of H. Recall, that a set $F \subseteq E(G)$ with $w(F) \leq k$ is said to be a *solution-forest* for Steiner Forest is for any $i \in \{1, \ldots, r\}$, $G[F]$ contains a (x_i, y_i)-path. Now notice that $F \subseteq E(G)$, of weight at most k, is a solution-forest to an instance $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ of Steiner Forest if and only if the vertices of S_i are in the same component of $G[F]$ for every $i \in \{1, \ldots, t\}$. We will use this correspondence to obtain an algorithm for Steiner Forest.

Now we bound the number of vertices in $V(H)$. Let F be a minimal forest-solution. First of all observe that since the weights on edges are positive, we have that $|F| \leq k$. The vertices of S_i must be in the same component of $G[F]$, thus we have that $|F| \geq \sum_{i=1}^{t}(|S_i| - 1)$. Hence, $\sum_{i=1}^{t}|S_i| \leq |F| + t$. If $\sum_{i=1}^{t}|S_i| > |F| + t$ we return that $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ is a no-instance. So from now onwards assume that $\sum_{i=1}^{t}|S_i| \leq |F| + t$. Furthermore, since F is a minimal forest-solution, each connected component of $G[F]$ has size at least 2 and thus $t \leq k$. Thus, we have an instance with $|V(H)| \leq 2k$ and $t \leq k$.

For $I \subseteq \{1, \ldots, t\}$, let $W(I)$ denote the minimum weight of a Steiner tree for the set of terminals $\cup_{i \in I} S_i$. We assume that $W(I) = +\infty$ if such a tree does not exist. Furthermore, if the minimum weight of a Steiner tree is at least $k+1$ then also we assign $W(I) = +\infty$. All the 2^t values of $W(I)$ corresponding to $I \subseteq \{1, \ldots, t\}$ can be computed in time $2^{|V(H)|} \cdot n^{\mathcal{O}(1)} = 4^k \cdot n^{\mathcal{O}(1)}$ using the results of [3] or [33].

For $J \subseteq \{1, \ldots, t\}$, let $W'(J)$ denote the minimum weight of a solution-forest for the sets S_j, where $j \in J$. That is, $W'(J)$ is assigned the minimum weight of a set $F \subseteq E(G)$ such that the vertices of S_j for $j \in J$ are in the same component of $G[F]$. Furthermore, if such a set F does not exist or the weight is at least $k+1$ then $W'(J)$ is assigned $+\infty$. Clearly, $W'(\emptyset) = 0$. Notice that $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ is a yes-instance for Steiner Forest if and only if $W'(\{1, \ldots, t\}) \leq k$. Next, we give the recurrence relation for the dynamic programming algorithm to compute the values of $W'(J)$.

$$W'(J) = \min_{\substack{I \subseteq J \\ I \neq \emptyset}} \Big\{ W'(J \setminus I) + W(I) \Big\}. \qquad (6.1)$$

We claim that the above recurrence holds for every $J \subseteq \{1, \ldots, t\}$. To prove the forward direction of the claim, assume that $F \subseteq E(G)$ is a set of edges of minimum weight such that the vertices in S_j, $j \in J$, are in the same component of $G[F]$. Let X be a set of vertices of an arbitrary component of $G[F]$ and L denote the set of edges of this component. Let $I = \{i \in J \mid S_i \subseteq X\}$. Notice that by the minimality of F, $I \neq \emptyset$. Since $W(I) \leq w(L)$ and $W'(J \setminus I) \leq w(F \setminus L)$, we have that

$$W'(J) = w(F) = w(F \setminus L) + w(L) \geq W'(J \setminus I) + W(I) \geq \min_{\substack{I \subseteq J \\ I \neq \emptyset}} \Big\{ W'(J \setminus I) + W(I) \Big\}.$$

To show the opposite inequality, consider a nonempty set $I \subseteq J$, and let L be the set of edges of a Steiner tree of minimum weight for the set of terminals $\cup_{i \in I} S_i$ and let F be the set of edges of a Steiner forest of minimum weight for the sets of terminals S_j for $j \in J \setminus I$. Then we have that for $F' = L \cup F$, the vertices of S_i are in the same component of $G[F']$ for each $i \in J$. Hence,

$$W'(J) \leq w(L) + w(F) = W'(J \setminus I) + W(I). \tag{6.2}$$

Because (6.2) holds for any nonempty $I \subseteq J$, we have that

$$W'(J) \leq \min_{\substack{I \subseteq J \\ I \neq \emptyset}} \Big\{ W'(J \setminus I) + W(I) \Big\}.$$

We compute the values for $W'(J)$ in the increasing order of the sizes of $J \subseteq \{1, \ldots, t\}$. Towards this we use Equation 6.1 and the fact that $W'(\emptyset) = 0$. Each entry of $W'(J)$ can be computed by taking a minimum over $2^{|J|}$ pre-computed entries in W' and W. Thus, the total time to compute W' takes $(\sum_{i=0}^{t} \binom{t}{i} 2^i) \cdot n^{\mathcal{O}(1)} = 3^t \cdot n^{\mathcal{O}(1)} = 3^k \cdot n^{\mathcal{O}(1)}$. Having computed W', we return yes or no based on whether $W'(\{1, \ldots, t\}) \leq k$. This completes the proof. ∎

6.2.2. An algorithm for Space Cover **on graphic matroids.** Now using the algorithm for Steiner Forest mentioned in Lemma 6.2, we design an algorithm for Space Cover on graphic matroids.

Lemma 6.3. Space Cover *can be solved in time* $4^k \cdot ||M||^{\mathcal{O}(1)}$ *on graphic matroids.*

Proof. Let (M, w, T, k) be an instance of Space Cover where M is a graphic matroid. First, we exhaustively apply Reduction Rules 5.1-5.5. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance, where M has no loops and the weights of nonterminal elements are positive. To simplify notation, we also denote the reduced instance by (M, w, T, k). Observe that M remains to be graphic. It is well-known that given a graphic matroid, in polynomial time one can find a graph G such that M is isomorphic to the cycle matroid $M(G)$ [39]. Assume that $T = \{x_1y_1, \ldots, x_ry_r\}$ is the set of edges of G corresponding to the terminals of the instance of Space Cover. We define the graph $G' = G - T$. Recall that $F \subseteq E(G) \setminus T$ spans T if and only if for each $e \in T$, there is a cycle C of G such that $e \in C \subseteq F \cup \{e\}$. Clearly, the second condition can be rewritten as follows: for any $i \in \{1, \ldots, r\}$, $G[F]$ contains an (x_i, y_i)-path. It means that the instance $(G', w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ of Steiner Forest is equivalent to the instance (M, w, T, k) of Space Cover. Now we apply Lemma 6.2 on the instance $(G', w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, k)$ of Steiner Forest to solve the problem. ∎

6.2.3. An Algorithm for Restricted Space Cover on graphic matroids. Besides solving Space Cover, we need to solve Restricted Space Cover on graphic matroids. In fact, Restricted Space Cover can be reduced to Steiner Forest. On the other hand, we can solve this problem by modifying the algorithm for Steiner Forest from Lemma 6.2, this provides a better running time.

Lemma 6.4. Restricted Space Cover *can be solved in time* $6^k \cdot ||M||^{\mathcal{O}(1)}$ *on graphic matroids.*

Proof. Let (M, w, T, k, e^*, t^*) be an instance of Restricted Space Cover, where M is a graphic matroid. First, we exhaustively apply Reduction Rules 5.3 and 5.6-5.9. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance. Notice that it can happen that e^* is deleted by Reduction Rules 5.3 and 5.6-5.9. For example, if e^* is a loop then it can be deleted by Reduction Rule 5.3. In this case we obtain an instance of Space Cover and can solve it using Lemma 6.3. From now onwards we assume that e^* is not deleted by our reduction rules.

To simplify notation, we use (M, w, T, k, e^*, t^*) to denote the reduced instance. If we started with graphic matroid then it remains so even after applying Reduction Rules 5.3 and 5.6-5.9. Furthermore, given M, in polynomial time we can find a graph G such that M is isomorphic to the cycle matroid $M(G)$ [39]. Let $T = \{x_1y_1, \ldots, x_ry_r\}$ denote the set of edges of G corresponding to the terminals of the instance of Restricted Space Cover. Without loss of generality assume that $t^* = x_1y_1$. Let G' and G^*_e denote the graphs $G - T$ and $G - \{e^*\}$, respectively. Recall that, $F \subseteq E(G) \setminus T$ spans T if and only if for each $e \in T$, there is a cycle C of G that contains e and all the edges in C are contained in $F \cup \{e\}$. Clearly, the second condition can be rewritten as follows: for every $i \in \{1, \ldots, r\}$, $G[F]$ contains a (x_i, y_i)-path. For Restricted Space Cover, we additionally have the condition that $F \setminus \{e^*\}$ spans t^*. That is, $G[F]$ contains a (x_1, y_1)-path that does not contain e^*. In terms of graphs, we obtain a special variant of Steiner Forest. We solve the problem by slightly modifying the algorithm of Dreyfus and Wagner [12] and Lemma 6.2.

As in the proof of Lemma 6.2, we consider the auxiliary graph H with $V(H) = \cup_{i=1}^{r}\{x_i, y_i\}$ and $E(H) = \{x_1, y_1\}, \ldots, \{x_r, y_r\}$. Let $S_1, \ldots, S_t$ denote the sets of vertices of the connected components of H. Without loss of generality we assume that $x_1, y_1 \in S_1$. Let F be a minimal solution. It is clear that $G[F]$ is a forest. Notice that $F \subseteq E(G) - T$, of weight at most k, is a minimal solution to an instance $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, e^*, t^*, k)$ of Restricted Space Cover if and only if the vertices of S_i are in the same component of $G[F]$ for every $i \in \{1, \ldots, t\}$ and the unique path between x_1 and y_1 in the component containing S_1 does not contain e^*. We will use this correspondence to obtain an algorithm for the special variant of Steiner Forest and hence Restricted Space Cover.

Now we bound the number of vertices in $V(H)$. Let F be a minimal solution. First of all observe that since the weights on edges are positive, with an exception of e^*, we have that $|F| \leq k+1$. The vertices of S_i must be in the same component of $G[F]$, thus we have that $|F| \geq \sum_{i=1}^{t}(|S_i| - 1)$. Hence, $\sum_{i=1}^{t}|S_i| \leq |F| + t$. If $\sum_{i=1}^{t}|S_i| > |F| + t$ we return that $(G, w, \{x_1, y_1\}, \ldots, \{x_r, y_r\}, e^*, t^*, k)$ is a no-instance. So from now onwards assume that $\sum_{i=1}^{t}|S_i| \leq |F| + t$. Furthermore, since F is a minimal solution each connected component of $G[F]$ has size at least 2 and thus $t \leq k + 1$. Thus, we have an instance with $|V(H)| \leq 2k + 1$ and $t \leq k + 1$.

Given $I \subseteq \{1, \ldots, t\}$, by Z_I, we denote $\cup_{i \in I} S_i$. For $I \subseteq \{1, \ldots, t\}$, let $W(I)$ denote the minimum weight of a tree R in G' such that $Z_I \subseteq V(R)$ and if $x_1, y_1 \in Z_I$, then the (x_1, y_1)-path in R does not contain e^*. We assume that $W(I) = +\infty$ if such a tree does not exist. Furthermore, if the minimum weight of such a tree R is at least $k+1$ then also we assign $W(I) = +\infty$. Notice that if $|Z_I| > k+2$, then $W(I) \geq k+1$ as any tree that contains Z_I has at least $|Z_I| - 1 > k+1$ edges and only e^* can have weight 0. In this case we can safely set $W(I) = +\infty$, because we are interested in trees of weight at most k. Thus from now onwards we can assume that $|Z_I| \leq k+2$. We compute the values of $I \subseteq \{1, \dots, t\}$ such that $1 \in I$ by modifying the algorithm of Dreyfus and Wagner [12]. Next we present this modified algorithm to compute the values of W.

For a vertex $v \in V(G)$ and $X \subseteq Z_I$, let $c(v, X, \ell)$ be the minimum weight of a subtree R' of G' with at most ℓ edges such that

i) $X \subseteq V(R')$,
ii) $v \in V(R)$,
iii) if $x_1, y_1 \in X$, then the (x_1, y_1)-path in R' does not contain e^*,
iv) if $x_1 \in X$ and $y_1 \notin X$, then the (x_1, v)-path in R' does not contain e^*, and
v) if $y_1 \in X$ and $x_1 \notin X$, then the (y_1, v)-path in R' does not contain e^*.

We assume that $c(v, X, \ell) = +\infty$ if such a tree R' does not exist.

Initially we set

$$c(v, X, 0) = \begin{cases} 0 & \text{if } \{v\} = X, \\ +\infty & \text{if } \{v\} \neq X. \end{cases}$$

We compute $c(v, X, \ell)$ using the following auxiliary recurrences. For an ordered pair of vertices (u, v) such that $uv \in E(G')$,

$$c'(u, v, X, \ell) = \begin{cases} +\infty & \text{if } uv = e^* \text{ and } |X \cap \{x_1, y_1\}| = 1, \\ c(v, X, \ell - 1) + w(uv) & \text{otherwise.} \end{cases}$$

For an ordered pair of vertices (u, v) such that $uv \in E(G')$, a nonempty $Y \subseteq X$, and two nonnegative integers ℓ_1 and ℓ_2 such that $\ell = \ell_1 + \ell_2 + 1$,

$$c''(u, v, X, Y, \ell_1, \ell_2) = \begin{cases} +\infty & \text{if } uv = e^* \text{ and} \\ & |Y \cap \{x_1, y_1\}| = 1, \\ c(u, X \setminus Y, \ell_1) & \\ +c(v, Y, \ell_2) + w(uv) & \text{otherwise.} \end{cases}$$

Finally,

$$c(u, X, \ell) = \min \Big\{ c(u, X, \ell - 1), \min_{v \in N_{G'}(u)} c'(u, v, X), \min_{v \in N_{G'}(u)} \Big\{ c''(u, v, X \setminus Y, Y, \ell_1, \ell_2) \mid \emptyset \neq Y \subseteq X, \ell_1 + \ell_2 = \ell - 1 \Big\} \Big\}.$$

For all $v \in V(G)$, we fill the table $c(v, \cdot, \cdot)$ as follows. We iteratively consider the values of ℓ starting from 1 and ending at k and for each value of ℓ we consider the subsets of Z_I in the increasing order of their size. If there is a vertex $v \in V(G)$ with $c(v, Z_I, k+1) \leq k$ then we set $W(I) = c(v, Z_I, k+1)$, else, we set $W(I) = +\infty$.

The correctness of the computation of $W(I)$ can be proved by standard dynamic programming arguments. In fact, it essentially follows along the lines of the proof of Dreyfus and Wagner [12]. The only difference is that we have to take into account the conditions *iii*) to *v*) that are used to ensure that the (x_1, y_1)-path in the obtained tree avoids e^*. Since $|Z| \leq k+2$, the computation of $W(I)$ for a given I can be done in time $3^k \cdot n^{\mathcal{O}(1)}$. Thus, all the 2^t values of $W(I)$ corresponding to $I \subseteq \{1,\ldots,t\}$ such that $1 \in I$ can be computed in time $6^k \cdot n^{\mathcal{O}(1)}$.

Next, we show how we can compute $W(I)$ for $I \subseteq \{2,\ldots,t\}$. Recall that $x_1, y_1 \in S_1$ and thus for $I \subseteq \{2,\ldots,t\}$, $W(I)$ just denotes the minimum weight of a Steiner tree for the set of terminals Z_I in the graph G'. Hence, for $I \subseteq \{2,\ldots,t\}$, we can compute $W(I)$ by using the algorithm of Dreyfus and Wagner [12] without modification. We could also compute $W(I)$ using the results of [3] or [33]. Thus, we can compute all the 2^t values of $W(I)$ corresponding to $I \subseteq \{1,\ldots,t\}$ in $6^k \cdot n^{\mathcal{O}(1)}$ time.

Now we use the table W to solve the instance (M, w, T, k, e^*, t^*) of RESTRICTED SPACE COVER. As in the proof of Lemma 6.2, for each $J \subseteq \{1,\ldots,t\}$, denote by $W'(J)$ the minimum weight of a set $F \subseteq E(G')$ such that the vertices of Z_J are in the same component of $G'[F]$ and if $1 \in J$ then the (x_1,y_1)-path in $G'[F]$ avoids e^*. Furthermore, if such a set F does not exist or has weight at least $k+1$ then we set $W'(J) = +\infty$.

Clearly, $W'(\emptyset) = 0$. Notice that (M, w, T, k, e^*, t^*) is a yes-instance for RESTRICTED SPACE COVER if and only if $W'(\{1,\ldots,t\}) \leq k$. Next we give the recurrence relation for the dynamic programming algorithm to compute the values of $W'(J)$.

$$W'(J) = \min_{\substack{I \subseteq J \\ I \neq \emptyset}} \Big\{ W'(J \setminus I) + W(I) \Big\}. \tag{6.3}$$

The proof of the correctness of the recurrence given in Equation 6.3 is verbatim same to the proof of recurrence given in Equation 6.1 in the proof of Lemma 6.2.

We compute the values for $W'(J)$ in the increasing order of size of $J \subseteq \{1,\ldots,t\}$. Towards this we use Equation 6.3 and the fact that $W'(\emptyset) = 0$. Each entry of $W'(J)$ can be computed by taking a minimum over $2^{|J|}$ pre-computed entries in W' and W. Thus, the total time to compute W' takes $(\sum_{i=0}^{t} \binom{t}{i} 2^i) \cdot n^{\mathcal{O}(n)} = 3^t \cdot n^{\mathcal{O}(1)} = 3^k \cdot n^{\mathcal{O}(1)}$. Having computed W' we return yes or no based on whether $W'(\{1,\ldots,t\}) \leq k$. This completes the proof. ∎

6.3. (RESTRICTED) SPACE COVER **for cographic matroids.** In this section we design algorithms for (RESTRICTED) SPACE COVER on co-graphic matroids. By the results of Xiao and Nagamochi [45], SPACE COVER can be solved in time $2^{\mathcal{O}(k \log k)} \cdot ||M||^{\mathcal{O}(1)}$, but to obtain a single-exponential in k algorithm we use a different approach based on the enumeration of *important separators* proposed by Marx in [31]. However, for our purpose we use the similar notion of *important cuts* and we follow the terminology given in [5] to define these objects.

To introduce this technique, we need some additional definitions. Let G be a graph and let $X, Y \subseteq V(G)$ be disjoint. A set of edges S is an $X-Y$ *separator* if S separates X and Y in G, i.e., every path that connects a vertex of X with a vertex of Y contains an edge of S. If X is a single element set $\{u\}$, we simply write that S is a $u-Y$ separator. An $X-Y$-separator is *minimal* if it is an inclusion minimal $X-Y$ separator. It will be convenient to look at minimal (X,Y)-cuts from a different perspective, viewing them as edges on the boundary of a certain set of vertices. If G is an undirected graph and $R \subseteq V(G)$ is a set of vertices, then we denote by $\Delta_G(R)$ the set of edges with exactly one endpoint in R, and we denote $d_G(R) = |\Delta_G(R)|$ (we omit the subscript G if it is clear from the context). We say that a vertex y is *reachable* from a vertex x in a graph G if G has an (x,y)-path. For a set X, a vertex y

is reachable from X if it is reachable from a vertex of X. Let S be a minimal (X,Y)-cut in G and let $R_G(X)$ be the set of vertices reachable from X in $G \setminus S$; clearly, we have $X \subseteq R_G(X) \subseteq V(G) \setminus Y$. Then it is easy to see that S is precisely $\Delta(R_G(X))$. Indeed, every such edge has to be in S (otherwise a vertex of $V(G) \setminus R$ would be reachable from X) and S cannot have an edge with both endpoints in $R_G(X)$ or both endpoints in $V(G) \setminus R_G(X)$, as omitting any such edge would not change the fact that the set is an (X,Y)-cut, contradicting minimality. When the context is clear we omit the subscript and the set X while defining R.

Proposition 6.5 ([5]). *If S is a minimal (X,Y)-cut in G, then $S = \Delta_G(R)$, where R is the set of vertices reachable from X in $G \setminus S$.*

Therefore, we may always characterize a minimal (X,Y)-cut S as $\Delta(R)$ for some set $X \subseteq R \subseteq V(G) \setminus Y$.

Definition 6.6. [5, Definition 8.6] [Important cut] *Let G be an undirected graph and let $X, Y \subseteq V(G)$ be two disjoint sets of vertices. Let $S \subseteq E(G)$ be an (X,Y)-cut and let R be the set of vertices reachable from X in $G \setminus S$. We say that S is an* important (X,Y)-cut *if it is inclusion-wise minimal and there is no (X,Y)-cut S' with $|S'| \leq |S|$ such that $R \subset R'$, where R' is the set of vertices reachable from X in $G \setminus S'$.*

Theorem 6.7. [30, 32], [5, Theorems 8.11 and 8.13] *Let $X, Y \subseteq V(G)$ be two disjoint sets of vertices in graph G and let $k \geq 0$ be an integer. There are at most 4^k important (X,Y)-cuts of size at most k. Furthermore, the set of all important (X,Y)-cuts of size at most k can be enumerated in time $\mathcal{O}(4^k \cdot k \cdot (n+m))$.*

For a partition (X,Y) of the vertex set of a graph G, we denote by $E(X,Y)$ the set of edges with one end vertex in X and the other in Y. For a set of bridges B of a graph G and a bridge $uv \in B$, we say that u is a *leaf with respect to B*, if the component of $G - B$ that contains u has no end vertex of any edge of $B \setminus \{uv\}$. Clearly, for any set of bridges, there is a leaf with respect to it. Also we can make the following observation.

Observation 6.1. *For the bond matroid $M^*(G)$ of a graph G and $T \subseteq E(G)$, a set $F \subseteq E(G) \setminus T$ spans T if and only if the edges of T are bridges of $G - F$.*

6.3.1. An algorithm for Space Cover **on cographic matroids.** For our purpose we need a slight modification to the definition of important cuts. We start by defining the object we need and proving a combinatorial upper bound on it.

Definition 6.1. *Let G be a graph $s \in V(G)$ be a vertex and $T \subseteq V(G) \setminus \{s\}$ be a subset of terminals. We say that a set $W \subseteq V(G)$ is interesting if (a) $G[W]$ is connected, (b) $s \in W$ and $|T \cap W| \leq 1$.*

Next we define a partial order on all interesting sets of a graph.

Definition 6.2. *Let G be a graph $s \in V(G)$ be a vertex and $T \subseteq V(G) \setminus \{s\}$ be a subset of terminals. Given two interesting sets W_1 and W_2 we say that W_1 is better than W_2 and denote by $W_2 \preceq W_1$ if (a) $W_2 \subseteq W_1$, $|\Delta(W_1)| \leq |\Delta(W_2)|$ and $T \cap W_1 \subseteq T \cap W_2$.*

Definition 6.3. *Let G be a graph $s \in V(G)$ be a vertex, $T \subseteq V(G) \setminus \{s\}$ be a subset of terminals and k be a nonnegative integer. We say that an interesting set W is a (s,T,k)-*semi-important set *if $|\Delta(W)| \leq k$ and there is no set W' such that $W \preceq W'$. That is, W is a maximal set under the relation $\preceq$. Furthermore, $\Delta(W)$ corresponding to a (s,T,k)-semi-important set is called a (s,T,k)-*semi-important cut.

Now we have all the necessary definitions to state our lemma that upper bounds the number of semi-important sets and semi-important cuts.

LEMMA 6.8. *For every graph G, a vertex $s \in V(G)$, a subset $T \subseteq V(G) \setminus \{s\}$ and a nonnegative integer k, there are at most $4^k(1+4^{k+1})$ (s,T,k)-semi-important cuts with $|\Delta(W)| = k$. Moreover, all such sets can be listed in time $16^k n^{\mathcal{O}(1)}$.*

Proof. Observe that (s,T,k)-semi-important cuts and (s,T,k)-semi-important sets are in bijective correspondence and thus bounding one implies a bound on the other. In what follows we upper bound the number of (s,T,k)-semi-important sets. Let $\mathcal{F}$ denote the set of all (s,T,k)-semi-important sets. There are two kinds of (s,T,k)-semi-important sets, those that do not contain any vertex of T and those that contain exactly one vertex of T. We denote the set of (s,T,k)-semi-important sets of first kind by $\mathcal{F}_0$ and the second kind by $\mathcal{F}_1$. We first bound the size of $\mathcal{F}_0$. We claim that for every set $W \in \mathcal{F}_0$, $\Delta(W)$ is an important (s,T)-cut of size k in G. For a contradiction assume that there is a set $W \in \mathcal{F}_0$ such that $\Delta(W)$ is not an important (s,T)-cut of size k in G. Then there exists a set W' such that $W \subsetneq W'$, $s \in W'$, $W' \cap T = \emptyset$ and $|\Delta(W')| \leq |\Delta(W)|$. However, this implies that $W \preceq W'$ – a contradiction. Thus, for every set $W \in \mathcal{F}_0$, $\Delta(W)$ is an important (s,T)-cut of size k in G and thus, by Theorem 6.7 we have that $|\mathcal{F}_0| \leq 4^k$.

Now we bound the size of $\mathcal{F}_1$. Towards this we first modify the given graph G and obtain a new graph G'. We first add a vertex $t \notin V(G)$ as a sink terminal. Then for every vertex $v_i \in T$ we add $k+1$ new vertices $Z_i = \{v_i^1, \ldots, v_i^{k+1}\}$ and add an edge $v_i z$, for all $z \in Z_i$. Now for every vertex $v_i^j \in Z_i$ we make $2k+3$ new vertices $Z_i^j = \{v_i^{j_1}, \ldots, v_i^{j_{2k+3}}\}$ and add an edge tz, for all $z \in Z_i^j$. Now we claim that for every set $W \in \mathcal{F}_1$, $\Delta(W)$ is an important (s,t)-cut of size $2k+1$ in G'. For a contradiction assume that there is a set $W \in \mathcal{F}_1$ such that $\Delta(W)$ is not an important (s,t)-cut of size $2k+1$ in G'. Then there exists a set W' such that $W \subsetneq W'$, $s \in W'$, $W' \cap \{t\} = \emptyset$ and $|\Delta(W')| \leq |\Delta(W)|$. That is, $\Delta(W')$ is an important cut dominating $\Delta(W)$. Since $W \in \mathcal{F}_1$, there exists a vertex (exactly one) say $w \in T$ such that $w \in W$. Observe that W' can not contain (a) any vertex but w from T and (b) any vertex from the set Z_i, $v_i \in T$. If it does then $|\Delta(W')|$ will become strictly more than $2k+1$. This together with the fact that $G[W']$ is connected we have that it does not contain any newly added vertex. That is, $W' \subseteq V(G)$ and contains only w from T. However, this implies that $W \preceq W'$ – a contradiction. Thus, for every set $W \in \mathcal{F}_1$, $\Delta(W)$ is an important (s,t)-cut of size $2k+1$ in G' and thus, by Theorem 6.7 we have that $|\mathcal{F}_1| \leq 4^{2k+1}$. Thus, $|\mathcal{F}_0| + |\mathcal{F}_1| \leq 4^k + 4^{2k+1}$. This concludes the proof. ◻

LEMMA 6.9. *Let $M^*(G)$ be the bond matroid of G, $T \subseteq E(G)$, and suppose that $F \subseteq E(G) \setminus T$ spans T. Let also x be an end vertex of an edge xy of T such that x is either in a leaf block or in a degree two block in $G - F$, Y is the set of end vertices of the edges of T distinct from x, $G' = G - T$ and let $W = R_{G'-F}(x)$. Then there is a (x,Y,k)-semi-important set W' such that $|\Delta_{G'}(W')| \leq |\Delta_{G'}(W)|$ and $F' = (F \setminus \Delta_{G'}(W)) \cup \Delta_{G'}(W')$ spans T in $M^*(G)$.*

Proof. It is clear that W is an interesting set. If W is a semi-important set and $\Delta_{G'}(W)$ is a (x,Y,k)-semi-important cut of G', then the claim holds for $W' = W$. Assume that $\Delta_{G'}(W)$ is not a (x,Y,k)-semi-important cut. Then there is a (x,Y,k)-semi-important set W' of G' such that $W \preceq W'$. Recall that this implies that (a) $G'[W']$ is connected, (b) $W \subsetneq W'$, (c) $s \in W'$, (d) $|Y \cap W'| \leq 1$ and $|\Delta_{G'}(W')| \leq |\Delta_{G'}(W)|$. Since G' does not have any edge of T we have that $\Delta_{G'}(W') \cap T = \emptyset$. Hence, $F' = (F \setminus \Delta_{G'}(W)) \cup \Delta_{G'}(W')$ is disjoint from T. That is, $F' \subseteq E(G) \setminus T$.

To prove that F' spans T, it is sufficient to show that for every $uv \in T$, there is a minimal cut-set C^*_{uv} of G such that $uv \in C^*_{uv} \subseteq F' \cup \{uv\}$. Let $uv \in T \setminus \{xy\}$. To obtain a contradiction, suppose there is no *minimal* cut-set $\hat{C}_{uv}$ in G such that $uv \in \hat{C}_{uv} \subseteq F' \cup \{uv\}$. Then, there is a (u, v)-path P in G such that P has no edge of $F' \cup \{uv\}$. On the other hand G has a cut-set C_{uv} such that $uv \in C_{uv} \subseteq F \cup \{uv\}$. This implies that every path between u and v that exists in $G - (F' \cup \{uv\})$, including P, must contain an edge of C_{uv} such that it is present in $\Delta_{G'}(W)$ (these are the only edges we have removed from F). By our assumption we know that P does not contain any edge from $\Delta_{G'}(W)$ (else we will be done). Now we know that W can contain at most one vertex from Y. Since W does not contain both end-points of an edge in T we have that at most one of u or v belongs to W. First let us assume that $W \cap \{u, v\} = \emptyset$. Thus by the definition of semi-important set, $W' \cap Y \subseteq W \cap Y$, we have that u, v is outside of W'. However, we know that $\Delta_{G'}(W)$ contains an edge of P and thus contains a vertex $z \in W$ that is on P. Since $W \subsetneq W'$ we have that $\Delta_G(W')$ contains at least two edges of P. However, none of these edges are present in $\Delta_{G'}(W')$. The only edges G' misses are those in T and thus the edges present in $\Delta_G(W') \cap E(P)$ must belong to T. Let Z denote the set of end-points of edges in $\Delta_G(W') \cap E(P)$. Observe that, $Z \cap S' = Z \cap S$. Let z_1 denote the first vertex on P belonging to W' (or W) and z_2 denote the last vertex on P belonging to W' (or W), respectively, when we walk along the path P starting from u. Since z_1 and z_2 belongs to W and $G[W]$ is connected we have that there is a path $Q_{z_1 z_2}$ in $G[W]$. Let P_{uz_1} denote the subpath of P between u and z_1 and let $P_{z_2 v}$ denote the subpath of P between z_2 and v. This implies that the path P' between u and v obtained by concatenating $P_{uz_1} Q_{z_1 z_2} P_{z_2 v}$ does not intersect $\Delta_{G'}(W)$. Observe that P' does not contain any edge of $\Delta_{G'}(W)$ and $F' \cup \{uv\}$. This is a contradiction to our assumption that every path between u and v that exists in $G - (F' \cup \{uv\})$ must contain an edge of C_{uv} such that it is present in $\Delta_{G'}(W)$.

Now we consider the case when $|W \cap \{u, v\}| = 1$ and say $W \cap \{u, v\}$ is u. We know that $\Delta_{G'}(W)$ contains an edge of P. Since $W \subsetneq W'$ we have that $\Delta_G(W')$ also contains at least one edge of P. However, none of these edges are present in $\Delta_{G'}(W')$. The only edges G' misses are those in T and thus the edges present in $\Delta_G(W') \cap E(P)$ must belong to T. Let Z denote the set of end-points of edges in $\Delta_G(W') \cap E(P)$. Observe that, $Z \cap S' = Z \cap S$. Let z_1 denote the first vertex on P belonging to W' (or W) when we walk along the path P starting from v. Since z_1 and u belongs to W and $G[W]$ is connected we have that there is a path Q_{uz_1} in $G[W]$. Let $P_{w_1 v}$ denote the subpath of P between w_2 and v. This implies that the path P' between u and v obtained by concatenating $P_{uz_1} P_{z_1 v}$ does not intersect $\Delta_{G'}(W)$. Observe that P' does not contain any edge of $\Delta_{G'}(W)$ and $F' \cup \{uv\}$. This is a contradiction to our assumption that every path between u and v that exists in $G - (F' \cup \{uv\})$ must contain an edge of C_{uv} such that it is present in $\Delta_{G'}(W)$. This completes the proof.□

Lemma 6.10. Space Cover *can be solved in time* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ *on cographic matroids.*

Proof. Let (M, w, T, k) be an instance of Space Cover, where M is a cographic matroid.

First, we exhaustively apply Reduction Rules 5.1-5.5. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance, where M has no loops, the weights of nonterminal elements are positive and $|T| \leq k$. To simplify notation, we also denote the reduced instance by (M, w, T, k). Observe that M remains to be cographic. It is well-known that given a cographic matroid, in polynomial time one can find a graph G such that M is isomorphic to the bond matroid $M^*(G)$ [39].

Next, we replace the weighted graph G by the unweighted graph G' as follows. For any nonterminal edge uv, we replace uv by $w(uv)$ parallel edges with the same end vertices u and v if $w(uv) \leq k$, and we replace uv by $k+1$ parallel edges if $w(uv) > k$. There is $F \subseteq E(G) \setminus T$ of weight at most k such that F spans T in G if and only if there is $F' \subseteq E(G') \setminus T$ of size at most k such that F' spans T in G'. In other words, we have that $I = (M^*(G'), w', T, k)$, where $w'(e) = 1$ for $e \in E(G')$, is an equivalent instance of the problem. Notice that Reduction Rule 5.2 (**Terminal circuit reduction rule**) for $M^*(G')$ can be restated as follows: if there is a minimal cut-set $R \subseteq T$, then contract any edge $e \in R$ in the graph G'.

It is well known that if H is a forest on n vertices then there are at least $\frac{n}{2}$ vertices of degree at most two. Suppose that I is a yes-instance, and $F \subseteq E(G') \setminus T$ of size at most k spans T. We know that in $G' - F$ every edge of T is a bridge and we let the degree of a connected component C of $G' - F - T$, denoted by $d^*(C, G' - F - T)$, be equal to the number of edges of T it is incident to. Notice that if we shrink each connected component to a single vertex then we get a forest on at most $|T|+1 \leq k+1$ vertices and thus there are at least $|T|/2$ components such that $d^*(C, G' - F - T)$ is at most two. Let $I = (M^*(G'), w', T, k)$ denote our instance. Let Q denote the set of end vertices of edges in T and $Z \subseteq Q$. We assume by guessing all possibilities in Step 3 that Z has the following property: If I is a yes-instance with a solution $F \subseteq E(G') \setminus T$, then Z is the set of end vertices of terminals that are in the connected components C of $G - F - T$ such that $d^*(C, G' - F - T) \leq 2$. Initially $Z = \emptyset$.

Algorithm ALG-CGM takes as instance (I, Q, Z) and executes the following steps.

1. While there is a minimal cut-set $R \subseteq T$ of G do the following. Denote by $Z_1 \subseteq Z$ the set of $z \in Z$ such that z is incident to exactly one $t \in T$, and let $Z_2 \subseteq Z$ be the set of $z \in Z$ such that z is incident to two edges of T. Clearly, Z_1 and Z_2 form a partition of Z. Find a minimal cut-set $R \subseteq T$ and select $xy \in R$. Contract xy and denote the contracted vertex by z. Set $T = T \setminus \{xy\}$ and recompute Q. If $x, y \in Z_1$ or if $x \notin Z$ or $y \notin Z$, then set $Z = Z \setminus \{x, y\}$. Otherwise, if $x, y \in Z$ and $\{x, y\} \cap Z_2 \neq \emptyset$, set $Z = (Z \setminus \{x, y\}) \cup \{z\}$.
2. If Z is empty go to next step. Else, pick a vertex $s \in Z$ and finds all the (s, Y, k) semi-important set W in $G' - T$ such that $\Delta(W) \leq k$, where $Y = W \setminus \{s\}$, using Lemma 6.8. For each such semi-important set W, we call the algorithm ALG-CGM on $(M^*(G' - \Delta(W)), w', T, k - |\Delta(W)|)$, W and Z. By Lemma 6.9, I is a yes-instance if and only if one of the obtained instances is a yes-instance of Space Cover.
3. Guess a subset $Z \subseteq Q$ with the property that if I is a yes-instance with a solution $F \subseteq E(G') \setminus T$, then Z is the set of end vertices of terminals that are in the connected components C of $G - F - T$ such that $d^*(C, G' - F - T) \leq 2$. In particular, we do not include in Z the vertices that are incident to at least 3 edges of T. Now call ALG-CGM on (I, Q, Z). By the properties of the forest we know that the size of $|Z| \geq \frac{|T|}{2}$.

Notice that because on Step 2 there are no minimal cut-sets $R \subseteq T$, for each considered semi-important set W, $\Delta(W)$ is not empty. It means that the parameter decreases in each recursive call. Moreover, by considering semi-important cuts of size i for $i = \{1, \dots, k\}$, we decrease the parameter by at least i. Let $\ell = |Q| - |Z|$. Because there are at most $4^i(1 + 4^{i+1})$ semi-important sets of size i, we have the following recurrences for the algorithm:

$$T(\ell,k) \le 2^{\ell} T\left(\ell - \frac{\ell}{4}, k\right) \tag{6.4}$$

$$T(\ell,k) \le \sum_{i=1}^{k} (4^i(1+4^{i+1}))T(\ell, k-i) \tag{6.5}$$

By induction hypothesis we can show that the above recurrences solve to $16^{\ell}84^k$. Since $\ell \le 2k$ we get that the above algorithm runs in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. This completes the proof. ∎

6.3.2. An algorithm for Restricted Space Cover**.** For Restricted Space Cover we need the following variant of Lemma 6.9.

Lemma 6.11. *Let $M^*(G)$ be the bond matroid of G, $T \subseteq E(G)$, $t^* \in T$, $e^* = uv \in E(G)$. Suppose that $F \subseteq E(G) \setminus T$ spans T and $F \setminus \{e^*\}$ spans t^*. Let also x be an end vertex of an edge xy of T such that x is either in a leaf block or in a degree two block in $G - F$, Y is the set of end vertices of the edges of T distinct from x, $G' = G - T$ and let $W = R_{G'-F}(x)$. If $u, v \notin R_{G'-F}(x)$, then there is a $(x, Y \cup \{u,v\}, k)$-semi-important set W' such that $|\Delta_{G'}(W')| \le |\Delta_{G'}(W)|$ and for $F' = (F \setminus \Delta_{G'}(W)) \cup \Delta_{G'}(W')$, it holds that $u, v \notin R_{G'-F'}(x)$, F' spans T in $M^*(G)$ and $F' \setminus \{e^*\}$ spans t.*

The proof of Lemma 6.11 uses exactly the same arguments as the proof of Lemma 6.9. The only difference is that we have to find a $(x, Y \cup \{u,v\}, k)$-semi-important set W' that separates x and $\{u,v\}$. To guarantee it, we can replace e^* by $k+1$ parallel edges for $k = |\Delta_{G'}(W')|$ with the end vertices being u and v and use a $(x, Y \cup \{u,v\}, k)$-semi-important set in the obtained graph. Modulo this modification, the proof is analogous to Lemma 6.9 and hence omitted. Next we give the algorithm for Restricted Space Cover on cographic matroids.

Lemma 6.12. Restricted Space Cover *can be solved in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ on cographic matroids.*

Proof. The proof uses the same arguments as the proof of Lemma 6.10. Hence, we only sketch the algorithm here.

Let (M, w, T, k, e^*, t^*) be an instance of Restricted Space Cover, where M is a cographic matroid. First, we exhaustively apply Reduction Rules 5.3 and 5.6-5.9. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance, where M has no loops, the weights of nonterminal elements are positive and $|T| \le k+1$. Notice that it can happen that e^* is deleted by Reduction Rules 5.3 and 5.6-5.9. For example, if e^* is a loop then it can be deleted by Reduction Rule 5.3. In this case we obtain an instance of Space Cover and can solve it using Lemma 6.10. From now onwards we assume that e^* is not deleted by our reduction rules.

To simplify notation, we use (M, w, T, k, e^*, t^*) to denote the reduced instance. If we started with cographic matroid then it remains so even after applying Reduction Rules 5.3 and 5.6-5.9. Furthermore, given M, in polynomial time we can find a graph G such that M is isomorphic to the bond matroid $M^*(G)$ [39]. Let $e^* = pq$.

Next, we replace the weighted graph G by the unweighted graph G' as follows. For any nonterminal edge $uv \ne e^*$, if $w(uv) \le k$ then we replace uv by $w(uv)$ parallel edges with the same end vertices u and v. On the other hand if $w(uv) > k$ then we replace uv by $k+1$ parallel edges. Recall that $w(e^*) = 0$. Nevertheless, we replace e^*

by $k+1$ parallel edges with the end vertices p and q to forbid including pq to a set that spans t^*.

Suppose that (M, w, T, k, e^*, t^*) is a yes-instance and let $F \subseteq E(G) \setminus T$ is a solution. Recall that in $G - F$ every edge of T is a bridge and the degree of a connected component C of $G' - F - T$, denoted by $d^*(C, G - F - T)$, is equal to the number of edges of T it is incident to. Notice that if we shrink each connected component to a single vertex then we get a forest on at most $|T| + 1 \leq k + 1$ vertices and thus there are at least $|T|/2$ components such that $d^*(C, G - F - T)$ is at most two. Only two components can contain p or q. Hence, there are at least $|T|/2 - 2$ such components that do not include p, q. Moreover, there is at least one such component, because $F \setminus \{e\}$ spans t^*. Let Q denote the set of end vertices of edges in T and $Z \subseteq Q$. Initially $Z = \emptyset$, but we assume that Z is the set of end vertices of terminals that are in the connected components C of degree one of the graph obtained from G' by deleting the edges of a solution and the terminals and, moreover, $p, q \notin C$.

Our algorithm ALG-CGM-restricted takes as instance (G', T, k, Q, Z) and proceeds as follows.

1. While there is a minimal cut-set $R \subseteq T$ of G do the following. Denote by $Z_1 \subseteq Z$ the set of $z \in Z$ such that z is incident to exactly one $t \in T$, and let $Z_2 \subseteq Z$ be the set of $z \in Z$ such that z is incident to two edges of T. Clearly, Z_1 and Z_2 form a partition of Z. Find a minimal cut-set $R \subseteq T$ and select $xy \in R$ such that $xy \neq t^*$ if $R \neq \{t^*\}$ and let $xy = t^*$ otherwise. Contract xy and denote the obtained vertex z. Set $T = T \setminus \{xy\}$ and recompute W. If $x, y \in Z_1$ or if $x \notin Z$ or $y \notin Z$, then set $Z = Z \setminus \{x, y\}$. Otherwise, if $x, y \in Z$ and $\{x, y\} \cap Z_2 \neq \emptyset$, set $Z = (Z \setminus \{x, y\}) \cup \{z\}$.
2. If $t^* \notin T$, then delete the edges pq. Notice that $t^* \notin T$ only if we already constructed a set that spans t^*. Hence, it is safe to get rid of e^* of weight 0.
3. If Z is empty go to the next step. Else, pick a vertex $s \in Z$ and find all the (s, Y, k) semi-important sets W in $G' - T$ such that $\Delta(W) \leq k$, where
$$Y = \begin{cases} (Q \setminus \{s\}) \cup \{p, q\}, & \text{if } t^* \in T, \\ Q \setminus \{s\}, & \text{if } t^* \notin T, \end{cases}$$
using Lemma 6.8. Notice that if $t^* \in T$, then there are $k+1$ copies of pq. Hence, W separates s from p and q. For each such semi-important set W, we call the algorithm ALG-CGM-restricted on $(G' - \Delta(W), T, k - |\Delta(W)|, Q, Z)$. We use Lemma 6.11 to argue that the branching step is safe.
4. Guess a subset $Z \subseteq Q$ with the property that Z is the set of end vertices of terminals that are in the connected components C of degree at most two of the graph obtained from G' by the deletion of edges of a solution and the terminals and, moreover, $p, q \notin C$. In particular, we do not include in Z the vertices that are incident to at least 3 edges of T. Now call ALG-CGM-restricted on (G', T, k, W, Z). Notice, that by the properties of the forest we know that $Z \neq \emptyset$ and the size of $|Z| \geq \frac{|T|}{2} - 2$.

Notice that because of Step 3 there are no minimal cut-sets $R \subseteq T$ and thus for each considered semi-important set W, $\Delta(W)$ is not empty. It means that the parameter decreases in each recursive call. Moreover, by considering semi-important cuts of size i for $i = \{1, \ldots, k\}$, we decrease the parameter by at least i. Let $\ell = |Q| - |Z|$. Because there are at most $4^i(1 + 4^{i+1})$ semi-important sets of size i, we have the following recurrences for the algorithm:

$$T(\ell,k) \leq 2^{\ell} T\left(\ell - \frac{\ell}{4} + 2, k\right) \tag{6.6}$$

$$T(\ell,k) \leq \sum_{i=1}^{k} (4^i(1+4^{i+1}))T(\ell, k-i) \tag{6.7}$$

As in the proof of Lemma 6.10 using induction hypothesis we can show that the above recurrences solve to $16^{\ell}84^{k}$. Since $\ell \leq 2k+1$ we get that the above algorithm runs in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. This completes the proof. ∎

7. Solving Space Cover for regular matroids. In this section we conjure all that have developed so far and design an algorithm for Space Cover on regular matroids, running in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. To give a clean presentation of our algorithm we have divided the section into three parts. We first give some generic steps, followed by steps when matroid in consideration is either graphic or cographic and ending with a result that ties them all.

Let (M, w, T, k) be the given instance of Space Cover. First, we exhaustively apply Reduction Rules 5.1-5.5. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance, where M has no loops and the weights of nonterminal elements are positive. To simplify notation, we also denote the reduced instance by (M, w, T, k). We say that a matroid M is *basic* if it can be obtained from R_{10} by adding parallel elements or M is graphic or cographic. If M is a basic matroid then we can solve Space Cover using Lemmas 6.1, or 6.3 or 6.10 respectively in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. This results in the following lemma.

Lemma 7.1. *Let (M, w, T, k) be an instance of* Space Cover. *If M is a basic matroid then* Space Cover *can be solved in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$.*

From now onwards we assume that the matroid M in the instance (M, w, T, k) is not basic. Now using Corollary 4.4, we find a conflict tree $\mathcal{T}$. Recall that the set of nodes of $\mathcal{T}$ is the collection of basic matroids $\mathcal{M}$ and the edges correspond to 1-, 2− and 3-sums. The key observation is that M can be constructed from $\mathcal{M}$ by performing the sums corresponding to the edges of $\mathcal{T}$ in an arbitrary order. Our algorithm is based on performing *bottom-up* traversal of the tree $\mathcal{T}$. We select an arbitrarily *node r as the root* of $\mathcal{T}$. Selection of r, as the root, defines the natural parent-child, descendant and ancestor relationship on the nodes of $\mathcal{T}$. We say that u is a *sub-leaf* if its children are leaves of $\mathcal{T}$. Observe that there always exists a sub-leaf in a tree on at least two nodes. Just take a node which is not a leaf and is farthest from the root. Clearly, this node can be found in polynomial time.

Throughout, this section we fix a sub-leaf of $\mathcal{T}$ – a basic matroid M_s.
We say that a child of M_s is a 1-, 2- *or* 3-leaf*, respectively, if the edge between M_s and the leaf corresponds to* 1-, 2- *or 3-sum, respectively.*

We first modify the decomposition by an exhaustive application of the following rule.

Reduction Rule 7.1 (**Terminal flipping rule).** *If there is a child M_ℓ of a sub-leaf M_s such that there is $e \in E(M_s) \cap E(M_\ell)$ that is parallel to a terminal $t \in E(M_\ell) \cap T$ in M_ℓ, then delete t from M_ℓ and add t to M_s as an element parallel to e.*

The safeness of Reduction Rule 7.1 follows from the following observation.

OBSERVATION 7.1 ([8]). *Let $M = M_1 \oplus M_2$. Suppose that there is $e' \in E(M_2) \setminus E(M_1)$ such that e' is parallel to $e \in E(M_1) \cap E(M_2)$. Then $M = M_1' \oplus M_2'$, where M_1' is obtained from M_1 by adding a new element e' parallel e and M_2' is obtained from M_2' by the deletion of e'.*

Proof of Observation 7.1 is implicit in [8]. Furthermore Reduction Rule 7.1 can be applied in polynomial time. Notice also allowed to a matroid obtained from R_{10} by adding parallel elements to be a basic matroid of a decomposition. Thus, we get the following lemma.

LEMMA 7.2. *Reduction Rule 7.1 is safe and can be applied in polynomial time.*

From now we assume that there is no child M_ℓ of M_s such that there exists an element $e \in E(M_s) \cap E(M_\ell)$ that is parallel to a terminal $t \in E(M_\ell) \cap T$ in M_ℓ. In what follows we do a bottom-up traversal of $\mathcal{T}$ and at each step we delete one of the child of M_s. A child of M_s is deleted either because of an application of a reduction rule or because of recursively solving the problem on a smaller sized tree. It is possible that, while recursively solving the problem, we could possibly modify (or replace) M_s to encode some auxiliary information that we have already computed while solving the problem. We start by giving some generic steps that do not depend on the types of either M_s or its child. *Throughout the section, given the conflict tree $\mathcal{T}$, we denote by $M_{\mathcal{T}}$ the matroid defined by $\mathcal{T}$.*

7.1. Few generic steps. We start by giving a reduction rule that is useful when we have 1-leaf. The reduction rule is as follows.

REDUCTION RULE 7.2 (**1-Leaf reduction rule).** *If there is a child M_ℓ of M_s that is a 1-leaf, then do the following.*

(i) If $E(M_\ell) \cap T = \emptyset$, then delete M_ℓ from $\mathcal{T}$.

(ii) If $E(M_\ell) \cap T \neq \emptyset$, then find the minimum $k' \leq k$ such that $(M_\ell, w_\ell, T \cap E(M_\ell), k')$ is a yes-instance of SPACE COVER *using Lemmas 6.1, or 6.3 or 6.10, respectively, depending on which primary matroid M_ℓ is. Here, w_ℓ is the restriction of w on $E(M_\ell)$. If $(M_\ell, w_\ell, T \cap E(M_\ell), k')$ is a no-instance for every $k' \leq k$ then we return no. Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by deleting the node M_ℓ. Furthermore, for simplicity, let $M_{\mathcal{T}'}$ be denoted by M', restriction of w to $E(M_{\mathcal{T}'})$ by w' and $T \cap E(M_{\mathcal{T}'})$ be denoted by T'. Our new instance is $(M', w', T', k - k')$.*

Safeness of the reduction rule follows by the definition of 1-sum, and it can be applied in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. Thus we get the following result.

LEMMA 7.3. *Reduction Rule 7.2 is safe and can be applied in $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time.*

7.1.1. Handling 2-leaves. For 2-leaves, we either reduce a leaf or apply a recursive procedure based on whether the leaf contains a terminal or not.

REDUCTION RULE 7.3 (**2-Leaf reduction rule).** *If there is a child M_ℓ of M_s that is a 2-leaf with $E(M_s) \cap E(M_\ell) = \{e\}$ and $T \cap E(M_\ell) = \emptyset$, then find the minimum $k' \leq k$ such that $(M_\ell, w_\ell, \{e\}, k')$ is a yes-instance of* SPACE COVER *using Lemmas 6.1, or 6.3 or 6.10, respectively, depending on which primary matroid M_ℓ is. Here, $w_\ell(e') = w(e')$ for $e' \in E(M_\ell) \setminus \{e\}$ and $w_\ell(e) = 0$. If $(M_\ell, w_\ell, \{e\}, k')$ is a no-instance for every $k' \leq k$ then we set $k' = k + 1$. Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by deleting the node M_ℓ. Furthermore, for simplicity, let $M_{\mathcal{T}'}$ be denoted by M'. We define w' on $E(M')$ as follows: for every $e^* \in E(M_{\mathcal{T}'})$, $e^* \neq e$, set $w'(e^*) = w(e^*)$ and let $w'(e) = k'$. Our new instance is (M', w', T, k).*

LEMMA 7.4. *Reduction Rule 7.3 is safe and can be applied* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ *time.*

Proof. To show that the rule is safe, denote by M' the matroid defined by $\mathcal{T}' = \mathcal{T} - M_\ell$ and let $w'(e') = w(e')$ for $e' \in E(M') \setminus \{e\}$ and $w'(e) = k'$. By **2-Leaf reduction rule**, there is a cycle C of M_ℓ such that $e \in C$ and the weight $w(C \setminus \{e\}) = k'$ is minimum among all cycles that include e.

Suppose that (M, w, T, k) is a yes-instance of SPACE COVER. Let $F \subseteq E(M) \setminus T$ be a set of weight at most k that spans T. If $F \cap E(M_\ell) = \emptyset$, then F spans T in M' and because $e \notin F$, the weight of F is the same as before. Hence, (M', w', T, k) is a yes-instance. Assume that $F \cap E(M_\ell) \neq \emptyset$. Let $F' = (F \cap E(M') \cup \{e\}$. For each $t \in T$, there is a circuit C_t of M such that $t \in C_t \subseteq F \cup \{t\}$. Because $F \cap E(M_\ell) \neq \emptyset$, there is $t \in T$ such that $C_t \cap E(M_\ell) \neq \emptyset$. By the definition of 2-sums, there are cycles C'_t of M' and C''_t of M_ℓ such that $C_t = C'_t \triangle C''_t$ and we have that $e \in C'_t \cap C''_t$, because C_t is a circuit, i.e., an inclusion-minimal nonempty cycle. Since $w(C''_t \setminus \{e\}) \geq w(C \setminus \{e\})$, we have that $w(F') \leq k$. To show that F' spans T, consider $t \in T$ and a cycle C_t of M such that $t \in C_t \subseteq F \cup \{t\}$. If $C_t \subseteq E(M')$, then $C_t \subseteq F' \cup \{t\}$ and F' spans t in M'. If $C_t \cap E(M_\ell) \neq \emptyset$, then there are cycles C'_t of M' and C''_t of M_ℓ such that $e \in C'_t \cap C''_t$ and $C_t = C'_t \triangle C''_t$. Because $C'_t \subseteq F' \cup \{t\}$, we have that F' spans t.

Assume now that (M', w', T, k) is a yes instance. Let $F' \subseteq E(M') \setminus T$ be a set of weight at most k that spans T in M'. If $e \notin F'$, then F' spans T in M and (M, w, T, k) is a yes-instance. Suppose that $e \in F'$. Let $F = F' \triangle C$. Clearly, $w(F) = w(F') \leq k$. We have to show that F spans T. Let $t \in T$. There is a cycle C'_t in M' such that $t \in C'_t \subseteq F' \cup \{t\}$. If $e \notin C'$, then $C'_t \subseteq F \cup \{t\}$ and F spans t. If $e \in C'_t$, then for $C_t = C'_t \triangle C$, we have that $t \in C_t \subseteq F \cup \{t\}$ and it implies that F spans t.

The rule can be applied in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ by Lemma 7.1. In fact, it can be done in polynomial time, because we are solving SPACE COVER for the sets of terminal of size one. It is easy to see that if M_ℓ is graphic, then the problem can be reduced to finding a shortest path, and if M_ℓ is cographic, then we can reduce it to the minimum cut problem. ∎

Reduction Rule 7.3 takes care of the case when M_ℓ has no terminal. If it has a terminal then we recursively solve the problem as described below in Branching Rule 7.1, and if any of these recursive calls of the algorithm returns yes then we return that the given instance is a yes-instance. Recall that $F \subseteq E(M) \setminus T$ is a solution for (M, w, T, k) if and only if for every $t \in T$, there is a circuit C_t such that $t \in C_t \subseteq F \cup \{t\}$. The three branches in the rule correspond to the structure of these circuits C_t in a potential solution with respect to $M_s \oplus_2 M_\ell$: (i) there is $t \in T \cap E(M_\ell)$ such that C_t contains elements of both M_ℓ and M_s, (ii) there is $t \in T \cap E(M_s)$ such that C_t contains elements of both M_ℓ and M_s, and (iii) for every $t \in T$, either $C_t \subseteq E(M_\ell)$ or $C_t \subseteq E(M_s)$.

BRANCHING RULE 7.1 (**2-Leaf branching).** *If there is a child M_ℓ of M_s that is a 2-leaf with $E(M_s) \cap E(M_\ell) = \{e\}$ and $T \cap E(M_\ell) = T_\ell \neq \emptyset$, then do the following. Let M' the matroid defined by $\mathcal{T}' = \mathcal{T} - M_\ell$ and let $T' = T \setminus T_\ell$. Consider the following three branches.*

(i) Let $w'(e') = w(e')$ for $e' \in E(M') \setminus \{e\}$ and $w'(e) = 0$. Define $w_\ell(e') = w(e')$ for $e' \in E(M_\ell) \setminus \{e\}$ and $w_\ell(e) = 0$. Find the minimum $k_1 \leq k$ such that $(M_\ell, w_\ell, T_\ell \cup \{e\}, k_1)$ is a yes-instance of SPACE COVER *using Lemmas 6.1, or 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell \cup \{e\}, k_1)$ is a no-instance for every $k_1 \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T', k - k_1)$.*

(ii) Let $w'(e') = w(e')$ for $e' \in E(M') \setminus \{e\}$ and $w'(e) = 0$. Define $w_\ell(e') = w(e')$

for $e' \in E(M_\ell) \setminus \{e\}$ *and* $w_\ell(e) = 0$. *Find the minimum* $k_2 \leq k$ *such that* $(M_\ell, w_\ell, T_\ell, k_2)$ *is a yes-instance of* Space Cover *using Lemmas 6.1, or 6.3 or 6.10, respectively, depending on the type of* M_ℓ. *If* $(M_\ell, w_\ell, T_\ell, k_2)$ *is a no-instance for every* $k_2 \leq k$, *then we return no and stop. Otherwise, solve the problem on the instance* $(M', w', T' \cup \{e\}, k - k_2)$.

(iii) Let $w'(e') = w(e')$ *for* $e' \in E(M') \setminus \{e\}$ *and* $w'(e) = k+1$. *Define* $w_\ell(e') = w(e')$ *for* $e' \in E(M_\ell) \setminus \{e\}$ *and* $w_\ell(e) = k+1$. *Find the minimum* $k_3 \leq k$ *such that* $(M_\ell, w_\ell, T_\ell, k_3)$ *is a yes-instance of* Space Cover *using Lemmas 6.1, or 6.3 or 6.10, respectively, depending on the type of* M_ℓ. *If* $(M_\ell, w_\ell, T_\ell, k_3)$ *is a no-instance for every* $k_3 \leq k$, *then we return no and stop. Otherwise, solve the problem on the instance* $(M', w', T', k - k_3)$.

Lemma 7.5. *Branching Rule 7.1 is exhaustive and in each recursive call the parameter strictly reduces. Each call of the rule takes* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ *time.*

Proof. To show correctness, assume first that (M, w, T, k) is a yes-instance of Space Cover. Let $F \subseteq E(M) \setminus T$ be a set of weight at most k that spans T. Without loss of generality we assume that F is inclusion-minimal and, therefore, F is independent by Observation 3.1. For each $t \in T$, there is a circuit C_t of M such that $t \subseteq C_t \subseteq F \cup \{t\}$. We have the following three cases.

Case 1. There is C_t such that $t \in T'$ and $C_t \cap E(M_\ell) \neq \emptyset$. Let $F_\ell = F \cap E(M_\ell)$ and $F' = (F \cap E(M')) \cup \{e\}$. We claim that F_ℓ spans $T_\ell \cup \{e\}$ in M_ℓ and F' spans T' in M'.

First, we show that F_ℓ spans $T_\ell \cup \{e\}$ in M_ℓ. Since there is a circuit C_t such that $t \in T'$ and $C_t \cap E(M_\ell) \neq \emptyset$, there are cycles C'_t of M' and C''_t of M_ℓ such that $C_t = C'_t \triangle C''_t$ and $e \in C'_t \cap C''_t$. Because $e \in C''_t$ and $C''_t \setminus \{e\} \subseteq F_\ell$, we have that F_ℓ spans e in M_ℓ. Let $t' \in T_\ell$. Since F spans t' in M, there is a cycle $C_{t'}$ of M such that $t' \in C_{t'} \subseteq F \cup \{t'\}$. If $C_{t'} \setminus t' \subseteq E(M_\ell)$, then F_ℓ spans t', because $C_{t'} \setminus \{t'\} \subseteq F_\ell$. Suppose that $C_{t'} \cap E(M') \neq \emptyset$. Then by the definition of 2-sum, there are cycles $C'_{t'}$ of M' and $C''_{t'}$ of M_ℓ such that $e \in C'_{t'} \cap C''_{t'}$ and $C_{t'} = C'_{t'} \triangle C''_{t'}$. Consider $C = C''_t \triangle C''_{t'}$. By Observation 3.4, C is a cycle. As $C \setminus \{e\} \subseteq F_\ell$, $e \in C''_{t'} \cap C''_t$ and $t' \notin C''_t$, we obtain that C is a cycle of M_ℓ and $t' \in C \subseteq F_\ell \cup \{t'\}$. Therefore, F_ℓ spans t'.

To prove that F' spans T' in M', consider $t' \in T'$. Since F spans t' in M, there is a circuit $C_{t'}$ of M such that $t' \in C_{t'} \subseteq F \cup \{t'\}$. If $C_{t'} \setminus t' \subseteq E(M')$, then F' spans t', because $C_{t'} \setminus \{t'\} \subseteq F'$. Suppose that $C_{t'} \cap E(M_\ell) \neq \emptyset$. Then by the definition of 2-sum, there are cycles $C'_{t'}$ of M' and $C''_{t'}$ of M_ℓ such that $e \in C'_{t'} \cap C''_{t'}$ and $C_{t'} = C'_{t'} \triangle C''_{t'}$. Observe that $C'_{t'} \setminus \{t'\} \subseteq F'$ and, therefore, F' spans t' in M'.

Since F_ℓ spans $T_\ell \cup \{e\}$ in M_ℓ, $w(F_\ell) \geq k_1$. Because $w(F') + w(F_\ell) = w(F) \leq k$ if the weight of e in M' is 0, $w(F') \leq k - k_1$ in this case. Hence, $(M', w', T', k - k_1)$ is a yes-instance for the first branch.

Case 2. There is C_t such that $t \in T_\ell$ and $C_t \cap E(M') \neq \emptyset$. This case is symmetric to Case 1, and by the same arguments, we show that $(M', w', T' \cup \{e\}, k - k_2)$ is a yes-instance for the second branch.

Otherwise, we have the remaining case.

Case 3. For any $t \in T'$, $C_t \subseteq E(M') \setminus \{e\}$, and for any $t \in T_\ell$, $C_t \subseteq E(M_\ell) \setminus \{e\}$. Let $F_\ell = F \cap E(M_\ell)$ and $F' = (F \cap E(M'))$. Observe that F_ℓ spans T_ℓ in M_ℓ and F' spans T' in M'. In particular, $w(F_\ell) \geq k_3$. Since $w(F') + w(F_\ell) = w(F) \leq k$, $(M', w', T', k - k_3)$ is a yes-instance for the third branch.

Suppose now that we have a yes-answer for one of the branches. We consider 3 cases depending on the branch.

Case 1. $(M', w', T', k-k_1)$ is a yes-instance for the first branch. Let $F_\ell \subseteq E(M_\ell)$ be a set of weight at most k_1 that spans $T_\ell \cup \{e\}$ in M_ℓ and let F' be a set of weight at most $k-k_1$ that spans T' in M'. Consider $F = F' \triangle F_\ell$. Clearly, $w(F) \leq k$. We claim that F spans T. Let $t \in T$. Suppose that $t \in T_\ell$. Notice that $e \notin F_\ell$, as e is a terminal in the instance $(M_\ell, w_\ell, T_\ell \cup \{e\}, k_1)$. It implies that F_ℓ spans t in M. Assume now that $t \in T'$. Since F' spans t, there is a cycle C_t of M' such that $t \in C_t \subseteq F' \cup \{t\}$. If $e \notin C_t$, then $C_t \setminus \{t\}$ and, therefore, F spans t in M. Suppose that $e \in C_e$. The set F_ℓ spans e in M_ℓ. Hence, there is a cycle C_e of M_ℓ such that $e \in C_e \subseteq F_\ell \cup \{e\}$. Let $C'_t = C_t \triangle C_e$. By definition, C'_t is a cycle of M. Because $t \in C'_t$ and $e \notin C_{t'}$, we have that $C'_t \setminus \{t\}$ spans t. As $C'_t \subseteq F$, F spans t. Because F is a set of weight at most k that spans T, (M, w, T, k) is a yes-instance.

Case 2. $(M', w', T' \cup \{e\}, k-k_2)$ is a yes-instance for the second branch. This case is symmetric to Case 1, and we use the same arguments to show that (M, w, T, k) is a yes-instance.

Case 3. $(M', w', T', k-k_3)$ is a yes-instance for the third branch. Let $F_\ell \subseteq E(M_\ell)$ be a set of weight at most k_1 that spans T_ℓ in M_ℓ and let F' be a set of weight at most $k-k_1$ that spans T' in M'. Notice that $e \notin F_\ell$ and $e \notin F'$, because the weight of e is $k+1$ in M_ℓ and M'. Let $F = F' \cup F_\ell$. Clearly, $w(F) \leq k$. Let $t \in T$. Then F_ℓ spans t in M. If $t \in T'$, then F' spans t in M. Hence, F spans T. Therefore, (M, w, T, k) is a yes-instance.

Notice that M_ℓ has no nonterminal elements of zero weight for the first and third branches and the elements of T_ℓ are not loops, because of the application of the reduction rules. Hence, $k_1, k_3 \geq 1$. For the second branch, e has the zero weight, but F_ℓ has no terminals parallel to e, because of **Terminal flipping rule**, hence, $k_2 \geq 1$ as well. We conclude that all recursive calls are done for the parameters that are strictly lesser that k.

The claim that each call of the rule (without recursive steps) takes $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time follows from Lemma 7.1. ◻

7.1.2. Handling 3-leaves. In this section we assume that all the children of M_s are 3-leaves. The analysis of this cases is done along the same lines as for the case of 2-leaves. However, this case is significantly more complicated.

OBSERVATION 7.2. *Let M be a matroid obtained from R_{10} by adding some parallel elements. Then any circuit of M has even size.*

It immediately implies that M_s and its children are graphic or cographic matroids.

For 3-sums, it is convenient to make the following observation.

OBSERVATION 7.3. *Let $M = M_1 \oplus_3 M_2$. If C is a cycle of M, then there are cycles C_1 and C_2 of M_1 and M_2 respectively such that $C = C_1 \triangle C_2$ and either $C_1 \cap C_2 = \emptyset$ or $|C_1 \cap C_2| = 1$. Moreover, if C is a circuit of M, then either C is a circuit of M_1 or M_2, or there are circuits C_1 and C_2 of M_1 and M_2 respectively such that $C = C_1 \triangle C_2$ and $|C_1 \cap C_2| = 1$.*

Proof. Let $Z = C_1 \cap C_2$. Recall that Z is a circuit of M_1 and M_2. Let $C = C_1 \triangle C_2$ and $|C_1 \cap C_2| \geq 2$. Consider $C'_1 = C_1 \triangle Z$ and $C'_2 = C_2 \triangle Z$. We have that C'_1 and C'_2 are cycles of M_1 and M_2 respectively by Observation 3.4 and $|C'_1 \cap C'_2| \leq 1$. It remains to notice that $C = C'_1 \triangle C'_2$. The second claim immediately follows from the fact that a circuit is an inclusion-minimal nonempty cycle. ◻

We use Observation 7.3 to analyze the structure of a solution of SPACE COVER for matroid sums. If $M = M_1 \oplus_3 M_2$ and for $t \in T$, a circuit C such that $t \in C \subseteq F \cup \{t\}$ for a solution F has nonempty intersection with $E(M_1)$ and $E(M_2)$, then $C = C_1 \triangle C_2$ for cycles C_1 and C_2 of M_1 and M_2 respectively and, moreover, it could be assumed that C_1 and C_2 are circuits. By Observation 7.3, we can always assume that $C_1 \cap C_2 = \{e\}$ for $e \in E(M_1) \cap E(M_2)$. Using this assumption, we say that C *goes through* e in this case.

We also need the following observation about circuits of size 3.

OBSERVATION 7.4. *Let M be a binary matroid, $w\colon E(M) \to \mathbb{N}_0$. Let also $C = \{e_1, e_2, e_3\}$ be a circuit of M. Suppose that $F \subseteq E(M) \setminus C$ is a set of minimum weight such that M has circuits (cycles) C_1 and C_2 such that $e_1 \in C_1 \subseteq F \cup \{e_1\}$ and $e_2 \in C_2 \subseteq F \cup \{e_2\}$. Then F is a subset of $E(M)\setminus C$ of minimum weight such that for each $i \in \{1, 2, 3\}$, M has a circuit (cycle) C_i such that $e_i \in C_i \subseteq F \cup \{e_i\}$. Moreover, for any distinct $i, j \in \{1, 2, 3\}$, F is a subset of minimum weight of $E(M)\setminus C$ such that M has circuits (cycles) C_i and C_j such that $e_i \in C_i \subseteq F \cup \{e_i\}$ and $e_j \in C_j \subseteq F \cup \{e_j\}$.*

Proof. Let $C' = C_1 \triangle C_2 \triangle C$. Because M is binary, C' is a cycle by Observation 3.4. Since $\{e_1\} = C \cap C_1$, $\{e_2\} = C \cap C_2$ and $e_3 \notin C_1 \cup C_2 = F$, C' contains a circuit C_3 such that $e_3 \in C_3 \subseteq C' \subseteq F \cup \{e_3\}$. Hence, the first claim holds by symmetry. Also by symmetry, the second claim is fulfilled. ∎

If a child of M_s has terminals, then we recursively solve the problem as described below in Branching Rule 7.2 and if any of these recursive calls returns yes then we return that the given instance is a yes-instance. Similarly to Reduction Rule 7.1, each branch corresponds to the behavior of circuits C_t with the property that for $t \in T$, there is $t \in C_t \subseteq F \cup \{t\}$ for a potential solution F. Since for 3-sums the structure is more complicated, we obtain 15 branches of 6 types.

BRANCHING RULE 7.2 (**3-Leaf branching).** *If there is a child M_ℓ of M_s that is a 3-leaf with $E(M_s) \cap E(M_\ell) = Z$ and $T \cap E(M_\ell) = T_\ell \neq \emptyset$, then let M' the matroid defined by $\mathcal{T}' = \mathcal{T} - M_\ell$ and let $T' = T \setminus T_\ell$. We set $w'(e) = w(e)$ for $e \in E(M') \setminus Z$ and $w_\ell(e) = w(e)$ for $e \in E(M_\ell) \setminus Z$. We let $Z = \{e_1, e_2, e_3\}$ and consider the following branches of six types.*

(i) Let $w_\ell(e_h) = k + 1$ for $h \in \{1, 2, 3\}$. For each $i \in \{1, 2, 3\}$ do the following. Set $w'(e_i) = 0$ and $w'(e_h) = k + 1$ for $h \in \{1, 2, 3\}$ such that $h \neq i$. Find the minimum $k_i^{(1)} \leq k$ such that $(M_\ell, w_\ell, T_\ell \cup \{e_i\}, k_i^{(1)})$ is a yes-instance of SPACE COVER *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell \cup \{e_i\}, k_i^{(1)})$ is a no-instance for every $k_i^{(1)} \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T', k - k_i^{(1)})$.*

(ii) Let $w_\ell(e_h) = k + 1$ for $h \in \{1, 2, 3\}$. Set $w'(e_1) = w'(e_2) = 0$ and $w'(e_3) = k + 1$. Find the minimum $k^{(2)} \leq k$ such that $(M_\ell, w_\ell, T_\ell \cup \{e_1, e_2\}, k^{(2)})$ is a yes-instance of SPACE COVER *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell \cup \{e_1, e_2\}, k^{(2)})$ is a no-instance for every $k^{(2)} \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T', k - k^{(2)})$.*

(iii) For any two distinct $i, j \in \{1, 2, 3\}$, do the following. Let $h \in \{1, 2, 3\}$ such that $h \neq i, j$. Set $w_\ell(e_i) = 0$ and $w_\ell(e_j) = w_\ell(e_h) = k + 1$. Let $w'(e_j) = 0$ and $w'(e_i) = w'(e_h) = k + 1$. Find the minimum $k_{ij}^{(3)} \leq k$ such that $(M_\ell, w_\ell, T_\ell \cup \{e_j\}, k_{ij}^{(3)}, e_i, e_j)$ is a yes-instance of RESTRICTED SPACE COVER *using Lemmas 6.4 or 6.12, respectively, depending on the type of M_ℓ. If*

$(M_\ell, w_\ell, T_\ell \cup \{e_j\}, k^{(3)}_{ij}, e_i, e_j)$ is a no-instance for every $k^{(3)}_{ij} \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T' \cup \{e_i\}, k - k^{(3)}_{ij})$.

(iv) *Let $w'(e_h) = k+1$ for $h \in \{1,2,3\}$. For each $i \in \{1,2,3\}$ do the following. Set $w_\ell(e_i) = 0$ and $w_\ell(e_h) = k+1$ for $h \in \{1,2,3\}$ such that $h \neq i$. Find the minimum $k^{(4)}_i \leq k$ such that $(M_\ell, w_\ell, T_\ell, k^{(4)}_i)$ is a yes-instance of* Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell, k^{(4)}_i)$ is a no-instance for every $k^{(4)}_i \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T' \cup \{e_i\}, k - k^{(4)}_i)$.*

(v) *Let $w_\ell(e_1) = w_\ell(e_2) = 0$ and $w_\ell(e_3) = k+1$. Set $w'(e_1) = w'(e_2) = w'(e_3) = k+1$. Find the minimum $k^{(5)} \leq k$ such that $(M_\ell, w_\ell, T_\ell, k^{(5)})$ is a yes-instance of* Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell, k^{(5)})$ is a no-instance for every $k^{(5)} \leq k$ then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T' \cup \{e_1, e_2\}, k - k^{(5)})$.*

(vi) *Set $w_\ell(e_1) = w_\ell(e_2) = w_\ell(e_3) = k+1$ and $w'(e_1) = w'(e_2) = w'(e_3) = k+1$. Find the minimum $k^{(6)} \leq k$ such that $(M_\ell, w_\ell, T_\ell, k^{(6)})$ is a yes-instance of* Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, T_\ell, k^{(6)})$ is a no-instance for every $k^{(6)} \leq k$, then we return no and stop. Otherwise, solve the problem on the instance $(M', w', T', k - k^{(6)})$.*

Note that the branching of the third type is the only place of our algorithm where we are solving Restricted Space Cover.

Lemma 7.6. *Branching Rule 7.2 is exhaustive and in each recursive call the parameter strictly reduces. Each call of the rule takes $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time.*

Proof. To show correctness, assume first that (M, w, T, k) is a yes-instance of Space Cover. Let $F \subseteq E(M) \setminus T$ be a set of weight at most k that spans T. Without loss of generality we assume that F is inclusion minimal and, therefore, F is independent by Observation 3.1. For each $t \in T$, there is a circuit C_t of M such that $t \subseteq C_t \subseteq F \cup \{t\}$. We have the following five cases corresponding to the types of branches.

Case 1. There is $i \in \{1,2,3\}$ such that a)there is $t \in T'$ such that $C_t \cap E(M_\ell) \neq \emptyset$ and C_t goes through e_i, and b)for any $t \in T$, there is no circuit C_t that goes through $e_h \in Z$ for $h \neq i$. Let $F_\ell = F \cap E(M_\ell)$ and $F' = (F \cap E(M')) \cup \{e_i\}$. We claim that F_ℓ spans $T_\ell \cup \{e_i\}$ in M_ℓ and F' spans T' in M'.

First, we show that F_ℓ spans $T_\ell \cup \{e_i\}$ in M_ℓ. By a), there is $t \in T'$ such that $C_t \cap E(M_\ell) \neq \emptyset$ and C_t goes through e_i. Hence, there are cycles C'_t of M' and C''_t of M_ℓ respectively such that $C_t = C'_t \triangle C''_t$ and $C'_t \cap C''_t = \{e_i\}$. Because $C''_t \setminus \{e_i\} \subseteq F_\ell$, we obtain that F_ℓ spans e_i in M_ℓ. Let $t' \in T_\ell$. Since F spans t' in M, there is a circuit $C_{t'}$ of M such that $t' \in C_{t'} \subseteq F \cup \{t'\}$. If $C_{t'} \setminus t' \subseteq E(M_\ell)$, then F_ℓ spans t', because $C_{t'} \setminus \{t'\} \subseteq F_\ell$. Suppose that $C_{t'} \cap E(M') \neq \emptyset$. By b), $C_{t'}$ goes through e_i. Then there are cycles $C'_{t'}$ of M' and $C''_{t'}$ of M_ℓ such that $\{e_i\} = C'_{t'} \cap C''_{t'}$ and $C_{t'} = C'_{t'} \triangle C''_{t'}$. Consider $C = C''_t \triangle C''_{t'}$. By Observation 3.4, C is a cycle. As $C \setminus \{e_i\} \subseteq F_\ell$, $\{e_i\} = C''_{t'} \cap C''_t$ and $t' \notin C''_t$, we obtain that C is a cycle of M_ℓ and $t' \in C \subseteq F_\ell \cup \{t'\}$. Therefore, F_ℓ spans t'.

To prove that F' spans T' in M', consider $t' \in T'$. Since F spans t' in M, there is a circuit $C_{t'}$ of M such that $t' \in C_{t'} \subseteq F \cup \{t'\}$. If $C_{t'} \setminus t' \subseteq E(M')$, then F' spans t', because $C_{t'} \setminus \{t'\} \subseteq F'$. Suppose that $C_{t'} \cap E(M_\ell) \neq \emptyset$. Then by the definition of 3-sum and b), there are cycles $C'_{t'}$ of M' and $C''_{t'}$ of M_ℓ such that $\{e_i\} = C'_{t'} \cap C''_{t'}$ and $C_{t'} = C'_{t'} \triangle C''_{t'}$. Observe that $C'_{t'} \setminus \{t'\} \subseteq F'$ and, therefore, F' spans t' in M'.

Since F_ℓ spans $T_\ell \cup \{e_i\}$ in M_ℓ, $w(F_\ell) \geq k_i^{(1)}$. Because $w(F') + w(F_\ell) = w(F) \leq k$ if the weight of e_i in M' is 0, $w(F') \leq k - k_i^{(1)}$ in this case. Hence, $(M', w', T', k - k_i^{(1)})$ is a yes-instance for a branch of type (i).

Case 2. There are distinct $i, j \in \{1,2,3\}$ such that a)there is $t \in T'$ such that $C_t \cap E(M_\ell) \neq \emptyset$ and C_t goes through e_i, b) there is $t \in T'$ such that $C_t \cap E(M_\ell) \neq \emptyset$ and C_t goes through e_j. Let $F_\ell = F \cap E(M_\ell)$ and $F' = (F \cap E(M')) \cup \{e_1, e_2\}$. We claim that F_ℓ spans $T_\ell \cup \{e_1, e_2\}$ in M_ℓ and F' spans T' in M'.

We prove first that F_ℓ spans $T_\ell \cup \{e_i, e_j\}$ in M_ℓ. By a), there is $t \in T'$ such that $C_t \cap E(M_\ell) \neq \emptyset$ and C_t goes through e_i. Hence, there are cycles C'_t of M' and C''_t of M_ℓ respectively such that $C_t = C'_t \triangle C''_t$ and $C'_t \cap C''_t = \{e_i\}$. Because $C''_t \setminus \{e_i\} \subseteq F_\ell$, obtain that F_ℓ spans e_i in M_ℓ. By the same arguments and b), we have that F_ℓ spans e_j in M_ℓ. Let $h \in \{1,2,3\}$ such that $h \neq i, j$. Since F_ℓ spans e_i and e_j in M_ℓ, there are cycles C^i and C^j of M_ℓ such that $e_i \in C^i \subseteq F_\ell \cup \{e_i\}$ and $e_j \in C^j \subseteq F_\ell \cup \{e_j\}$. Consider $C = C^i \triangle C^j \triangle Z$. By Observation 3.4, C is a cycle of M_ℓ. Notice that $e_h \in C \subseteq F_\ell \cup \{e_h\}$. Hence, F_ℓ spans e_h. Because F_ℓ spans $Z = \{e_1, e_2, e_3\}$, in particular, F_ℓ spans e_1 and e_2. Let $t \in T_\ell$. Since F spans t in M, there is a circuit C_t of M such that $t \in C_t \subseteq F \cup \{t\}$. If $C_t \setminus t \subseteq E(M_\ell)$, then F_ℓ spans t, because $C_t \setminus \{t\} \subseteq F_\ell$. Suppose that $C_t \cap E(M) \neq \emptyset$. We have that C_t goes through e_h for some $h \in \{1,2,3\}$. Then there are cycles C'_t of M' and C''_t of M_ℓ such that $\{e_h\} = C'_t \cap C''_t$ and $C_t = C'_t \triangle C''_t$. Consider $C = C^h \triangle C''_{t'}$. By Observation 3.4, C is a cycle of M_ℓ. Notice that $t \in C \subseteq F_\ell \cup \{t\}$ and, therefore, F_ℓ spans t.

Now we show that F' spans T' in M'. Let $t \in T'$. Since F spans t in M, there is a circuit C_t of M such that $t \in C_t \subseteq F \cup \{t\}$. If $C_t \setminus t \subseteq E(M')$, then F' spans t, because $C_t \setminus \{t\} \subseteq F'$. Suppose that $C_t \cap E(M_\ell) \neq \emptyset$. Then there are cycles C'_t of M' and C''_t of M_ℓ such that $\{e_h\} = C'_t \cap C''_t$ for some $h \in \{1,2,3\}$ and $C_t = C'_{t'} \triangle C''_t$. If $h = 1$ or $h = 2$, then $C'_t \setminus \{t\} \subseteq F'$ and, therefore, F' spans t' in M'. Let $h = 3$. Consider $C = C'_t \triangle Z$. Now $t \in C \subseteq F' \cup \{t\}$. Because C is a cycle of M' by Observation 3.4, F' spans t in M'.

Since F_ℓ spans $T_\ell \cup \{e_1, e_2\}$ in M_ℓ, $w(F_\ell) \geq k^{(2)}$. Because $w(F') + w(F_\ell) = w(F) \leq k$, $w(F') \leq k - k^{(2)}$ in this case. Hence, $(M', w', T', k - k^{(2)})$ is a yes-instance for a branch of type (ii).

Case 3. There are distinct $i, j \in \{1,2,3\}$ such that a)there is $t \in T_\ell$ such that $C_t \cap E(M') \neq \emptyset$ and C_t goes through e_i, b) there is $t' \in T'$ such that $C_{t'} \cap E(M_\ell) \neq \emptyset$ and $C_{t'}$ goes through e_j, and c) for any $t'' \in T$, there is no circuit $C_{t''}$ that goes through $e_h \in Z$ for $h \neq i, j$. Let $F_\ell = (F \cap E(M_\ell)) \cup \{e_i\}$ and $F' = (F \cap E(M')) \cup \{e_j\}$. We claim that F_ℓ spans $T_\ell \cup \{e_j\}$ and $F_\ell \setminus \{e_i\}$ spans e_j in M_ℓ and F' spans $T' \cup \{e_i\}$ in M'.

We prove that F_ℓ spans $T_\ell \cup \{e_j\}$. By b), there is $t' \in T'$ such that $C_{t'} \cap E(M_\ell) \neq \emptyset$ and $C_{t'}$ goes through e_j. Then there are cycles $C'_{t'}$ and $C''_{t'}$ of M' and M_ℓ respectively such that $C_{t'} = C'_{t'} \triangle C''_{t'}$ and $C'_{t'} \cap C''_{t'} = \{e_j\}$. Because $e_j \in C''_{t'} \subseteq F_\ell \cup \{e_j\}$ and $e_i \notin C''_{t'}$, $F_\ell \setminus \{e_i\}$ spans e_j in M_ℓ. Let $t'' \in T_\ell$. There is a circuit $C_{t''}$ of M such that $t'' \in C_{t''} \subseteq F \cup \{t''\}$. If $C_{t''} \setminus \{t''\} \subseteq E(M_\ell)$, then $C_{t''} \setminus \{t''\} \subseteq F_\ell$ and F_ℓ spans t'' in M_ℓ. Assume that $C_{t''} \cap E(M') \neq \emptyset$. Then there are cycles $C'_{t''}$ and $C''_{t''}$ of M' and M_ℓ respectively such that $C_{t''} = C'_{t''} \triangle C''_{t''}$ and $C'_{t''} \cap C''_{t''} = \{e_h\}$ for some $h \in \{1,2,3\}$. By c), either $h = i$ of $h = j$. If $h = i$, then $e_h \in F_\ell$ and, therefore, $C''_{t''} \setminus \{t'\} \subseteq F_\ell$. Hence, F_ℓ spans t'' in this case. Assume that $h = j$ and consider $C = C''_{t''} \triangle C''_{t'}$.

Notice that C is a cycle of M_ℓ by Observation 3.4 and $t'' \in C \subseteq F_\ell \cup \{t''\}$. Hence, F_ℓ spans t''.

The proof of the claim that F' spans $T' \cup \{e_i\}$ in M' is done by the same arguments using symmetry.

Since F_ℓ spans $T_\ell \cup \{e_j\}$ in M_ℓ, $w(F_\ell) \geq k_{ij}^{(3)}$. Because $w(F') + w(F_\ell) = w(F) \leq k$, $w(F') \leq k - k_{ij}^{(3)}$ in this case. Hence, $(M', w', T' \cup \{e_i\}, k - k_{ij}^{(3)})$ is a yes-instance for a branch of type (iii).

Case 4. There is $i \in \{1,2,3\}$ such that a)there is $t \in T_\ell$ such that $C_t \cap E(M') \neq \emptyset$ and C_t goes through e_i, and b)for any $t \in T$, there is no circuit C_t that goes through $e_h \in Z$ for $h \neq i$. Notice that this case is symmetric to Case 1. Using the same arguments, we prove that $(M', w', T' \cup \{e_i\}, k - k_i^{(4)})$ is a yes-instance for a branch of type (iv).

Case 5. There are distinct $i, j \in \{1,2,3\}$ such that a)there is $t \in T_\ell$ such that $C_t \cap E(M') \neq \emptyset$ and C_t goes through e_i, b) there is $t \in T'$ such that $C_t \cap E(M') \neq \emptyset$ and C_t goes through e_j. This case is symmetric to Case 2. Using the same arguments, we obtain that $(M', w', T' \cup \{e_1, e_2\}, k - k^{(5)})$ is a yes-instance for a branch of type (v).

If the conditions of Cases 1–5 are not fulfilled, we get the last case.

Case 6. For any $t \in T$, either $C_t \subseteq E(M_\ell)$ or $C_t \subseteq E(M')$. Let $F_\ell = F \cap E(M_\ell)$ and $F' = F \cap E(M')$. We have that F_ℓ spans T_ℓ and F' spans T'. Notice that $w(F_\ell) \geq k^{(6)}$. Because $w(F') + w(F_\ell) = w(F) \leq k$, we have that $(M', w', T', k - k^{(6)})$ is a yes-instance for a branch of type (vi).

Assume now that for one of the branches, we get a yes-answer. We show that the original instance (M, w, T, k) is a yes-instance. To do it, we consider 6 cases corresponding to the types of branches. We use essentially the same arguments in all the cases: we take a solution F' for the instance obtained in the corresponding branch and combine it with a solution F_ℓ of the instance for M_ℓ to obtains a solution for the original instance.

Case 1. $(M', w', T', k - k_i^{(1)})$ is a yes-instance of a branch of type (i). Let $F_\ell \subseteq E(M_\ell) \setminus (T_\ell \cup \{e_i\})$ with $w_\ell(F_\ell) \leq k_i^{(1)}$ be a set that spans $T_\ell \cup \{e_i\}$ in M_ℓ. Clearly, $k_i^{(1)} \leq k$. Consider $F' \subseteq E(M') \setminus T'$ with $w'(F') \leq k - k_i^{(1)}$ that spans T' in M'. Let $F = (F' \setminus \{e_i\}) \cup F_\ell$. Notice that $Z \cap F_\ell = \emptyset$, because $w_\ell(e_h) = k+1$ for $h \in \{1,2,3\}$. Similarly, $e_h \notin F'$ for $h \in \{1,2,3\}$ such that $h \neq i$, because $w'(e_h) = k+1$. Hence, $F \subseteq E(M) \setminus T$. It is easy to see that $w(F) \leq k$. We show that F spans T in M.

Let $t \in T$. Suppose first that $t \in T_\ell$. There is a circuit C_t of M_ℓ such that $t \in C_t \subseteq F_\ell \cup \{t\}$. It is sufficient to notice that C_t is a cycle of M and, therefore, F spans t in M. Let $t \in T'$. There is a circuit C_t of M' such that $t \in C_t \subseteq F' \cup \{t\}$. If $C_t \setminus \{t\} \subseteq F$, i.e., $e_i \notin C_t$, then F' spans t. Suppose that $e_i \in C_t$. Recall that F_ℓ spans e_i in M_ℓ. Hence, there is a cycle $C^{(i)}$ of M_ℓ such that $e_i \in C^{(i)} \subseteq F_\ell \cup \{e_i\}$. Let $C'_t = C_t \triangle C^{(i)}$. By the definition of 3-sums, C'_t is a cycle of M. We have that $t \in C'_t \subseteq F \cup \{t\}$ and, therefore, F spans t.

Case 2. $(M', w', T', k - k(2))$ is a yes-instance of a branch of type (ii). Let $F_\ell \subseteq E(M_\ell) \setminus (T_\ell \cup \{e_1, e_2\})$ with $w_\ell(F_\ell) \leq k_i^{(1)}$ be a set that spans $T_\ell \cup \{e_1, e_2\}$ in M_ℓ. Clearly, $k^{(2)} \leq k$. Consider $F' \subseteq E(M') \setminus T'$ with $w'(F') \leq k = k^{(2)}$ that spans T' in M'. Let $F = (F' \setminus \{e_1, e_2\}) \cup F_\ell$. Notice that $Z \cap F_\ell = \emptyset$, because $w_\ell(e_h) = k+1$ for

$h \in \{1,2,3\}$. Similarly, $e_3 \notin F'$, because $w'(e_3) = k+1$. Hence, $F \subseteq E(M) \setminus T$. It is easy to see that $w(F) \leq k$. We show that F spans T in M.

Let $t \in T$. Suppose first that $t \in T_\ell$. There is a circuit C_t of M_ℓ such that $t \in C_t \subseteq F_\ell \cup \{t\}$. It is sufficient to notice that C_t is a cycle of M and, therefore, F spans t in M. Let $t \in T'$. There is a circuit C_t of M' such that $t \in C_t \subseteq F' \cup \{t\}$. If $C_t \setminus \{t\} \subseteq F$, i.e., $e_1, e_2 \notin C_t$, then F' spans t. Suppose that $e_1 \in C_t$ and $e_2 \notin C_t$. Recall that F_ℓ spans e_1 in M_ℓ. Hence, there is a cycle $C^{(1)}$ of M_ℓ such that $e_1 \in C^{(1)} \subseteq F_\ell \cup \{e_1\}$. Let $C'_t = C_t \triangle C^{(1)}$. By the definition of 3-sums, C'_t is a cycle of M. We have that $t \in C'_t \subseteq F \cup \{t\}$ and, therefore, F spans t. If $e_1 \notin C_t$ and $e_2 \in C_t$, then we observe that F_ℓ spans e_2 in M_ℓ and there is a cycle $C^{(2)}$ of M_ℓ such that $e_2 \in C^{(2)} \subseteq F_\ell \cup \{e_1\}$. Then we conclude that F spans t using the same arguments as before using symmetry. Suppose that $e_1, e_2 \in C_t$. Consider the cycle $C'_t = C_t \triangle C^{(1)} \triangle C^{(2)}$ of M. We have that $t \in C'_t \subseteq F \cup \{t\}$ and, therefore, F spans t.

Case 3. $(M', w', T' \cup \{e_i\}, k - k_{ij}(3))$ is a yes-instance of a branch of type (iii). Let $F_\ell \subseteq E(M_\ell) \setminus (T_\ell \cup \{e_j\})$ with $w_\ell(F_\ell) \leq k^{(3)}_{ij}$ be a set that spans $T_\ell \cup \{e_j\}$ in M_ℓ such that $F \setminus \{e_i\}$ spans e_j. Clearly, $k^{(3)}_{ij} \leq k$. Consider $F' \subseteq E(M') \setminus (T' \cup \{e_i\})$ with $w'(F') \leq k - k^{(3)}_{ij}$ that spans $T' \cup \{e-i\}$ in M'. Let $F = (F' \setminus \{e_j\}) \cup (F_\ell \setminus \{e_i\})$. Notice that $e_h \notin F_\ell = \emptyset$ for $h \in \{1,2,3\}$ such that $h \neq i$, because $w_\ell(e_h) = k+1$, and $e_h \notin F' = \emptyset$ for $h \in \{1,2,3\}$ such that $h \neq j$, because $w'(e_h) = k+1$. Hence, $F \subseteq E(M) \setminus T$. It is straightforward that $w(F) \leq k$. We show that F spans T in M.

Let $t \in T$. Suppose first that $t \in T_\ell$. There is a circuit C_t of M_ℓ such that $t \in C_t \subseteq F_\ell \cup \{t\}$. If $e_i \notin F_\ell$, then $C_t \setminus \{t\} \subseteq F$ and, therefore, F spans t in M. Suppose that $e_i \in C_t$. Because F' spans e_i in M', there is a cycle $C^{(i)}$ of M' such that $e_i \in C^{(i)} \subseteq F' \cup \{e_i\}$. Suppose that $e_j \notin C^{(i)}$. Let $C'_t = C_t \triangle C^{(i)}$. We have that C'_t is a cycle of M and $t \in C'_t \subseteq F \cup \{t\}$. Hence, F spans t. Suppose now that $e_j \in C^{(i)}$. Since $F_\ell \setminus \{e_i\}$ spans e_j, there is a cycle $C^{(j)}$ of M_ℓ such that $e_j \subseteq C^{(j)} \subseteq (F_\ell \setminus \{e_i\}) \cup \{e_j\}$. Let $C'_t = C_t \triangle C^{(i)} \triangle C^{(j)}$. We obtain that C'_t is a cycle of M and $t \in C'_t \subseteq F \cup \{t\}$. Hence, F spans t. The proof for the case $t \in T'$ uses the same arguments using symmetry.

Case 4. $(M', w', T' \cup \{e_i\}, k - k_i(4))$ is a yes-instance of a branch of type (iv). This case is symmetric to Case 1 and is analyzed in the same way. We consider a set $F_\ell \subseteq E(M_\ell) \setminus T_\ell$ with $w_\ell(F_\ell) \leq k^{(4)}_i$ that spans T_ℓ in M_ℓ and $F' \subseteq E(M') \setminus T'$ with $w'(F') \leq k - k^{(4)}_i$ that spans $T' \cup \{e_i\}$ in M'. Let $F = F' \cup (F_\ell \setminus \{e_i\})$. We have that $F \subseteq E(M) \setminus T$ has weight at most k and spans T in M.

Case 5. $(M', w', T' \cup \{e_1, e_2\}, k - k^{(5)})$ is a yes-instance of a branch of type (v). This case is symmetric to Case 2 and is analyzed in the same way. We consider a set $F_\ell \subseteq E(M_\ell) \setminus T_\ell$ with $w_\ell(F_\ell) \leq k^{(5)}$ that spans T_ℓ in M_ℓ and $F' \subseteq E(M') \setminus T'$ with $w'(F') \leq k - k^{(5)}$ that spans $T' \cup \{e_1, e_2\}$ in M'. Let $F = F' \cup (F_\ell \setminus \{e_1, e_2\})$. We have that $F \subseteq E(M) \setminus T$ has weight at most k and spans T in M.

It remains to consider the last case.

Case 6. $(M', w', T', k - k^{(6)})$ is a yes-instance of a branch of type (v). Let $F_\ell \subseteq E(M_\ell) \setminus T_\ell$ with $w_\ell(F_\ell) \leq k_{(6)}$ be a set that spans T_ℓ in M_ℓ and let $F' \subseteq E(M') \setminus T'$ be a set with $w'(F') \leq k - k^{(6)}$ that spans T' in M'. Notice that for $i \in \{1,2,3\}$, $e_i \notin F_\ell$ and $e_i \notin F'$, because $w_\ell(e_i) = w'(e_i) = k+1$. Consider $F = F'_F \cup F_\ell$. Clearly, $w(F) \leq k$. We show that F spans T in M.

Let $t \in T$. If $t \in T_\ell$, then there is a circuit C_t of M_ℓ such that $t \in C_t \subseteq F_\ell \cup \{t\}$. Since $C_t \subseteq E(M_\ell)$, we have that F_ℓ spans t in M. If $t \in T'$, then by the same arguments, F' spans t not only in M' but also in M.

Since we always have that $k_i^{(1)}, k^{(2)}, k_{ij}^{(3)}, k_i^{(4)}, k^{(5)}, k^{(6)} \geq 1$, the recursive calls are done for the parameters that are strictly less than k. This completes the proof.

The claim that each call of the rule (without recursive steps) takes $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time follows from Lemmas 6.4, 6.12 and 7.1. ∎

From now onwards we assume that there is no child of M_s with terminals. Recall that M_s is either a graphic or cographic matroid. The subsequent steps depend on the type of M_s and are considered in separate sections.

7.2. The case of a graphic sub-leaf. Throughout this section we assume that M_s is a graphic matroid. Let G be a graph such that its cycle matroid $M(G)$ is isomorphic to M_s. We assume that $M(G) = M_s$. Recall that the circuits of $M(G)$ are exactly the cycles of G. We reduce leaves in this case by the following reduction rule. In this reduction rule we first solve a few instances of Space Cover and later use the solutions to these instances to reduce the graph and re-define the weight function.

Reduction Rule 7.4 (**Graphic 3-leaf reduction rule).** *For a child M_ℓ of M_s with $T \cap E(M_\ell) = \emptyset$, do the following. Let $Z = \{e_1, e_2, e_3\} = E(M_s) \cap E(M_\ell)$. Set $w_\ell(e) = w(e)$ for $e \in E(M_\ell) \setminus Z$, $w_\ell(e_1) = w_\ell(e_2) = w_\ell(e_3) = k + 1$.*

(i) For each $i \in \{1, 2, 3\}$, find the minimum $k_i \leq k$ such that $(M_\ell, w_\ell, \{e_i\}, k_i)$ is a yes-instance of Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, \{e_i\}, k_i)$ is a no-instance for every $k_i \leq k$, then we set $k_i = k + 1$.*

(ii) Find the minimum $k' \leq k$ such that $(M_\ell, w_\ell, \{e_1, e_2\}, k')$ is a yes-instance of Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, \{e_1, e_2\}, k')$ is a no-instance for every $k' \leq k$, then we set $k' = k + 1$. If $k' \leq k$, then we find an inclusion minimal set $F_\ell \subseteq E(M_\ell) \setminus Z$ of weight k' that spans e_1 and e_2. Observe that Lemmas 6.3 or 6.10 are only for decision version. However, we can apply standard self reducibility tricks to make them output a solution also. There are circuits C_1 and C_2 of M_ℓ such that $e_1 \in C_1 \subseteq F_\ell \cup \{e_1\}$, $e_2 \in C_2 \subseteq F_\ell \cup \{e_2\}$ and $F_\ell = (C_1 \setminus \{e_1\}) \cup (C_2 \setminus \{e_2\})$. Notice that C_1 and C_2 can be found by finding inclusion minimal subsets of F_ℓ that span e_1 and e_2, respectively.*

Recall that Z induces a cycle of G. Denote by v_1, v_2, and v_3 the vertices of the cycle. Furthermore, let v_1, v_2, and v_3 be incident to e_3, e_1, e_1, e_2 and e_2, e_3, respectively. We construct the graph G' by adding a new vertex u and making it adjacent to v_1, v_2 and v_3. Notice that because the circuits of $M(G)$ are cycles of G, any circuit of $M(G)$ is also a circuit of $M(G')$. Let M' the matroid defined by the conflict tree $\mathcal{T}' = \mathcal{T} - M_\ell$ and where M_s is replaced by $M(G')$. The weight function $w' \colon E(M') \to \mathbb{N}$ is defined by setting $w'(e) = w(e)$ for $e \in E(M') \setminus (Z \cup \{v_1u, v_2u, v_3u\})$, $w'(e_1) = k_1$, $w'(e_2) = k_2$, and $w'(e_3) = k_3$. If if $k' \leq k$ then we set $w'(v_1u) = w(C_1 \setminus (C_2 \cup \{e_1\}))$, $w'(v_3u) = w(C_1 \setminus (C_2 \cup \{e_2\}))$ and $w'(v_1u) = w(C_1 \cap C_2)$; else we set $w'(v_1u) = w'(v_2u) = w'(v_3u) = k + 1$. The reduced instance is denoted by (M', w', T, k).

The construction of G' and Observation 7.4 immediately imply the following observation.

Observation 7.5. *For any distinct* $i, j \in \{1, 2, 3\}$,

$$w'(e_i) + w'(e_j) = k_i + k_j \geq k' = w'(v_1u) + w'(v_2u) + w'(v_3u)$$

and if $k' \leq k$ *then* $w'(v_iu) + w'(v_ju) \geq w'(v_iv_j)$. *Also, if* $w'(e_i) + w'(e_j) \leq k$ *for some distinct* $i, j \in \{1, 2, 3\}$, *then* $k' \leq k$.

We use Observation 7.5 to prove that the rule is safe.

Lemma 7.7. *Reduction Rule 7.4 is safe and can be applied in* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ *time.*

Proof. Denote by M'' the matroid defined by $\mathcal{T}' = \mathcal{T} - M_\ell$. To prove that the rule is safe, first assume that (M, w, T, k) is a yes-instance. Then there is an inclusion minimal set $F \subseteq E(M) \setminus T$ of weight at most k that spans T. If $F \cap E(M_\ell) = \emptyset$, then F spans T in M' as well and (M', w', T, k) is a yes-instance. Suppose from now that $F \cap E(M_\ell) \neq \emptyset$.

For each $t \in T$, there is a circuit C_t of M such that $t \in C \subseteq F \cup \{t\}$. If $C_t \cap E(M_\ell) \neq \emptyset$, $C_t = C'_t \triangle C''_t$, where C'_t is a cycle of M'' and C''_t is a cycle of M_ℓ. By Observation 7.3, we can assume that $C'_t \cap C''_t$ contains the unique element e_i, i.e., C_t goes through e_i. To simplify notation, it is assumed that $v_4 = v_1$. We consider the following three cases.

Case 1. There is a unique $e_i \in Z$ such that for any $t \in T$, either $C_t \subseteq E(M'')$ or C_t goes through e_i. Let $F' = (F \cap E(M'')) \cup \{e_i\}$.

We show that F' spans T in M'. Let $t \in T$. If $C_t \subseteq E(M'')$, then $t \in C_t \subseteq (F \cap E(M'')) \cup \{t\}$ and, therefore, F' spans t in M'. Suppose that $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t = C'_t \triangle C''_t$, where C'_t is a cycle of M'', C''_t is a cycle of M_ℓ and $C'_t \cap C''_t = \{e_i\}$. We have that $t \in C'_t \cup \{t\}$ and $C'_t \setminus \{t\} \subseteq F'$ spans t.

Because $F \cap E(M_\ell) \neq \emptyset$ and F is inclusion minimal spanning set, there is $t \in T$ such that C_t goes through e_i. Let $C_t = C'_t \triangle C''_t$, where C'_t is a cycle of M'', C''_t is a cycle of M_ℓ and $C'_t \cap C''_t = \{e_i\}$. Notice that $C''_t \setminus \{e_i\}$ spans e_i in M_ℓ. Hence, $w_\ell(C''_t \setminus \{e_i\}) \geq k_i$. Because $w'(e_i) = k_i$, we conclude that $w'(F') \leq w(F)$.

Since $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 2. There are two distinct $e_i, e_j \in Z$ such that for any $t \in T$, either $C_t \subseteq E(M'')$, or C_t goes through e_i, or C_t goes through e_j, and at least one C_t goes through e_i and at least one C_t goes through e_j. Let $F' = (F \cap E(M'')) \cup \{v_1u, v_2u, v_3u\}$.

We claim that F' spans T in M'. Let $t \in T$. If $C_t \subseteq E(M'')$, then $t \in C_t \subseteq (F \cap E(M'')) \cup \{t\}$ and, therefore, F' spans t in M'. Suppose that $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t = C'_t \triangle C''_t$, where C'_t is a cycle of M'', C''_t is a cycle of M_ℓ and either $C'_t \cap C''_t = \{e_i\}$ or $C'_t \cap C''_t = \{e_j\}$. By symmetry, let $C'_t \cap C''_t = \{e_i\}$. Because $e_i, v_iu, v_{i+1}u$ induce a cycle of the graph G', $\{e_i, v_iu, v_{i+1}u\}$ is a circuit of M' and $C'''_t = C'_t \triangle \{e_i, v_iu, v_{i+1}u\}$ is a cycle of M'. We have that $t \in C'''_t \cup \{t\}$ and $C'''_t \setminus \{t\} \subseteq F'$ spans t.

Because $F \cap E(M_\ell) \neq \emptyset$, there is $t \in T$ such that C_t goes through e_i and there is $t' \in T$ such that $C_{t'}$ goes through e_j. Let $C_t = C'_t \triangle C''_t$ and $C_{t'} = C'_{t'} \triangle C''_{t''}$, where $C'_t, C'_{t'}$ are cycles of M'', $C''_t, C''_{t'}$ are cycles of M_ℓ and $C'_t \cap C''_t = \{e_i\}$, $C'_{t'} \cap C''_{t'} = \{e_j\}$. Notice that $C''_t \setminus \{e_i\}$ spans e_i in M_ℓ and $C''_{t'} \setminus \{e_j\}$ spans e_j. Hence, $w_\ell((C''_t \setminus \{e_i\}) \cup (C''_{t'} \setminus \{e_j\})) \geq w_\ell(F_\ell) = k'$ by Observation 7.4. Because $w'(\{v_1u, v_2u, v_3u\}) = k'$, $w'(F') \leq w(F)$.

Since $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 3. For each $i \in \{1,2,3\}$, there is $t \in T$ such that C_t goes through e_i. As in Case 1, we set $F' = (F \cap E(M'')) \cup \{v_1u, v_2u, v_3u\}$ and use the same arguments to show that $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M'.

Assume now that the reduced instance (M', w', T, k) is a yes-instance. Let $F' \subseteq E(M') \setminus T$ be an inclusion minimal set of weight at most k that spans T in M'. Let $S = \{e_1, e_2, e_3, v_1u, v_2u, v_3u\}$. If $F' \cap S = \emptyset$, then $F' \subseteq E(M)$ and, therefore, F' spans T in M as well. Assume from now that $F' \cap S \neq \emptyset$. By Observation 3.1 and because $\{v_1, v_2, v_3\}$ separates u from $V(G) \setminus \{v_1, v_2, v_3\}$ in G', the edges of $F' \cap S$ induce a tree in G'. Moreover, u is incident to either 2 or 3 edges of this tree. We consider the following cases depending on the structure of the tree.

Case 1. One one the following holds: i) $v_1u, v_2u, v_3u \in F'$ or ii) $|\{v_1u, v_2u, v_3u\} \cap F'| = 2$ and $\{e_1, e_2, e_3\} \cap F' \neq \emptyset$ or iii)$|\{e_1, e_2, e_3\} \cap F'| \geq 2$. We define $F = (F' \setminus S) \cup F_\ell$. Clearly, $F \subseteq E(M) \setminus T$. Notice also that $w'(F \cap S) \geq k'$ by Observation 7.5 and, therefore, $w(F) \leq k$. To show that (M, w, T, k) is a yes-instance, we prove that F spans T in M.

Let $t \in T$. Since F' spans t in M', there is a circuit C_t of M' such that $t \in C_t \subseteq F' \cup \{t\}$. If $C_t \cap S = \emptyset$, then $C_t \setminus \{t\}$ spans t in M. Suppose that $C_t \cap S \neq \emptyset$. As S induces a complete graph on 4 vertices in G' and $\{v_1, v_2, v_3\}$ separate u from $V(G) \setminus \{v_1, v_2, v_3\}$, we conclude that there is $i \in \{1,2,3\}$ such that $C'_t = (C_t \setminus S) \cup \{e_i\}$ is a cycle of M'. Notice that C'_t is also a cycle of M''. By the definition of F_ℓ and Observation 7.4, there is a cycle C''_t of M_ℓ such that $e_i \in C''_t \subseteq F_\ell \cup \{e_i\}$. Consider the cycle $C'''_t = C'_t \triangle C''_t$ of M. We have that $t \in C'''_t \subseteq F$ and, therefore, F spans t.

If the conditions i)–iii) of Case 1 are not fulfilled, then $F' \cap S = \{e_i\}$ for some $i \in \{1,2,3\}$.

Case 2. $F' \cap S = \{e_i\}$ for some $i \in \{1,2,3\}$. By the definition of $w'(e_i) = k_i$, there is a circuit C of M_ℓ such that $e_i \in C \subseteq (E(M_\ell) \setminus Z) \cup \{e_i\}$ and $w_\ell(C \setminus \{e_i\}) = k_i$. Let $F = F' \triangle C$. Clearly, $w(F) \leq k$. We show that F spans T.

Let $t \in T$. Since F' spans t in M', there is a circuit C_t of M' such that $t \in C_t \subseteq F' \cup \{t\}$. If $C_t \cap S = \emptyset$, then C_t spans t in M. Suppose that $C_t \cap S \neq \emptyset$, i.e., $C_t \cap S = \{e_i\}$. Notice that C_t is also a cycle of M''. Consider the cycle $C'_t = C_t \triangle C$. Since $t \in C'_t \subseteq F \cup \{t\}$, F spans t.

From the description of Reduction Rule 7.4 and Lemma 7.1, it can be deduced that Reduction Rule 7.4 can be applied in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. ☐

7.3. The case of a cographic sub-leaf. Now we have reached the final step of our algorithm. Throughout this section we assume that M_s is a cographic matroid. Let G be a graph such that the bond matroid of G is isomorphic to M_s. The algorithm that constructs a good $\{1,2,3\}$-decomposition could be also be used to output the graph G Without loss of generality, we can assume that G is connected. Also, recall that the circuits of the bond matroid $M^*(G)$ are exactly minimal cut-sets of G.

The isomorphism between M_s and $M^*(G)$ is not necessarily unique. We could choose any isomorphism between M_s and $M^*(G)$ that is beneficial for our algorithmic purposes. Indeed, in what follows we fix an isomorphism that is useful in designing our algorithm. Let $M_\ell^{(1)}, \ldots, M_\ell^{(s)}$ denote those leaves of the conflict tree $\mathcal{T}$ that are also the children of M_s. Let $Z_i = E(M_s) \cap E(M_\ell^{(i)})$, $i \in \{1, \ldots, s\}$. If M_s has a parent M^* in $\mathcal{T}$ and $E(M_s) \cap E(M^*) \neq \emptyset$, then let Z^* denote $Z^* = E(M_s) \cap E(M^*)$; we *emphasize* that Z^* may not exist. Next we define the notion of *clean cut*.

Definition 7.8. *We say that* $\alpha(Z_i) \subseteq E(G)$ *is a* clean cut *with respect to an isomorphism* $\alpha\colon M_s \to M^*(G)$, *if there is a component* H *of* $G - \alpha(Z_i)$ *such that*

(i) H *has no bridge,*

(ii) $E(H) \cap \alpha(Z_j) = \emptyset$ *for* $j \in \{1, \ldots, s\}$, *and*

(iii) $E(H) \cap \alpha(Z^*) = \emptyset$ *if* Z^* *exists.*

We call H *a* clean component *of* $G - \alpha(Z_i)$.

Next we show that given any isomorphism between M_s and $M^*(G)$, we can obtain another isomorphism between M_s and $M^*(G)$ with respect to which we have at least one clean component.

Lemma 7.9. *There is an isomorphism* $\alpha\colon M_s \to M^*(G)$ *and a child* $M_\ell^{(i)}$ *of* M_s *such that* $\alpha(Z_i)$ *is a clean cut with respect to* α. *Moreover, given any arbitrary isomorphism from* M_s *to* $M^*(G)$, *one can obtain such an isomorphism and a clean cut together with a clean component in polynomial time.*

Proof. We prove the lemma first assuming that Z^* exists. Let $\alpha\colon M_s \to M^*(G)$ be an isomorphism. Clearly α maps $E(M_s)$ to the edges of G. Suppose that there is $p \in \{1, \ldots, s\}$ such that there is a component H of $G - \alpha(Z_p)$ with $E(H) \cap \alpha(Z^*) = \emptyset$. Then we set $\alpha_0 = \alpha$, $H^{(0)} = H$ and $i_0 = p$. Otherwise, let $p \in \{1, \ldots, s\}$. Denote by H_1 and H_2 the components of $G - \alpha(Z_p)$. Because $|Z^*| \leq 3$, $E(H_1) \cap \alpha(Z^*) \neq \emptyset$ and $E(H_2) \cap \alpha(Z^*) \neq \emptyset$, there is H_j for $j \in \{1, 2\}$ such that $|E(H_j) \cap \alpha(Z^*)| = 1$. Let $\{e\} = E(H_j) \cap \alpha(Z^*)$. Since $\alpha(Z^*)$ is a cut-set, e is a bridge of H_j. By the minimality of $\alpha(Z^*)$, every component of $H - e$ contains an end vertex of an edge of $\alpha(Z_p)$. Since $|\alpha(Z_p)| = 3$, we obtain that there is $e' \in \alpha(Z_p)$ such that $\{e, e'\}$ is a minimal cut-set of G. Let $\alpha'(x) = \alpha(x)$ for $x \in E(M_s) \setminus \{\alpha^{-1}(e), \alpha^{-1}(e')\}$, $\alpha'(\alpha^{-1}(e)) = e'$ and $\alpha'(\alpha^{-1}(e')) = e$. By Observation 3.2, α' is an isomorphism of M_s to $M^*(G)$. Notice that now we have a component H of $G - \alpha'(Z_p)$ with $E(H) \cap \alpha'(Z^*) = \emptyset$. Respectively, we set $\alpha_0 = \alpha'$, $H^{(0)} = H$ and $i_0 = p$.

Assume inductively that we have a sequence $(\alpha_0, i_0, H^{(0)}), \ldots, (\alpha_q, i_q, H^{(q)})$, where $\alpha_0, \ldots, \alpha_q$ are isomorphisms of M_s to $M^*(G)$, $i_0, \ldots, i_q \in \{1, \ldots, s\}$, $H^{(j)}$ is a component of $G - \alpha_j(Z_{i_j})$ for $j \in \{1, \ldots, q\}$, $Z^* \cap E(H^{(j)}) = \emptyset$ for $j \in \{1, \ldots, s\}$, and $V(H^{(0)}) \supset \ldots \supset V(H^{(q)})$.

If $\alpha(Z_{i_q})$ is a clean cut with respect to α_q, the algorithm returns $(\alpha_q, i_q, H^{(q)})$ and stops. Suppose that $\alpha(Z_{i_q})$ is not clean cut with respect to α_q. We show that we can extend the sequence in this case. To do it, we consider the following three cases.

Case 1. $H^{(q)}$ has a bridge e. Because loops of M are deleted by **Loop reduction rule**, e is not a bridge of G. Hence, each of the two components of $H^{(q)}$ contains an end vertex of an edge of $\alpha_q(Z_{i_q})$. Since $|Z_{i_q}| = 3$, there is a component H' of $H^{(q)} - e$ that contains an end vertex of a unique edge e' of $\alpha_q(Z_{i_q})$ and the other component $H^{(q+1)}$ contains end vertices of two edges of $\alpha_q(Z_{i_q})$. We obtain that $\{e, e'\}$ is a minimal cut-set of G. Let $\alpha_{q+1}(x) = \alpha_q(x)$ for $x \in E(M_s) \setminus \{\alpha_q^{-1}(e), \alpha_q^{-1}(e')\}$, $\alpha_{q+1}(\alpha_q^{-1}(e)) = e'$ and $\alpha_{q+1}(\alpha_q^{-1}(e')) = e$. By Observation 3.2, α_{q+1} is an isomorphism of M_s to $M^*(G)$. Clearly, $H^{(q+1)}$ is a component of $G - \alpha_{q+1}(Z_{i_q})$ and $V(H^{(q+1)}) \subset V(H^{(q)})$. Hence, we can extend the sequence by $(\alpha_{q+1}, i_{q+1}, H^{(q+1)})$ for $i_{q+1} = i_q$.

Case 2. There is $i_{q+1} \in \{1, \ldots, s\}$ such that $\alpha_q(Z_{i_{q+1}}) \subseteq E(H^{(q)})$. Because $\alpha_q(Z_{i_{q+1}})$ is a minimal cut-set of G, we obtain that there is a component $H^{(q+1)}$ of $G - \alpha_q(Z_{i_{q+1}})$ such that $V(H^{(q+1)}) \subset V(H^{(q)})$. We extend the sequence by $(\alpha_{q+1}, i_{q+1}, H^{(q+1)})$ for $\alpha_{q+1} = \alpha_q$.

Case 3. There is $i_{q+1} \in \{1, \ldots, s\}$ such that $\alpha_q(Z_{i_{q+1}}) \cap E(H^{(q)}) \neq \emptyset$ but $|\alpha_q(Z_{i_{q+1}}) \cap$

$E(H^{(q)})| \leq 2$. If $|\alpha_q(Z_{i_{q+1}}) \cap E(H^{(q)})| = 1$, then the unique edge $e \in \alpha_q(Z_{i_{q+1}}) \cap E(H^{(q)})$ is a bridge of $H^{(q)}$, because $\alpha_q(Z_{i_{q+1}})$ is a minimal cut-set. Hence, we have Case 1. Assume that $|\alpha_q(Z_{i_{q+1}}) \cap E(H^{(q)})| = 1$. Let H' be the component of $G - \alpha_q(Z_{i_q})$ distinct from $H^{(q)}$. Since $|Z_{i_{q+1}}| = 3$, we have that $|\alpha_q(Z_{i_{q+1}}) \cap E(H')| = 1$, then the unique edge $e \in \alpha_q(Z_{i_{q+1}}) \cap E(H')$ is a bridge of H'. By the same arguments as in Case 1, there is $e' \in \alpha_q(Z_{i_q})$ such that $\{e, e'\}$ is a minimal cut-set of G. Using Observation 3.2, we construct the isomorphism α_{q+1} of M_s to $M^*(G)$ by defining $\alpha_{q+1}(x) = \alpha_q(x)$ for $x \in E(M_s) \setminus \{\alpha_q^{-1}(e), \alpha_q^{-1}(e')\}$, $\alpha_{q+1}(\alpha_q^{-1}(e)) = e'$ and $\alpha_{q+1}(\alpha_q^{-1}(e')) = e$. It remains to observe that $G - \alpha_{q+1}(Z_{i_{q+1}})$ has a component $H^{(q+1)}$ such that $V(H^{(q+1)}) \subset V(H^{(q)})$ and extend the sequence by $(\alpha_{q+1}, i_{q+1}, H^{(q+1)})$.

For each $j \geq 1$ we have that $V(H^{(j)} \subset V(H^{(j-1)})$. This implies that the sequence

$$(\alpha_0, i_0, H^{(0)}), \ldots, (\alpha_q, i_q, H^{(q)})$$

has length at most n. Hence, after at most n iteration we obtain an isomorphism α and a clean cut with respect to α together with a clean component. Since every step in the iterative construction of the sequence $(\alpha_0, i_0, H^{(0)}), \ldots, (\alpha_q, i_q, H^{(q)})$ can be done in polynomial time, the algorithm is polynomial.

Recall that in the beginning we assume that Z^* is present. The case when Z^* is absent is more simpler and could be proved as in the case when Z^* is present and thus it is omitted. ∎

Using Lemma 7.9, we can always assume that we have an isomorphism of M_s to $M^*(G)$ such that for a child M_ℓ of M_s in (T), $Z = E(M_s) \cap E(M_\ell)$ is mapped to a clean cut. To simplify notation, we assume that $M_s = M^*(G)$ and Z is a clean cut with respect to this isomorphism. Denote by H the clean component. Let $Z = \{e_1, e_2, e_3\}$ and let $e_i = x_i y_i$ for $i \in \{1, 2, 3\}$, where $y_1, y_2, y_3 \in V(H)$. Notice that some y_1, y_2, y_3 can be the same. We first handle the case when $E(H) \cap T = \emptyset$.

7.3.1. Cographic sub-leaf: $E(H) \cap T = \emptyset$**.** In this case we give a reduction rule that reduces the leaf M_ℓ. Recall that $E(M_\ell) \cap T = \emptyset$. Now we are ready to give a reduction rule analogous to the one for graphic matroid.

Reduction Rule 7.5 (**Cographic 3-leaf reduction rule).** *If $E(H) \cap T = \emptyset$, then do the following. Set $w_\ell(e) = w(e)$ for $e \in E(M_\ell) \setminus Z$, $w_\ell(e_1) = w_\ell(e_2) = w_\ell(e_3) = k + 1$.*

(i) For each $i \in \{1, 2, 3\}$, find the minimum $k_i^{(1)} \leq k$ such that $(M_\ell, w_\ell, \{e_i\}, k_i^{(1)})$ is a yes-instance of Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, \{e_i\}, k_i^{(1)})$ is a no-instance for every $k_i^{(1)} \leq k$, then we set $k_i^{(1)} = k + 1$.*

(ii) Find the minimum $p^{(1)} \leq k$ such that $(M_\ell, w_\ell, \{e_1, e_2\}, p^{(1)})$ is a yes-instance of Space Cover *using Lemmas 6.3 or 6.10, respectively, depending on the type of M_ℓ. If $(M_\ell, w_\ell, \{e_1, e_2\}, p^{(1)})$ is a no-instance for every $p^{(1)} \leq k$), then we set $p^{(1)} = k + 1$. If $p^{(1)} \leq k$, then we find an inclusion minimal set $F_\ell \subseteq E(M_\ell) \setminus Z$ of weight $p^{(1)}$ that spans e_1 and e_2. Observe that Lemmas 6.3 or 6.10 are only for decision version. However, we can apply standard self reducibility tricks to make them output a solution also. There are circuits C_1 and C_2 of M_ℓ such that $e_1 \in C_1 \subseteq F_\ell \cup \{e_1\}$, $e_2 \in C_2 \subseteq F_\ell \cup \{e_2\}$ and $F_\ell = (C_1 \setminus \{e_1\}) \cup (C_2 \setminus \{e_2\})$. Notice that C_1 and C_2 can be found by finding inclusion minimal subsets of F_ℓ that span e_1 and e_2 respectively. Let*

$p_1^{(1)} = w_\ell(C_1 \setminus (C_2 \cup \{e_1\}))$, $p_2^{(1)} = w_\ell(C_2 \setminus (C_1 \cup \{e_2\}))$ *and* $p_3^{(1)} = w_\ell(C_1 \cap C_2)$. *If* $p^{(1)} = k+1$, *we set* $p_1^{(1)} = p_2^{(1)} = p_3^{(1)} = k+1$.

Construct an auxiliary graph H' *from* H *by adding a vertex* u *and edges* e'_1, e'_2, e'_3, *where* $e'_i = uy_i$ *for* $i \in \{1,2,3\}$*; notice that this could result in multiple edges. Set* $w_h(e) = w(e)$ *for* $e \in E(H)$ *and set* $w_h(e'_1) = w_h(e'_2) = w_h(e'_3) = k+1$.

(iii) For each $i \in \{1,2,3\}$, *find the minimum* $k_i^{(2)} \leq k$ *such that* $(M^*(H'), w_h, \{e'_i\}, k_i^{(2)})$ *is a yes-instance of* Space Cover *using Lemma 6.10. If* $(M^*(H'), w_h, \{e'_i\}, k_i^{(2)})$ *is a no-instance for every* $k_i^{(1)} \leq k$, *then we set* $k_i^{(2)} = k+1$.

(iv) Find the minimum $p^{(2)} \leq k$ *such that* $(M^*(H'), w_h, \{e'_1, e'_2\}, p^{(2)})$ *is a yes-instance of* Space Cover *using Lemma 6.10. If* $(M^*(H'), w_h, \{e'_1, e'_2\}, p^{(2)})$ *is a no-instance for every* $p^{(2)} \leq k$, *then we set* $p^{(2)} = k+1$ *. If* $p^{(2)} \leq k$, *then we find an inclusion minimal set* $F_h \subseteq E(H') \setminus Z$ *of weight* $p^{(2)}$ *that spans* e'_1 *and* e'_2*. Observe that Lemma 6.10 is only for decision version. However, we can apply standard self reducibility tricks to make it output a solution also. There are circuits* C_1 *and* C_2 *of* $M^*(H')$ *such that* $e'_1 \in C_1 \subseteq F_h \cup \{e'_1\}$, $e_2 \in C_2 \subseteq F_h \cup \{e'_2\}$ *and* $F_h = (C_1 \setminus \{e'_1\}) \cup (C_2 \setminus \{e'_2\})$*. Notice that* C_1 *and* C_2 *can be found by finding inclusion minimal subsets of* F_h *that span* e'_1 *and* e'_2 *respectively. Let* $p_1^{(2)} = w_h(C_1 \setminus (C_2 \cup \{e'_1\}))$, $p_2^{(2)} = w_h(C_2 \setminus (C_1 \cup \{e'_2\}))$ *and* $p_3^{(3)} = w_h(C_1 \cap C_2)$*. If* $p^{(2)} = k+1$, *we set* $p_1^{(2)} = p_2^{(2)} = p_3^{(2)} = k+1$.

Construct the graph G' *from* $G - V(H)$ *by adding three pairwise adjacent vertices* z_1, z_2, z_3 *and edges* x_1z_1, x_2z_2, x_3z_3*. Let* M' *the matroid defined by* $\mathcal{T}' = \mathcal{T} - M_\ell$, *where* M_s *is replaced by* $M^*(G')$*. The weight function* $w' \colon E(M') \to \mathbb{N}$ *is defined by setting* $w'(e) = w(e)$ *for* $e \in E(M') \setminus \{x_1z_1, x_2z_2, x_2z_3, z_1z_2, z_2z_3, z_1z_3\}$, $w'(x_iz_i) = \min\{k_i^1, k_i^2\}$ *for* $i \in \{1,2,3\}$*. If* $p^{(1)} \leq p^{(2)}$, *then* $w'(z_1z_3) = p_1^{(1)}$, $w'(z_2z_3) = p_2^{(1)}$ *and* $w'(z_1z_2) = p_3^{(1)}$, *and* $w'(z_1z_3) = p_1^{(2)}$, $w'(z_2z_3) = p_2^{(2)}$ *and* $w'(v_1v_2) = p_3^{(2)}$ *otherwise. The reduced instance is* (M', w', T, k).

Similarl to Observation 7.5, we observe the following using Observation 7.4.

Observation 7.6. *For each* $i \in \{1,2,3\}$, *and* $j, q \in \{1,2,3\} \setminus \{i\}$ *we have that* $w'(z_iz_j) + w'(z_iz_q) \geq w'(x_iz_i)$*. Also, for any distinct* $i, j \in \{1,2,3\}$ *and* $q \in \{1,2\}$, *if* $k_i^{(q)} + k_j^{(q)} \leq k$, *then* $p^{(q)} \leq k_i^{(q)} + k_j^{(q)}$.

The next lemma proves the safeness of the Reduction Rule 7.5.

Lemma 7.10. *Reduction Rule 7.5 is safe and can be applied in* $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ *time.*

Proof. Denote by M'' the matroid defined by $\mathcal{T}' = \mathcal{T} - M_\ell$. To prove that the rule is safe, assume first that (M, w, T, k) is a yes-instance. Then there is an inclusion minimal set $F \subseteq E(M) \setminus T$ of weight at most k that spans T.

Suppose that $F \cap E(M_\ell) = \emptyset$ and $F \cap E(H) = \emptyset$. By the definition of G', any minimal cut-set C of G such that $C \cap Z$ and $C \cap E(H) = \emptyset$ is a minimal cut-set of G', because H is a connected graph. We obtain that F spans T in M' and (M', w', T, k) is a yes-instance.

Assume that $F \cap E(M_\ell) \neq \emptyset$ and $F \cap E(H) = \emptyset$. The proof for this case is, in fact, almost identical to the proof for **Graphic 3-leaf Reduction Rule**.

For each $t \in T$, there is a circuit C_t of M such that $t \in C \subseteq F \cup \{t\}$. If $C_t \cap E(M_\ell) \neq \emptyset$, $C_t = C'_t \triangle C''_t$, where C'_t is a cycle of M'' and C''_t is a cycle of M_ℓ. By Observation 7.3, we can assume that C'_t and C''_t are circuits of M'' and M_ℓ

respectively and $C'_t \cap C''_t$ contains the unique element e_i, i.e., C_t goes through e_i. Notice that every $(C'_t \setminus \{e_i\}) \cup \{x_i z_i\}$ is a minimal cut-set of G' and, therefore, a circuit of $M^*(G')$. We consider the following three cases.

Case 1. There is a unique $e_i \in Z$ such that for any $t \in T$, either $C_t \subseteq E(M'')$ or C_t goes through e_i. Let $F' = (F \cap E(M'')) \cup \{x_i z_i\}$.

We show that F' spans T in M'. Let $t \in T$. If $C_t \subseteq E(M'')$, then $t \in C_t \subseteq (F \cap E(M'')) \cup \{t\}$ and, therefore, F' spans t in M'. Suppose that $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t = C'_t \triangle C''_t$, where C'_t is a circuit of M'', C''_t is a circuit of M_ℓ and $C'_t \cap C''_t = \{e_i\}$. We have that $t \in C'_t \cup \{t\}$ and $((C'_t \setminus \{e_i\}) \cup \{x_i z_i\}) \setminus \{t\} \subseteq F'$ spans t.

Because $F \cap E(M_\ell) \neq \emptyset$ and F is inclusion minimal spanning set, there is $t \in T$ such that C_t goes through e_i. Let $C_t = C'_t \triangle C''_t$, where C'_t is a circuit of M'', C''_t is a circuit of M_ℓ and $C'_t \cap C''_t = \{e_i\}$. Notice that $C''_t \setminus \{e_i\}$ spans e_i in M_ℓ. Hence, $w_\ell(C''_t \setminus \{e_i\}) \leq k_i^{(1)}$. Because $w'(x_i z_i) \leq k_i^{(1)}$, we conclude that $w'(F') \leq w(F)$.

Since $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 2. There are two distinct $e_i, e_j \in Z$ such that for any $t \in T$, either $C_t \subseteq E(M'')$, or C_t goes through e_i, or C_t goes through e_j, and at least one C_t goes through e_i and at least one C_t goes through e_j. Let $F' = (F \cap E(M'')) \cup \{z_1 z_2, z_2 z_3, z_1 z_3\}$.

We claim that F' spans T in M'. Let $t \in T$. If $C_t \subseteq E(M'')$, then $t \in C_t \subseteq (F \cap E(M'')) \cup \{t\}$ and, therefore, F' spans t in M'. Suppose that $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t = C'_t \triangle C''_t$, where C'_t is a circuit of M'', C''_t is a circuit of M_ℓ and either $C'_t \cap C''_t = \{e_i\}$ or $C'_t \cap C''_t = \{e_j\}$. By symmetry, let $C'_t \cap C''_t = \{e_i\}$. Because $\{x_i z_i, z_i z_{i-1}, z_i z_{i+1}\}$ (here and further it is assumed that $z_0 = z_3$ and $z_4 = z_1$) is a minimal cut-set of G, $\{x_i z_i, z_i z_{i-1}, z_i z_{i+1}\}$ is a circuit of M' and $C'''_t = ((C'_t \setminus \{e_i\}) \cup \{x_i z_i\}) \triangle \{x_i z_i, z_i z_{i-1}, z_i z_{i+1}\}$ is a cycle of M'. We have that $t \in C'''_t \cup \{t\}$ and $C'''_t \setminus \{t\} \subseteq F'$ spans t.

Because $F \cap E(M_\ell) \neq \emptyset$, there is $t \in T$ such that C_t goes through e_i and there is $t' \in T$ such that $C_{t'}$ goes through e_j. Let $C_t = C'_t \triangle C''_t$ and $C_{t'} = C'_{t'} \triangle C''_{t''}$, where $C'_t, C'_{t'}$ are cycles of M'', $C''_t, C''_{t'}$ are cycles of M_ℓ and $C'_t \cap C''_t = \{e_i\}$, $C'_{t'} \cap C''_{t'} = \{e_j\}$. Notice that $C''_t \setminus \{e_i\}$ spans e_i in M_ℓ and $C''_{t'} \setminus \{e_j\}$ spans e_j. Hence, $w_\ell((C''_t \setminus \{e_i\}) \cup (C''_{t'} \setminus \{e_j\})) \geq w_\ell(F_\ell) = p^{(1)}$ by Observation 7.4. Because $w'(\{z_1 z_2, z_2 z_3, z_1 z_3\}) \geq p^{(1)}$, $w'(F') \leq w(F)$.

Since $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 3. For each $i \in \{1, 2, 3\}$, there is $t \in T$ such that C_t goes through e_i. As in Case 2, we set $F' = (F \cap E(M'')) \cup \{z_1 z_2, z_2 z_3, z_1 z_3\}$ and use the same arguments to show that $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M'.

Suppose that $F \cap E(M_\ell) = \emptyset$ and $F \cap E(H) \neq \emptyset$.

For each $t \in T$, there is a circuit C_t of M such that $t \in C \subseteq F \cup \{t\}$. By the definition of 1, 2 and 3-sums and Observation 7.3, we have that $C_t = C'_t \triangle C^{(1)} \triangle \ldots \triangle C^{(q)}$, where C'_t is a circuit of M_s and each $C^{(1)}, \ldots, C^{(q)}$ is a circuit of child of M_s in $\mathcal{T}$ or a circuit in the matroid defined by the conflict tree $\mathcal{T}''$ obtained from $\mathcal{T}$ by the deletion of M_s and its children. Notice that if $C_t \cap E(H) \neq \emptyset$, then $C_t \cap E(H)$ is a minimal cut-set of H. Moreover, each component of $H - C_t \cap E(H)$ contains a vertex from the set $\{y_1, y_2, y_3\}$.

We consider the following three cases.

Case 1. There is a unique $i \in \{1, 2, 3\}$ such that for any $t \in T$, either $C_t \cap E(H) = \emptyset$

or y_i is in one component of $H - C_t \cap E(H)$ and y_{i-1}, y_{i+1} are in the other. Let $F' = (F \setminus E(H)) \cup \{x_i z_i\}$.

We show that F' spans T in M'. Let $t \in T$. If $C_t \cap E(H) = \emptyset$, then F' spans t in M', because C_t is a circuit of $M^*(G')$ as H is connected. Suppose that $C_t \cap E(H) \neq \emptyset$. Consider $C''_t = (C_t \setminus (C_t \cap E(H))) \cup \{x_i z_i\}$. Since $(C'_t \setminus (C_t \cap E(H))) \cup \{x_i z_i\}$ is a minimal cut-set of G, we obtain that $C''_t \setminus \{t\} \subseteq F'$ spans t in M'.

Because $F \cap E(H) \neq \emptyset$, there is $t \in T$ such that $C_t \cap E(H) \neq \emptyset$. Observe that $w(C_t \cap E(H)) \geq k_i^{(2)} \geq w'(x_i z_i)$. Hence, $w'(F') \leq w(F)$.

Since $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 2. There are two distinct $i, j \in \{1, 2, 3\}$ such that for any $t \in T$, either i) $C_t \cap E(H) = \emptyset$ or ii) y_i is in one component of $H - C_t \cap E(H)$ and y_{i-1}, y_{i+1} are in the other or iii) y_j is in one component of $H - C_t \cap E(H)$ and y_{j-1}, y_{j+1} are in the other, and for at least one t, ii) holds and for at least one t iii) is fulfilled. Let $F' = (F \setminus E(H)) \cup \{z_1 z_2, z_2 z_3, z_1 z_3\}$.

We claim that F' spans T in M'. Let $t \in T$. If $C_t \cap E(H) = \emptyset$, then F' spans t in M', because C'_t is a circuit of $M^*(G')$ as H is connected. Suppose that $C_t \cap E(H) \neq \emptyset$. By symmetry, assume without loss of generality that ii) is fulfilled for C_t. Consider $C''_t = (C_t \setminus (C_t \cap E(H))) \cup \{z_i z_{i-1}, z_i z_{i+1}\}$. Since $(C'_t \setminus (C_t \cap E(H))) \cup \{x_i z_i\}$ is a minimal cut-set of G, we obtain that $C''_t \setminus \{t\} \subseteq F'$ spans t in M'.

Because there are distinct $i, j \in \{1, 2, 3\}$ such that ii) holds for some $t \in T$ iii) for some $t' \in T$, we have that $w(C_t \cap E(H)) + w(C_{t'} \cap E(H)) \geq k^2 \geq w'(\{z_1 z_2, z_2 z_3, z_1 z_3\})$. Hence, $w'(F') \leq w(F)$. As $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M', (M', w', T, k) is a yes-instance.

Case 3. For each $i \in \{1, 2, 3\}$, there is $t \in T$ such that y_i is in one component of $H - C_t \cap E(H)$ and y_{i-1}, y_{i+1} are in the other. As in Case 2, we set $F' = (F \setminus E(H)) \cup \{z_1 z_2, z_2 z_3, z_1 z_3\}$ and use the same arguments to show that $F' \subseteq E(M') \setminus T$ spans T and has the weight at most k in M'.

Finally, assume that $F \cap E(M_\ell) \neq \emptyset$ and $F \cap E(H) \neq \emptyset$. For each $t \in T$, there is a circuit C_t of M such that $t \in C \subseteq F \cup \{t\}$. Then there is $i \in \{1, 2, 3\}$ such that $C_t = C'_t \triangle C''_t$, where C'_t and C''_t are circuits of M'' and M_ℓ, and C_t goes through e_i, i.e, $C'_t \cap C''_t = \{e_i\}$. Also there is $j \in \{1, 2, 3\}$ such that y_j is in one component of $H - C_t \cap E(H)$ and y_{j-1}, y_{j+1} are in the other. Notice that $i \neq j$, as otherwise F contains a dependent set $(C_t \cap E(H)) \cup \{e_i\}$, where y_i is in one component of $H - C_t \cap E(H)$ and y_{i-1}, y_{i+1} are in the other, contradicting minimality of F. Let $F' = ((F \cap E(M'')) \setminus E(H)) \cup \{x_i z_i, x_j z_j\}$. Denote by $q \in \{1, 2, 3\}$ the element of the set distinct from i and j.

We claim that F' spans T in M'. Let $t \in T$.

If $C_t \cap E(H) = \emptyset$ and $C_t \subseteq E(M'')$, then it is straightforward to verify that $C_t \setminus \{t\}$ spans t in M' and, therefore, F' spans t.

Suppose that $C_t \cap E(H) \neq \emptyset$ and $C_t \subseteq E(M'')$. Then $C_t \cap E(H)$ is a minimal cut-set of H such that a vertex y_f is in one component of $H - C_t \cap E(H)$ and y_{f-1}, y_{f+1} are in the other. If $f = i$ or $f = j$, then in the same way as in the case, where $F \cap E(M_\ell) = \emptyset$ and $F \cap E(H) \neq \emptyset$, we have that $((C_t \setminus E(H)) \cup \{x_f z_f\}) \setminus \{t\}$ spans t. Suppose that $f = q$. Then we observe that $((C_t \setminus E(H)) \cup \{x_i z_i, x_j y_j\}) \setminus \{t\}$ spans t. Hence, F' spans t.

Suppose that $C_t \cap E(H) = \emptyset$ and $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t = C'_t \triangle C''_t$, where C'_t and C''_t are cycles of M'' and M_ℓ respectively, and C_t goes through some e_f

for $f \in \{1,2,3\}$. If $f = i$ or $f = j$, then in the same way as in the case, where $F \cap E(M_\ell) \neq \emptyset$ and $F \cap E(H) = \emptyset$, we have that $((C'_t \setminus \{e_f\}) \cup \{x_f z_f\}) \setminus \{t\} \subseteq F'$ spans t. Suppose that $f = q$. Then we observe that $((C'_t \setminus \{e_f\}) \cup \{x_i z_i, x_j y_j\}) \setminus \{t\} \subseteq F'$ spans t, because $\{x_1z_1, x_2z_2, x_3z_3\}$ is a circuit of M'.

Suppose now that $C_t \cap E(H) \neq \emptyset$ and $C_t \cap E(M_\ell) \neq \emptyset$. Then $C_t \cap E(H)$ is a minimal cut-set of H such that a vertex y_f is in one component of $H - C_t \cap E(H)$ and y_{f-1}, y_{f+1} are in the other. Also $C_t = C'_t \triangle C''_t$, where C'_t and C''_t are circuits of M'' and M_ℓ respectively, and C_t goes through some e_g for $g \in \{1,2,3\}$. Notice that $f \neq g$, as otherwise C'_t contains a dependent set $(C_t \cap E(H)) \cup \{e_f\}$ contradicting minimality of circuits. If $\{f,g\} = \{i,j\}$, we obtain that $(((C'_t \setminus E(H)) \setminus \{e_f\}) \cup \{x_f z_f, x_g z_g\}) \setminus \{t\} \subseteq F'$ spans t by the same arguments as in previous cases. If $\{f,g\} \neq \{i,j\}$, then let $q' \in \{1,2,3\}$ be distinct form f,g. Clearly, $q' \in \{i,j\}$. Then $(((C'_t \setminus E(H)) \setminus \{e_f\}) \cup \{x_{q'} z_{q'}\}) \setminus \{t\} \subseteq F'$ spans t spans t, because $\{x_1z_1, x_2z_2, x_3z_3\}$ is a circuit of M'.

Now we show that $w'(F) \leq k$. Recall that there is $C_t = C'_t \triangle C''_t$, where C'_t and C''_t are circuits of M'' and M_ℓ, and C_t goes through e_i. Observe that $w'(e_i) \leq k_i^{(1)} \leq w(C''_t \setminus \{e_i\})$. Recall also that there is C_t such that $C_t \cap E(H) \neq \emptyset$ and y_j is in one component of $H - C_t \cap E(H)$ and y_{j-1}, y_{j+1} are in the other. We have that $w'(x_j z_j) \leq k_j^{(2)} \leq w(C_t \cap E(H))$. It implies that $w'(F) \leq k$.

We considered all possible cases and obtained that if the original instance (M, w, T, k) is a yes-instance, then the reduced instance (M', w', T, k) is also a yes-instance. Assume now that the reduced instance (M', w', T, k) is a yes-instance. Let $F' \subseteq E(M') \setminus T$ be an inclusion minimal set of weight at most k that spans T in M'.

Let $S = \{x_1z_1, x_2z_2, x_3z_3, z_1z_2, z_2z_3, z_1z_3\}$. If $F' \cap S = \emptyset$, then we have that F' spans T in M as well. Assume from now that $F' \cap S \neq \emptyset$.

Notice that $|F' \cap \{z_1z_2, z_2z_3, z_1z_3\}| \neq 1$, because z_1z_2, z_2z_3, z_1z_3 induce a cycle in C'. Observe also that if $F' \cap \{z_1z_2, z_2z_3, z_1z_3\} = \{z_{i-1}z_i, z_i z_{i+1}\}$ for some $i \in \{1,2,3\}$, then by Observation 7.6 we can replace $z_{i-1}z_i, z_i z_{i+1}$ by $x_i z_i$ in F using the fact that $z_{i-1}z_i, z_i z_{i+1}, x_i z_i$ is a cut-set of G'. Hence, without loss of generality we assume that either $F' \cap \{z_1z_2, z_2z_3, z_1z_3\} = \emptyset$ or $z_1z_2, z_2z_3, z_1z_3 \in F'$. We have that $|F' \cap \{x_1z_1, x_2z_2, x_3z_3\}| \leq 2$, because $\{x_1z_1, x_2z_2, x_3z_3\}$ is a minimal cut-set of G', and if $z_1z_2, z_2z_3, z_1z_3 \in F'$, then $F' \cap \{x_1z_1, x_2z_2, x_3z_3\} = \emptyset$ by the minimality of F'. We consider the cases according to these possibilities.

Case 1. $z_1z_2, z_2z_3, z_1z_3 \in F'$.

If $p^{(1)} \leq p^{(2)}$, then recall that $(M_\ell, w_\ell, \{e_1, e_2\}, p^{(1)})$ of is a yes-instance of Space Cover. Let F_ℓ be a set of weight at most $p^{(1)}$ in that spans e_1 and e_2 in M_ℓ. Notice that F_ℓ spans e_3 by Observation 7.4. Notice also that $e_1, e_2, e_3 \notin F_\ell$. We define $F = (F' \setminus \{z_1z_2, z_2z_3, z_1z_3\}) \cup F_\ell$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{z_1z_2, z_2z_3, z_1z_3\}) = p^{(1)}$. We claim that F spans T in M. Consider $t \in T$. There is a circuit C'_t of M' such that $t \in C'_t \subseteq F' \cup \{t\}$. If $C'_t \cap \{z_1z_2, z_2z_3, z_1z_3\} = \emptyset$, then $C'_t \setminus \{t\}$ spans t in M. Suppose that $C'_t \cap \{z_1z_2, z_2z_3, z_1z_3\} \neq \emptyset$. Notice that because z_1z_2, z_2z_3, z_1z_3 form a triangle in G', C'_t contains exactly two elements of $\{z_1z_2, z_2z_3, z_1z_3\}$. By symmetry, assume without loss of generality that $z_1z_2, z_2z_3 \in C'_t$. There is a circuit C of M_ℓ such that $e_1 \in C \subseteq F_\ell \cup \{e_1\}$. Observe that for any $X \subseteq E(G')$ such that $X \cap S = \{z_1z_2, z_1z_3\}$, X is a minimal cut-set of G' if and only if $(X \setminus \{z_1z_2, z_1z_3\}) \cup \{e_1\}$ is a minimal cut-set of G. It implies that $C_t = (C'_t \setminus \{z_1z_2, z_1z_3\}) \cup (C \setminus \{e_1\}) \subseteq F$ is a cycle of M. Hence, F spans t.

Suppose that $p^{(2)} < p^{(1)}$. Recall that $(M^*(H'), w_h, \{e'_1, e'_2\}, p^{(2)})$ is a yes-instance of Space Cover. Let F_h be a set of weight at most $p^{(2)}$ in that spans e'_1 and e'_2 in $M^*(H')$. Notice that F_h spans e'_3 by Observation 7.4. Notice also that $e'_1, e'_2, e'_3 \notin F_h$. We define $F = (F' \setminus \{z_1z_2, z_2z_3, z_1z_3\}) \cup F_h$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{z_1z_2, z_2z_3, z_1z_3\}) = p^{(2)}$. We claim that F spans T in M. Consider $t \in T$. There is a circuit C'_t of M' such that $t \in C'_t \subseteq F' \cup \{t\}$. If $C'_t \cap \{z_1z_2, z_2z_3, z_1z_3\} = \emptyset$, then $C'_t \setminus \{t\}$ spans t in M. Suppose that $C'_t \cap \{z_1z_2, z_2z_3, z_1z_3\} \neq \emptyset$. Notice that because z_1z_2, z_2z_3, z_1z_3 form a triangle in G', C'_t contains exactly two elements of $\{z_1z_2, z_2z_3, z_1z_3\}$. By symmetry, assume without loss of generality that $z_1z_2, z_2z_3 \in C'_t$. There is a circuit C of M_h such that $e'_1 \in C \subseteq F_h \cup \{e'_1\}$. Notice that for any $X \subseteq E(G')$ such that $X \cap S = \{z_1z_2, z_1z_3\}$, X is a minimal cut-set of G' if and only if $(X \setminus \{z_1z_2, z_1z_3\}) \cup Y$ is a minimal cut-set of G for a minimal cut-set Y of H such that y_1 is in one component of $H - Y$ and y_2, y_3 are in the other. It implies that $C_t = (C'_t \setminus \{z_1z_2, z_1z_3\}) \cup (C \setminus \{e'_1\}) \subseteq F$ is a cycle of M. Hence, F spans t.

Case 2. $F' \cap S = \{x_iz_i\}$ for $i \in \{1, 2, 3\}$.

Suppose first that $k_i^{(1)} \leq k_i^{(2)}$. Then $(M_\ell, w_\ell, \{e_i\}, k_i^{(1)})$ is a yes-instance of Space Cover. Let F_ℓ be a set of weight at most $k_i^{(1)}$ in that spans e_i in M_ℓ. Notice $e_1, e_2, e_3 \notin F_\ell$. We define $F = (F' \setminus \{x_iz_i\}) \cup F_\ell$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(x_iz_i) = k_i^{(1)}$. We claim that F spans T in M. Consider $t \in T$. There is a circuit C'_t of M' such that $t \in C'_t \subseteq F' \cup \{t\}$. If $x_iz_i \notin C'_t$, then $C'_t \setminus \{t\}$ spans t in M. Suppose that $x_iz_i \in C'_t$. There is a circuit C of M_ℓ such that $e_i \in C \subseteq F_\ell \cup \{e_i\}$. Observe that for any $X \subseteq E(G')$ such that $X \cap S = \{x_iz_i\}$, X is a minimal cut-set of G' if and only if $(X \setminus \{x_iz_i\}) \cup \{e_i\}$ is a minimal cut-set of G. It implies that $C_t = (C'_t \setminus \{x_iz_i\}) \cup (C \setminus \{e_i\}) \subseteq F$ is a cycle of M. Hence, F spans t.

Assume that $k_i^{(2)} < k_i^{(1)}$. Recall that $(M^*(H'), w_h, \{e'_i\}, k_i^{(2)})$ is a yes-instance of Space Cover. Let F_h be a set of weight at most $k_i^{(2)}$ in that spans e'_i in $M^*(H')$. Notice that $e'_1, e'_2, e'_3 \notin F_h$. We define $F = (F' \setminus \{x_iz_i\}) \cup F_h$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{x_iz_i\}) = k_i^{(2)}$. We claim that F spans T in M. Consider $t \in T$. There is a circuit C'_t of M' such that $t \in C'_t \subseteq F' \cup \{t\}$. If $x_iz_i \notin C'_t$, then $C'_t \setminus \{t\}$ spans t in M. Suppose that $x_iz_i \in C'_t$. There is a circuit C of M_h such that $e'_i \in C \subseteq F_h \cup \{e'_i\}$. Observe that any $X \subseteq E(G')$ such that $X \cap S = \{x_iz_i\}$, X is a minimal cut-set of G' if and only if $(X \setminus \{x_iz_i\}) \cup Y$ is a minimal cut-set of G for a minimal cut-set Y of H such that y_i is in one component of $H - Y$ and y_{i-1}, y_{i+1} are in the other. It implies that $C_t = (C'_t \setminus \{xIz_i\}) \cup (C \setminus \{e'_i\}) \subseteq F$ is a cycle of M. Hence, F spans t.

Case 3. $F' \cap S = \{x_iz_i, x_jz_j\}$ for two distinct $i, j \in \{1, 2, 3\}$.

Suppose that $w'(x_iz_i) = k_i^{(1)}$ and $w'(x_jz_j) = k_j^{(1)}$. By Observation 7.6, $p^{(1)} \leq k_i^{(1)} + k_j^{(1)}$. We have that $(M_\ell, w_\ell, \{e_1, e_2\}, p^{(1)})$ is a yes-instance of Space Cover. Let F_ℓ be a set of weight at most $p^{(1)}$ in that spans e_1 and e_2 in M_ℓ. Notice that F_ℓ spans e_3 by Observation 7.4. Notice also that $e_1, e_2, e_3 \notin F_\ell$. We define $F = (F' \setminus \{x_iz_i, x_jz_j\}) \cup F_\ell$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{x_iz_i, x_jz_j\}) \geq p^{(1)}$. In the same way as in Case 1, we obtain that F spans T in M.

Assume that $w'(x_iz_i) = k_i^{(2)}$ and $w'(x_jz_j) = k_j^{(2)}$. By Observation 7.6, $p^{(2)} \leq k_i^{(2)} + k_j^{(2)}$. Recall that $(M^*(H'), w_h, \{e'_1, e'_2\}, p^{(2)})$ is a yes-instance of Space Cover. Let F_h be a set of weight at most $p^{(2)}$ in that spans e'_1 and e'_2 in $M^*(H')$. Notice that F_h spans e'_3 by Observation 7.4. Notice also that $e'_1, e'_2, e'_3 \notin F_h$. We define $F = (F' \setminus \{x_iz_i, x_jz_j\}) \cup F_h$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{x_iz_i, x_jz_j\}) \geq p^{(2)}$. By the same arguments as in Case 1, we have that F spans T in M.

Suppose now that $w'(x_iz_i) = k_i^{(1)}$ and $w'(x_jz_j) = k_j^{(2)}$ or, symmetrically, $w'(x_iz_i) = k_i^{(2)}$ and $w'(x_jz_j) = k_j^{(1)}$. Assume that $w'(x_iz_i) = k_i^{(1)}$ and $w'(x_jz_j) = k_j^{(2)}$, as the second possibility is analysed by the same arguments. We have that $(M_\ell, w_\ell, \{e_i\}, k_i^{(1)})$ is a yes-instance of Space Cover. Let F_ℓ be a set of weight at most $k_i^{(1)}$ in that spans e_i in M_ℓ. Notice $e_1, e_2, e_3 \notin F_\ell$. We have also that $(M^*(H'), w_h, \{e'_i\}, k_j^{(2)})$ is a yes-instance of Space Cover. Let F_h be a set of weight at most $k_j^{(2)}$ in that spans e'_j in $M^*(H')$. Notice that $e'_1, e'_2, e'_3 \notin F_h$. We define $F = (F' \setminus \{x_iz_i, x_jz_j\}) \cup F_\ell \cup F_h$. Clearly, $F \subseteq E(M) \setminus T$ and $w(F) \leq k$ as $w'(\{x_iz_i\}) \leq k_i^{(1)}$ and $w'(\{x_iz_i\}) \leq k_i^{(1)}$. We show that F spans T. Consider $t \in T$. There is a circuit C'_t of M' such that $t \in C'_t \subseteq F' \cup \{t\}$. There is a circuit C of M_ℓ such that $e_i \in C \subseteq F_\ell \cup \{e_i\}$, and there is a circuit C' of M_h such that $e'_j \in C \subseteq F_h \cup \{e'_j\}$. If $x_iz_i, x_jz_j \notin C'_t$, then $C'_t \setminus \{t\}$ spans t in M. Suppose that $x_iz_i \in C'_t$ but $x_jz_j \notin C'_t$. Then by the same arguments as were used to analyse the first possibility of Case 2, we show that $C_t = (C'_t \setminus \{x_iz_i\}) \cup (C \setminus \{e_i\})$ is a cycle of M such that $t \in C_t \subseteq F \cup \{t\}$. If $x_iz_i \notin C'_t$ and $x_jz_j \in C'_t$. Then by the same arguments as were used to analyse the second possibility of Case 2, we obtain that $C_t = (C'_t \setminus \{x_jz_j\}) \cup (C' \setminus \{e'_j\})$ is a cycle of M such that $t \in C_t \subseteq F \cup \{t\}$. Finally, if $x_iz_i, x_jz_j \in C'_t$, we consider $C_t = (C'_t \setminus \{x_jz_j\}) \cup (C \setminus \{e_i\}) \cup (C' \setminus \{e'_j\})$ and essentially by the same arguments as in Case 2, obtain that C_t is a cycle of M and $t \in C_t \subseteq F \cup \{t\}$. Hence, in all possible cases F spans t.

This completes the correctness proof. From the description of Reduction Rule 7.10 and Lemma 7.1, it follows that Reduction Rule 7.4 can be applied in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. ∎

7.3.2. Cographic sub-leaf: $E(H) \cap T \neq \emptyset$. From now onwards we assume that $E(H) \cap T \neq \emptyset$. We either reduce H or recursively solve the problem on smaller H. Rather than describing these steps, we observe that we can decompose M_s further and apply the already described Reduction Rule 7.2 (**1-Leaf reduction rule**) or Branching Rules 7.1 (**2-Leaf branching**) and 7.2 (**3-Leaf branching**).

We use the following fact about matroid decompositions (see [42]).Since we apply the decomposition theorem for the specific case of bond matroids, for convenience we state it in terms of graphs. Let G be a graph. A pair (X, Y) of nonempty subsets $X, Y \subset V(G)$ is a *separation* of G if $X \cup Y = V(G)$ and no vertex of $X \setminus Y$ is adjacent to a vertex of $Y \setminus Y$. For our convenience we assume that (X, Y) is an ordered pair. The next lemma can be derived from either the general results of [42, Chapter 8], or it can be proved directly using definitions of 1-, 2- and 3-sums and the fact that the circuits of the bond matroid of G are exactly the minimal cut-sets of G.

Lemma 7.11. *Let (X, Y) be a separation of a graph G, $H_1 = G[X]$ and $H_2 = G[Y] - E(G_1)$. Then the following holds.*

(i) If $|X \cap Y| = 1$, then $M^*(G) = M^*(H_1) \oplus_1 M^*(H_2)$.
(ii) If $|X \cap Y| = 2$, then $M^*(G) = M^*(H'_1) \oplus_2 M^*(H'_2)$, where H'_i is the graph obtained from H_i by adding a new edge e with its end vertices in the two vertices of $X \cap Y$ for $i = 1, 2$; $E(H'_1) \cap E(H'_2) = \{e\}$.
(iii) If $|X \cap Y| = 3$ and $X \cap Y = \{v_1, v_2, v_3\}$, then $M^*(G) = M^*(H''_1) \oplus_2 M^*(H''_2)$, where for $i = 1, 2$, H''_i is the graph obtained from H_i by adding a new vertex v and edges $e_j = vv_j$ for $j \in \{1, 2, 3\}$; $E(H'_1) \cap E(H'_2) = \{e_1, e_2, e_3\}$.

We use this lemma to decompose $M_s = M^*(G)$. Let Y be the set of end vertices of e_1, e_2, e_3 in $V(H)$. The set Y contains y_1, y_2, y_3, but some of these vertices could be the same. Let $X = (V(G) \setminus V(H)) \cup Y$. We have that $(V(H), X)$ is a separation of G. We apply Lemma 7.11 to this separation. Recall that Z is a clean cut of G. That means that no edge of H is an element of a matroid that is a node of $\mathcal{T}$ distinct from M_s. Therefore, in this way we obtain a good $\{1, 2, 3\}$-decomposition with the conflict tree $\mathcal{T}'$ that is obtained form $\mathcal{T}$ by adding a leaf adjacent to M_s. Then we either reduce the new leaf if it is a 1-leaf or branch on it is 2- or 3-leaf. More formally, we do the following.

- If $|Y| = 1$, then let $G' = G[X]$, decompose $M^*(G) = M^*(G') \oplus_1 M^*(H)$ and construct a new conflict tree $\mathcal{T}'$ for the obtained decomposition of M: we replace the node M_s in $\mathcal{T}$ by $M^*(G')$ that remains adjacent to the same nodes as M_s in $\mathcal{T}$ and then add a new child $M^*(H)$ of $M^*(G')$ that is a leaf of $\mathcal{T}'$. Thus we can apply Reduction Rule 7.2 (**1-Leaf reduction rule**) on the new leaf.
- If $|Y| = 2$, then let G' be the graph obtained from $G[X]$ by adding a new edge e with its end vertices being the two vertices of Y. Furthermore, let H' be the graph obtained from H by adding a new edge e with its end vertices being the two vertices of Y. Now decompose $M^*(G) = M^*(G') \oplus_2 M^*(H')$ and consider a new conflict tree $\mathcal{T}'$ for the obtained decomposition: M_s is replaced by $M^*(G')$ and a new leaf $M^*(H')$ that is a child of $M^*(G')$ is added. Notice that because H has no bridges, no terminal $t \in T \cap E(H)$ is parallel to e in $M^*(H')$. Thus we can apply Branching Rule 7.1 (**2-Leaf branching**) on the new leaf.
- If $|Y| = 3$, then $Y = \{y_1, y_2, y_3\}$. Let G' be the graph obtained from $G[X]$ by adding a new vertex v and the edges $e_1' = y_1v$, $e_2' = y_2v$, $e_3' = y_3v$. Let H' be the graph obtained from H by adding a new vertex v and the edges $e_1' = y_1v$, $e_2' = y_2v$, $e_3' = y_3v$. Then decompose $M^*(G) = M^*(G') \oplus_3 M^*(H')$ and consider a new conflict tree $\mathcal{T}'$ for the obtained decomposition: M_s is replaced by $M^*(G')$ and a new leaf $M^*(H')$ that is a child of $M^*(G')$ is added. Notice that because H has no bridges, no terminal $t \in T \cap E(H)$ is parallel to e_1', e_2', e_3' in $M^*(H')$. Thus we can apply Branching Rule 7.2 (**3-Leaf branching**) on the new leaf.

Lemma 7.11 together with Lemmas 7.5 and 7.6 imply the correctness of the above procedure. Furthermore, all the reduction and branching rules can be performed in $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time.

7.4. Proof of Theorem 1.1. Given an instance (M, w, T, k) of Space Cover we either apply a reduction rule or a branching rule and if any of these applications (reduction rule or branching rule) returns no, we return the same. Correctness of the answer follows from the correctness of the corresponding rules.

Let (M, w, T, k) be the given instance of Space Cover. First, we exhaustively apply elementary Reduction Rules 5.1-5.5. Thus, by Lemma 5.4, in polynomial time we either solve the problem or obtain an equivalent instance, where M has no loops and the weights of nonterminal elements are positive. To simplify notation, we also denote the reduced instance by (M, w, T, k). If M is a basic matroid (obtained from R_{10} by adding parallel elements or M is graphic or cographic) then we can solve Space Cover using Lemma 7.1 in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$.

From now onwards we assume that the matroid M in the instance (M, w, T, k) is not basic. Now using Corollary 4.4, we find a conflict tree $\mathcal{T}$. Recall that the set of nodes of $\mathcal{T}$ is the collection of basic matroids $\mathcal{M}$ and the edges correspond to 1-, 2− and 3-sums. The key observation is that M can be constructed from $\mathcal{M}$

by performing the sums corresponding to the edges of $\mathcal{T}$ in an arbitrary order. Our algorithm is based on performing *bottom-up* traversal of the tree $\mathcal{T}$. We select an arbitrarily *node r as the root* of $\mathcal{T}$. Selection of r, as the root, defines the natural parent-child, descendant and ancestor relationship on the nodes of $\mathcal{T}$. We say that u is a *sub-leaf* if its children are leaves of $\mathcal{T}$. Observe that there always exists a sub-leaf in a tree on at least two nodes. Just take a node which is not a leaf and is farthest from the root. Clearly, this node can be found in polynomial time. Rest of our argument is based on selection a sub-leaf M_s. We say that a child of M_s is a 1-, 2- or 3-*leaf*, respectively, if the edge between M_s and the leaf corresponds to 1-, 2- or 3-sum, respectively. If there is a child M_ℓ of M_s such that there is $e \in E(M_s) \cap E(M_\ell)$ that is parallel to a terminal $t \in E(M_\ell) \cap T$ in M_ℓ, then we apply Reduction Rule 7.1 (**Terminal flipping rule**). We apply Reduction Rule 7.1 exhaustively. Correctness of this step follows from Lemma 7.2.

From now we assume that there is no child M_ℓ of M_s such that there exists an element $e \in E(M_s) \cap E(M_\ell)$ that is parallel to a terminal $t \in E(M_\ell) \cap T$ in M_ℓ. Now given a sub-leaf M_s and a child M_ℓ of M_s, we apply the first rule (reduction or branching) among

- Reduction Rule 7.2 (1**-Leaf reduction rule**)
- Reduction Rule 7.3 (2**-Leaf reduction rule**)
- Branching Rule 7.1 (2**-Leaf branching**)
- Branching Rule 7.2 (3**-Leaf branching**)
- Reduction Rule 7.4 (**Graphic** 3**-leaf reduction rule**)
- Reduction Rule 7.5 (**Cographic** 3**-leaf reduction rule**)

which is applicable. If none of the above is applicable then we are in a specific subcase of M_s being cographic matroid. That is, the case which is being handled in Section 7.3.1. However, even in this case we modify our instance to fall into one of the cases above. Note that we we do not recompute the decompositions of the matroids obtained by the application of the rules but use the original decomposition modified by the rules. Observe additionally that the elementary Reduction Rules 5.1-5.5 also could be used to modify the decomposition. Clearly, graphic and cographic remain graphic and cographic respectively and we just modify the corresponding graphs but we can delete or contract an element of a copy R_{10}. For this case, observe that Lemma 6.1 still could be applied and these matroids are not participating in 3-sums. Each of the above rules reduces the $\mathcal{T}$ by one and thus these rules are only applied $\mathcal{O}(|E(M)|)$ times. The correctness of algorithm follows from Lemmas 7.3, 7.4, 7.5, 7.6, 7.7 and 7.10. The only thing that is remaining is the running time analysis.

Either we apply reduction rules in polynomial time or in $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time. So all the reduction rules can be carried out in $\mathcal{O}(|E(M)|)) \cdot 2^{\mathcal{O}(k)} = 2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time. By Lemmas 7.5 and 7.6 we know that when we apply Branching Rules 7.1 and 7.2 then the parameter reduces in each branch and thus the number of leaves in the search-tree is upper bounded by the recurrence, $T(k) \leq 15T(k-1)$, corresponding to the Branching Rule 7.2. Thus, the number of nodes in the search tree is upper bounded by 15^k and since at each node we take $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$ time, we have that the overall running time of the algorithm is upper bounded by $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. This completes the proof.

8. Reducing rank. In the h-Way Cut problem, we are given a connected graph G and positive integers h and k, the task is to find at most k edges whose removal increases the number of connected components by at least h. The problem has a simple formulation in terms of matroids: Given a graph G and an integers k,

h, find k elements of the cycle matroid of G whose removal reduces its rank by at least h. This motivated Joret and Vetta [26] to introduce the Rank h-Reduction problem on matroids. Here we define Rank h-Reduction on binary matroids.

> Rank h-Reduction **Parameter:** k
> **Input:** A binary matroid $M = (E, \mathcal{I})$ given together with its matrix representation over GF(2) and two positive integers h and k.
> **Question:** Is there a set $X \subseteq E$ with $|X| \leq k$ such that $r(M) - r(M - X) \geq h$?

As a corollary of Theorem 1.1, we show that on regular matroids Rank h-Reduction is FPT for any fixed h.

We use the following lemma.

Lemma 8.1. *Let M be a binary matroid and let $k \geq h$ be positive integers. Then M has a set $X \subseteq E$ with $|X| \leq k$ such that $r(M) - r(M - X) \geq h$ if and only if there are disjoint sets $F, T \subseteq E$ such that $|T| = h$, $|F| \leq k - h$, and $T \subseteq \text{span}(F)$ in M^*.*

Proof. Notice that deletion of one element cannot decrease the rank by more than one. Moreover, deletion of $e \in E$ decreases the rank if and only if e belongs to every basis of M. Recall that e belongs to every basis of M if and only if e is a coloop (see [36]). It follows that M has a set $X \subseteq E$ with $|X| \leq k$ such that $r(M) - r(M - X) \geq h$ if and only if there are disjoint sets $F, T \subseteq E$ such that $|T| = h$, $|F| \leq k - h$ and every $e \in T$ is a coloop of $M - F$. Switching to the dual matroid, we rewrite this as follows: M has a set $X \subseteq E$ with $|X| \leq k$ such that $r(M) - r(M - X) \geq h$ if and only if there are disjoint sets $F, T \subseteq E$ such that $|T| = h$, $|F| \leq k - h$ and every $e \in T$ is a loop of M^*/F. It remains to observe that every $e \in T$ is a loop of M^*/F if and only if $T \subseteq \text{span}(F)$ in M^*. ◻

For graphic matroids, when Rank h-Reduction is equivalent to h-Way Cut, the problem is FPT parameterized by k even if h is a part of the input [27]. Unfortunately, similar result does not hold for cographic matroids.

Proposition 8.2. Rank h-Reduction *is* W[1]*-hard for cographic matroids parameterized by $h + k$.*

Proof. Consider the bond matroid $M^*(G)$ of a simple graph G. By Lemma 8.1, $(M^*(G), h, k)$ is a yes-instance of Rank h-Reduction if and only if there are disjoint sets of edges $F, T \subseteq E(G)$ such that $|T| = h$ and $|F| \leq k - h$ and $T \subseteq \text{span}(F)$ in $M(G)$. Recall that $T \subseteq \text{span}(F)$ in $M(G)$ if and only if for every $uv \in T$, $G[F]$ has a (u, v)-path. Let $p \geq 3$ be an integer, $k = (p - 1)p/2$ and $h = (p - 1)(p - 2)/2$. It is easy to see that for this choice of h and k, G has disjoint sets of edges $F, T \subseteq E(G)$ such that $|T| = h$, $|F| \leq k - h$ and for every $uv \in T$, $G[F]$ has a (u, v)-path if and only if G has a clique with p vertices. Since it is well-know that it is W[1]-complete with the parameter p to decide whether a graph G has a clique of size p (see [10]), we conclude that Rank h-Reduction is W[1]-hard when parameterized by $h + k$. ◻

However, by Theorem 1.1, for fixed h, Rank h-Reduction is FPT parameterized by k on regular matroids.

Theorem 8.3. Rank h-Reduction *can be solved in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(h)}$ on regular matroids.*

Proof. Let (M, h, k) be an instance of Rank h-Reduction. By Lemma 8.1, (M, h, k) is a yes-instance if and only if there are disjoint sets $F, T \subseteq E$ such that $|T| = h$, $|F| \leq k - h$ and $T \subseteq \text{span}(F)$ in M^*. There are at most $||M||^h$ possibilities to choose T. For each choice, we check whether there is $F \subseteq E \setminus T$ such that $|F| \leq k-h$ and $T \subseteq \text{span}(F)$ in M^*. By Theorem 1.1, it can be done in time $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(1)}$. Then the total running time is $2^{\mathcal{O}(k)} \cdot ||M||^{\mathcal{O}(h)}$. ∎

9. Conclusion. In this paper, we used the structural theorem of Seymour for designing parameterized algorithm for Space Cover. While structural graph theory and graph decompositions serve as the most usable tools in the design of parameterized algorithms, the applications of structural matroid theory in parameterized algorithms are limited. There is a series of papers about width-measures and decompositions of matroids (see, in particular, [23, 24, 25, 29, 34, 35] and the bibliography therein) but, apart of this specific area, we are not aware of other applications except the works Gavenciak et al. [14] and our recent work [13]. In spite of the tremendous progress in understanding the structure of matroids representable over finite fields [18, 15, 16, 17], this rich research area still remains to be explored from the perspective of parameterized complexity.

As a concrete open problem, what about the parameterized complexity of Space Cover on any proper minor-closed class of binary matroids?

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