Split Contraction: The Untold Story*

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The edit operation that contracts edges, which is a fundamental operation in the theory of graph minors, has recently gained substantial scientific attention from the viewpoint of Parameterized Complexity. In this paper, we examine an important family of graphs, namely the family of split graphs, which in the context of edge contractions, is proven to be significantly less obedient than one might expect. Formally, given a graph G and an integer k, Split Contraction asks whether there exists $X \subseteq E(G)$ such that G/X is a split graph and $|X| \leq k$. Here, G/X is the graph obtained from G by contracting edges in X. Guo and Cai [Theoretical Computer Science, 2015] claimed that Split Contraction is fixed-parameter tractable. However, our findings are different. We show that Split Contraction, despite its deceptive simplicity, is W[1]-hard. Our main result establishes the following conditional lower bound: under the Exponential Time Hypothesis, Split Contraction cannot be solved in time $2^{o(\ell^2)} \cdot n^{O(1)}$ where ℓ is the vertex cover number of the input graph. We also verify that this lower bound is essentially tight. To the best of our knowledge, this is the *first* tight lower bound of the form $2^{o(\ell^2)} \cdot n^{O(1)}$ for problems parameterized by the vertex cover number of the input graph. In particular, our approach to obtain this lower bound borrows the notion of harmonious coloring from Graph Theory, and might be of independent interest.

CCS Concepts: • **Theory of computation → Graph algorithms analysis**; **Fixed parameter tractability**; **Problems, reductions and completeness**;

Additional Key Words and Phrases: Split Contraction, Parameterized Complexity, split graphs, edge contraction

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1 INTRODUCTION

Graph modification problems have been extensively studied since the inception of Parameterized Complexity in the early 90's. The input of a typical graph modification problem consists of a graph G and a positive integer k, and the objective is to edit k vertices (or edges) so that the resulting

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graph belongs to some particular family $\mathcal{F}$ of graphs. These problems are not only mathematically and structurally challenging, but have also led to the discovery of several important techniques in the field of Parameterized Complexity. It would be completely appropriate to say that solutions to these problems played a central role in the growth of the field. In fact, just over the course of the last couple of years, parameterized algorithms have been developed for Chordal Editing [10], Unit Interval Editing [8], Interval Vertex (Edge) Deletion [7, 9], Proper Interval Completion [3], Interval Completion [4] Chordal Completion [5, 25, 34], Cluster Editing [24], Threshold Editing [18], Chain Editing [18], Trivially Perfect Editing [19, 20] and Split Editing [26]. This list is not comprehensive but rather illustrative.

The focus of all of these papers, and in fact, of the vast majority of papers on parameterized graph editing problems, has so far been limited to edit operations that delete vertices, delete edges or add edges. Using a different terminology, these problems can also be phrased as follows. For some particular family of graphs, $\mathcal{F}$, we say that a graph G belongs to $\mathcal{F} + kv$, $\mathcal{F} + ke$ or $\mathcal{F} - ke$ if some graph in $\mathcal{F}$ can be obtained by deleting at most k vertices from G, deleting at most k edges from G or adding at most k edges to G, respectively. Recently, a methodology for proving lower bounds on running times of algorithms for such parameterized graph editing problems was proposed by Bliznets et al. [2]. Furthermore, a well-known result by Cai [5] states that in case $\mathcal{F}$ is a hereditary family of graphs with a finite set of forbidden induced subgraphs, then the graph modification problem defined by $\mathcal{F}$ and the aforementioned edit operations admit a simple FPT algorithm.

In recent years, a different edit operation has begun to attract significant scientific attention. This operation, which is arguably the most natural edit operation apart from deletions/insertions of vertices/edges, is the one that contracts an edge. Here, given an edge (u, v) in the input graph, we remove the edge from the graph and merge its two endpoints. Edge contraction is a fundamental operation in the theory of graph minors. Using our alternative terminology, we say that a graph G belongs to $\mathcal{F}/ke$ if some graph in $\mathcal{F}$ can be obtained by contracting at most k edges in G. Then, given a graph G and a positive integer k, $\mathcal{F}$-Edge Contraction asks whether G belongs to $\mathcal{F}/ke$. For several families of graphs $\mathcal{F}$, early papers by Watanabe et al. [43, 44] and Asano and Hirata [1] showed that $\mathcal{F}$-Edge Contraction is NP-complete. In the framework of Parameterized Complexity, these problems exhibit properties that are quite different from those of problems where we only delete or add vertices and edges. Indeed, for these problems, the result by Cai [5] does not hold. In particular, Lokshtanov et al. [38] and Cai and Guo [6] independently showed that if $\mathcal{F}$ is either the family of P_ℓ-free graphs for some $\ell \geq 5$ or the family of C_ℓ-free graphs for some $\ell \geq 4$, then $\mathcal{F}$-Edge Contraction is W[2]-hard.

To the best of our knowledge, Heggernes et al. [31] were the first to explicitly study $\mathcal{F}$-Edge Contraction from the viewpoint of Parameterized Complexity. They showed that in case $\mathcal{F}$ is the family of trees, $\mathcal{F}$-Edge Contraction is FPT but does not admit a polynomial kernel, while in case $\mathcal{F}$ is the family of paths, the corresponding problem admits a faster algorithm and an $O(k)$-vertex kernel. Golovach et al. [27] proved that if $\mathcal{F}$ is the family of planar graphs, then $\mathcal{F}$-Edge Contraction is again FPT. Moreover, Cai and Guo [6] showed that in case $\mathcal{F}$ is the family of cliques, $\mathcal{F}$-Edge Contraction is solvable in time $2^{O(k \log k)} \cdot n^{O(1)}$, while in case $\mathcal{F}$ is the family of chordal graphs, the problem is W[2]-hard. Heggernes et al. [32] developed an FPT algorithm for the case where $\mathcal{F}$ is the family of bipartite graphs. Later, a faster algorithm was proposed by Guillemot and Marx [29].

A recent paper by Cai and Guo [30] studied the case where $\mathcal{F}$ is the family of split graphs, which corresponds to the following problem.

Split Contraction	**Parameter:** k
Input: A graph G and an integer k.	
Question: Does there exist $X \subseteq E(G)$ such that G/X is a split graph and $\|X\| \leq k$?	

Cai and Guo [30] claimed to design an algorithm that solves Split Contraction in time $2^{O(k^2)} \cdot n^{O(1)}$, which proves that the problem is FPT. Our initial objective was to either speed-up their algorithm or obtain a tight conditional lower bound. In fact, it seemed plausible that Split Contraction, like $\mathcal{F}$-Edge Contraction where $\mathcal{F}$ is the family of cliques, is solvable in time $2^{O(k \log k)} \cdot n^{O(1)}$. The algorithm by Cai and Guo [30] first computes a set of vertices of small size whose removal renders the graph into a split graph. Then, it is based on case distinction. In case the remaining graph contains a large clique, the problem is solved in time $2^{O(k \log k)} \cdot n^{O(1)}$, and otherwise it is solved in time $2^{O(k^2)} \cdot n^{O(1)}$. In particular, in case the clique is small, the minimum size of a vertex cover of the input graph is small—it can be bounded by $O(k)$. Thus, the bottleneck of the proposed algorithm is captured by graphs having small vertex covers. Interestingly, our first main result, given in Section 3, proves that it is unlikely to overcome the difficulty imposed by such graphs.

Theorem 1.1. *Unless the* ETH *fails, Split Contraction parameterized by ℓ, the size of a minimum vertex cover of the input graph, does not have an algorithm running in time $2^{o(\ell^2)} \cdot n^{O(1)}$. Here, n is the number of vertices in the input graph.*

To the best of our knowledge, under the Exponential Time Hypothesis (ETH) [13, 33], this is the *first* tight lower bound of this form for problems parameterized by the vertex cover number of the input graph. Lately, there has been increasing scientific interest in the examination of lower bounds of forms other than $2^{o(s)} \cdot n^{O(1)}$ for some parameters s. For example, lower bounds that are "slightly super-exponential", i.e. of the form $2^{o(s \log s)} \cdot n^{O(1)}$ for various parameters s, have been studied in [37]. Cygan et al. [14] obtained a lower bound of the form $2^{2^{o(k)}} \cdot n^{O(1)}$, where k is the solution size, for the Edge Clique Cover problem. Very recently, Marx and Mitsoue [39] have further obtained lower bounds of the forms $2^{2^{o(w)}} \cdot n^{O(1)}$ and $2^{2^{2^{o(w)}}} \cdot n^{O(1)}$, where w is the treewidth of the input graph, for choosability problems.

In order to derive our main result, we make use of a partitioning of the vertex set $V(G)$ into independent sets $C_1, \ldots, C_t$ such that for each $i, j \in [t]$, $i \neq j$, $|E(G[C_i \cup C_j]) \cap E(G)| \leq 1$. Essentially, this coloring can be viewed as a proper coloring $f : V(G) \to [t]$ with the additional property that between any two color classes we have at most one edge. (Here, we use the standard notation $[t] = \{1, 2, \ldots, t\}$.) This kind of coloring, called *harmonious coloring* [21, 36, 40], has been studied extensively in the literature. We are not aware of uses of harmonious coloring in deriving lower bound results and believe that this approach could be of independent interest.

After we had established Theorem 1.1, we took a closer look at the algorithm by Cai and Guo [30], and were not able to verify some of their arguments (see Phase 2 of their algorithm). We next prove that unless FPT=W[1], the algorithm by Cai and Guo [30] is incorrect, as the problem is W[1]-hard (Section 4).

Theorem 1.2. *Split Contraction is* W[1]-hard *when parmeterized by the size of a solution.*

We find this result surprising: one might a priori expect that "contraction to split graphs" should be easy as split graphs have structures that seem relatively simple. Indeed, many NP-hard problems admit simple polynomial-time algorithms if restricted to split graphs. Consequently, our result can also be viewed as a strong evidence of the inherent complexity of the edit operation which contracts edges. Furthermore, some of the ideas underlying the constructions of this reduction, such as the exploitation of properties of a special case of the PERFECT CODE problem to analyze budget constraints involving edge contractions, might be used to establish other W[1]-hard results for problems of similar flavors. We remark that despite errors in the paper [30], it can be verified that the lower bound given by Theorem 1.1 is tight. For the sake of completeness, we give a standalone FPT algorithm for SPLIT CONTRACTION that runs in time $2^{O(\ell^2)} \cdot n^{O(1)}$.

2 PRELIMINARIES

We denote the set of natural numbers by $\mathbb{N}$. For $k \in \mathbb{N}$, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$.

We use standard terminology from the book of Diestel [15] for terms that are not explicitly defined here. We consider only finite simple graphs. For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and edge sets of the graph G, respectively. For a vertex $v \in V(G)$, we denote the degree of v, i.e the number of edges incident with v in G by $d_G(v)$. For $v \in V(G)$, we denote the set $\{u \in V(G) \mid (v,u) \in E(G)\}$ by $N_G(v)$. We drop the subscript G from $d_G(v)$ and $N_G(v)$ when the context is clear. For a vertex subset $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, i.e. the graph with the vertex set S and the edge set $\{(v,u) \in E(G) \mid v, u \in S\}$. We denote the graph $G[V(G) \setminus S]$ by $G - S$. We say that two disjoint vertex subsets, say $S, S' \subseteq V(G)$, are *adjacent* if there exist $v \in S$ and $v' \in S'$ such that $(v, v') \in E(G)$. Furthermore, an edge $(u,v) \in E(G)$ is *between* S, S' if $u \in S$ and $v \in S'$ (or $v \in S$ and $u \in S'$).

A *split graph* is a graph G whose vertex set $V(G)$ can be partitioned into two sets, A and B, such that $G[A]$ is a clique while B is an independent set, i.e. $G[B]$ is an edgeless graph. A graph G is called a *sub-cubic* graph if for each $v \in V(G)$, we have $d(v) \leq 3$. A *path* in a graph is a sequence of vertices $v_1, v_2, \ldots, v_l$ such that for all $i \in [l-1]$, $(v_i, v_{i+1}) \in E(G)$. Furthermore, we say that such a path is a path between v_1 and v_l. A graph is called *connected* if there is a path between every pair of vertices. A maximal connected-graph is called a *component* in a graph. A vertex subset $S \subseteq V(G)$ is said to *cover* an edge $(u,v) \in E(G)$ if $Y \cap \{u,v\} \neq \emptyset$. A vertex subset $S \subseteq V(G)$ is called a *vertex cover* in G if it covers all the edges in G. A *minimum vertex cover* is $S \subseteq V(G)$ such that S is a vertex cover and for all $S' \subseteq V(G)$ such that S' is a vertex cover, we have $|S| \leq |S'|$.

For $e = (v,u) \in E(G)$, the result of *contracting* the edge e in G is the graph obtained by the following operation. We add a vertex w_e and make it adjacent to the vertices in $(N(v) \cup N(u)) \setminus \{v,u\}$ and delete v, u from the graph. We often call such an operation *contraction* of the edge e. For $E' \subseteq E(G)$, the graph G/E' denotes the graph obtained by contracting the edges of E' in G. We note that the order in which the edges in E' are contracted is insignificant.

A graph G is *isomorphic* to a graph H if there exists a *bijective* function $\phi : V(G) \to V(H)$ such that for $v, u \in V(G)$, $(v,u) \in E(G)$ if and only if $(\phi(v), \phi(u)) \in E(H)$. A graph G is *contractible* to a graph H if there exists $E' \subseteq E(G)$ such that G/E' is *isomorphic* to H. In other words, G is contractible to H if there exists a *surjective* function $\varphi : V(G) \to V(H)$ with the following properties.

- For all $h, h' \in V(H)$, $(h, h') \in E(H)$ if and only if $W(h), W(h')$ are *adjacent* in G. Here, $W(h) = \{v \in V(G) \mid \varphi(v) = h\}$.
- For all $h \in V(H)$, $G[W(h)]$ is *connected*.

Let $\mathcal{W} = \{W(h) \mid h \in V(H)\}$. Observe that $\mathcal{W}$ defines a partition of the vertex set of G. We call $\mathcal{W}$ a *H-witness structure* of G. The sets in $\mathcal{W}$ are called *witness-sets*.

Parameterized Complexity. A parameterized problem Π is a subset of $\Gamma^* \times \mathbb{N}$, where Γ is a finite alphabet. An instance of a parameterized problem is a tuple (x,k), where x is a classical problem instance, and k is called the parameter. A central notion in parameterized complexity is *fixed-parameter tractability (FPT)* which means, for a given instance (x,k), decidability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial in the input size. On the one hand, to prove that a problem is FPT, it is possible to give an explicit algorithm, called a *parameterized algorithm*, which solves it in time $f(k) \cdot p(|x|)$. On the other hand, to show that a problem is unlikely to be FPT, it is possible to use polynomial-time reductions analogous to those employed in Classical Complexity. Here, the concept of W[1]-hardness replaces the one of NP-hardness, and we need not only construct an equivalent instance in FPT time, but also ensure that the size of the parameter in the new instance depends only on the size of the parameter in the original instance. For more details on Parameterized Complexity, we refer the reader to the books of Downey and Fellows [17], Flum and Grohe [23], Niedermeier [41], and the recent book by Cygan et al. [13].

3 LOWER BOUND FOR SPLIT-CONTRACTION PARAMETERIZED BY VERTEX COVER

In this section we show that unless the ETH fails, Split Contraction does not admit an algorithm running in time $2^{o(\ell^2)}n^{O(1)}$, where ℓ is the size of a minimum vertex cover of the input graph G on n vertices. We complement it by giving an algorithm in Section 5) for Split Contraction parameterized by ℓ, running in time $2^{O(\ell^2)}n^{O(1)}$.

To obtain our lower bound, we give an appropriate reduction from Vertex Cover on sub-cubic graphs. For this we utilise the fact that Vertex Cover on sub-cubic graphs does not have an algorithm running in time $2^{o(n)}n^{O(1)}$ unless the ETH fails [33, 35]. For the ease of presentation we split the reduction into two steps. The first step comprises of reducing a special case of Vertex Cover on sub-cubic graphs, which we call Sub-Cubic Partitioned Vertex Cover (Sub-Cubic PVC) to Split Contraction. In the second step we show that there does not exist an algorithm running in time $2^{o(n)}n^{O(1)}$ for Sub-Cubic PVC. We remark that the reduction from Vertex Cover on sub-cubic graphs (Sub-Cubic VC) to Sub-Cubic PVC is a Turing reduction.

3.1 Reduction from Sub-Cubic Partitioned Vertex Cover to Split Contraction

In this section we give a reduction from Sub-Cubic Partitioned Vertex Cover to Split Contraction. Next, we formally define Sub-Cubic Partitioned Vertex Cover.

Sub-Cubic Partitioned Vertex Cover (Sub-Cubic PVC)
Input: A sub-cubic graph G; an integer t; for $i \in [t]$, an integer $k_i \geq 0$; a partition $\mathcal{P} = \{C_1, \ldots, C_t\}$ of $V(G)$ such that $t \in O(\sqrt{|V(G)|})$ and for all $i \in [t]$, C_i is an independent set and $|C_i| \in O(\sqrt{|V(G)|})$. Furthermore, for $i, j \in [t], i \neq j$, $|E(G[C_i \cup C_j]) \cap E(G)| = 1$.
Question: Does G have a vertex cover X such that for all $i \in [t]$, $|X \cap C_i| \leq k_i$?

We first explain (informally) the ideas behind our reduction. Let X be a *hypothetical* vertex cover we are looking for. Recall that we assume the ETH holds and thus we are allowed to use $2^{o(n)}n^{O(1)}$ time to obtain our reduction. We will use this freedom to design our reduction and to construct an instance (G', k') of Split Contraction. For $i \in [t]$, in $V(G')$, we have a *vertex* corresponding to each possible intersection of X with C_i on at most k_i vertices. Furthermore, we have a vertex $c_i \in V(G')$ corresponding to each C_i, for $i \in [t]$. We want to make sure that for each $(u, v) \in E(G)$, we choose an edge of $E(G')$ (in the solution to Split Contraction) that is incident to a vertex which corresponds to a subset containing one of u or v and one of c_i or c_j. Furthermore, we want to force these selected vertices to be contracted to the clique side in the resulting split graph. We crucially exploit the fact that there is exactly one edge between every C_i, C_j pair, where $i, j \in [t], i \neq j$. Finally, we will add a clique, say Γ, of size $3t$ and make each of its vertices adjacent to many pendant vertices, which ensures that after contracting the solution edges, the vertices of Γ remain in the clique side. We will assign appropriate adjacencies between the vertices of Γ and c_i,

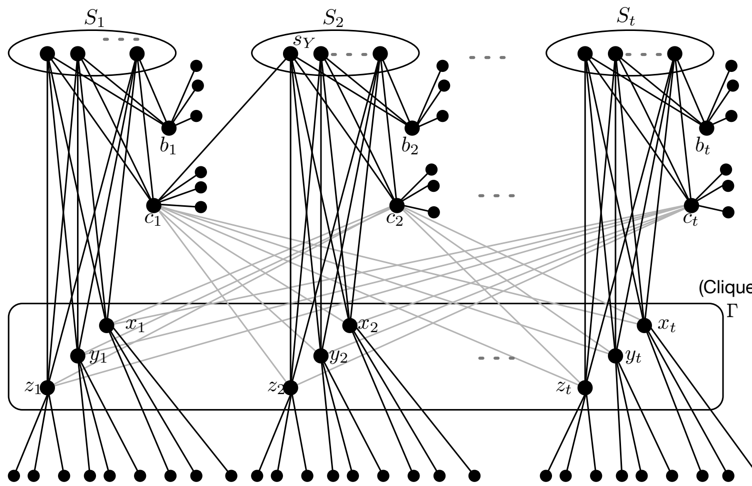

Fig. 1. Reduction from Sub-Cubic PVC to Split Contraction.

for $i \in [t]$. This will guide us in selecting edges for the solution of the contraction problem. We now move to the formal description of the construction used in the reduction.

Construction. Let $(G, \mathcal{P} = \{C_1, C_2, \ldots, C_t\}, k_1, \ldots, k_t)$ be an instance of Sub-Cubic PVC and $n = |V(G)|$. We create an instance of Split Contraction (G', k') as follows. For $i \in [t]$, let $S_i = \{v_Y \mid Y \subseteq C_i \text{ and } |Y| \leq k_i\}$. That is, S_i comprises of vertices corresponding to subsets of C_i of size at most k_i. For each $i \in [t]$, we add five vertices b_i, c_i, x_i, y_i, z_i to $V(G')$. The vertices $\{x_i, y_i, z_i \mid i \in [t]\}$ induce a clique (on $3t$ vertices) in G'. We add the edges $(b_i, s_Y), (c_i, s_Y), (x_i, s_Y), (y_i, s_Y), (z_i, s_Y)$ for all $s_Y \in S_i$ to $E(G')$. For $i, j \in [t]$, $i \neq j$, we add the edges $(c_i, x_j), (c_i, y_j), (c_i, z_j)$ to $E(G')$. For $i, j \in [t]$, $i \neq j$ and $s_Y \in S_j$, we add the edge (c_i, s_Y) in $E(G')$ if and only if Y covers the unique edge between C_i and C_j. For all $i \in [t]$, we add $4t + 2$ pendant vertices, $b_j'^i$, $j \in [4t + 2]$, to b_i. Similarly, for all $i \in [t]$, we add $4t + 2$ pendant vertices $c_j'^i$, $x_j'^i$, $y_j'^i$, and $z_j'^i$, $j \in [4t + 2]$, to c_i, x_i, y_i and z_i, respectively. The pendant vertices are added in order to make sure that the vertices resulting after the contraction of their witness sets belong to the clique side. This completes the construction of the graph G'. Observe that $\{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$ forms a minimum vertex cover of G' of size $5t$. Finally, we set $k' = 2t$. The resulting instance of Split Contraction is (G', k'). We refer the reader to Figure 1 for an illustration of the construction.

In the next few lemmas (Lemma 3.1 to 3.6) we prove certain properties of the instance (G', k') of Split Contraction. This will be helpful later for establishing the equivalence between the original instance $(G, \mathcal{P} = \{C_1, C_2, \ldots, C_t\}, k_1, \ldots, k_t)$ of Sub-Cubic PVC and the instance (G', k') of Split Contraction. In Lemma 3.1 to 3.6 we will use the following notations. We use T to denote a solution to Split Contraction in (G', k') and $H = G'/T$ with $\hat{C}$, $\hat{I}$ being a partition of $V(H)$ inducing a clique and an independent set, respectively, in H. We let $\varphi : V(G') \to V(H)$ be the surjective function defining the contractibility of G' to H, and $\mathcal{W}$ be the *H-witness structure* of G'.

In the following lemma we show that the vertices (or their contracted counterparts) b_i, c_i, x_i, y_i, z_i, for $i \in [t]$, always belong to the clique side (which is $\hat{C}$, in our case).

Lemma 3.1. *Let (G', k') be a* YES *instance of* Split Contraction. *Then, for all $v \in \{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$, we have $\varphi(v) \in \hat{C}$.*

Proof. Consider $v \in \{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$. Recall that there are $4t + 2 = 2k' + 2$ pendant vertices $v_j'^i$, for $j \in [2k' + 2]$ adjacent to v. At most k' edges in $\{(v_j'^i, v) \mid j \in [2k' + 2]\}$ can belong to T. Therefore, there exist $j_1, j_2 \in [2k' + 2]$, $j_1 \neq j_2$ such that no edge incident to $v_{j_1}'^i$ or $v_{j_2}'^i$ is in T. In other words, for $h_1 = \varphi(v_{j_1}'^i)$ and $h_2 = \varphi(v_{j_2}'^i)$, $W(h_1)$ and $W(h_2)$ are singleton sets. Since $\mathcal{W}$ is a H-witness structure of G', $(h_1, h_2) \notin E(H)$. Therefore, at least one of h_1, h_2 belongs to $\hat{I}$, say $h_1 \in \hat{I}$. This implies that $\varphi(v) \in \hat{C}$. □

Next lemma shows that for each $i \in [t]$, there is an edge $(b_i, s_{Y_i}) \in T$. This will be helpful in selecting of a subset of vertices of size at most k_i from S_i.

Lemma 3.2. *Let (G', k') be a* YES *instance of* Split Contraction*. Then, for all $i \in [t]$, there exists $s_{Y_i} \in S_i$ such that $(b_i, s_{Y_i}) \in T$.*

Proof. Towards a contradiction assume that there is $i \in [t]$ such that for all $s_Y \in S_i$, $(b_i, s_Y) \notin T$. Recall that $N_{G'}(b_i) = S_i \cup \{b_j'^i \mid j \in [4t + 2]\}$. Let $h = \varphi(b_i)$ and $A = \{b_j, c_j, x_j, y_j, z_j \mid j \in [t], j \neq i\}$. There exists $v \in A$ such that $|W(h')| = 1$, where $h' = \varphi(v)$. This follows from the fact that at most $2k' = 4t$ vertices in A can be incident to an edge in T, although $|A| = 5(t - 1) > 4t$, as t can be assumed to be larger than 6, else the graph has constantly many edges and we can solve the problem in polynomial time. From Lemma 3.1 it follows that $(h, h') \in E(H)$, but $W(h), W(h')$ are not adjacent in G', contradicting that $\mathcal{W}$ is an *H-witness structure* of G'. Hence the claim follows. □

For each $i \in [t]$, we arbitrarily choose a vertex $s^\star_{Y_i} \in S_i$ such that $(b_i, s^\star_{Y_i}) \in T$. The existence of such a vertex is guaranteed by Lemma 3.2. In Lemma 3.1 we proved that the contracted counterparts of vertices c_i and b_i, for $i \in [t]$ belong to $\hat{C}$. Our next goal will to show that for each $i \in [t]$, vertices b_i, c_i and $s^\star_{Y_i}$ belong to the same witness set, which does not contain any other vertex. The above is shown using Lemma 3.3 to 3.6.

Lemma 3.3. *Let (G', k') be a* YES *instance of* Split Contraction *and $(b_i, s^\star_{Y_i}) \in T$ for $i \in [t]$. Then, for $h_i = \varphi(s^\star_{Y_i})$, we have $|W(h_i)| \geq 3$. Furthermore, there is an edge in T incident to b_i or $s^\star_{Y_i}$ other than $(b_i, s^\star_{Y_i})$.*

Proof. Suppose there exists $i \in [t]$, $h_i = \varphi(s^\star_{Y_i})$ such that $|W(h_i)| < 3$. Recall that $|W(h_i)| \geq 2$, since $b_i \in W(h_i)$. Let $A = \{x_j, y_j, z_j \mid j \in [t], j \neq i\}$. From Lemma 3.2, it follows that for each $j \in [t]$, there is an edge $(b_j, s^\star_{Y_j}) \in T$, therefore the number of edges in T incident to a vertex in A is bounded by $k' - t = t$. But $|A| = 3t - 3 > 2t$, therefore, there exists $a \in A$ such that for $h_a = \varphi(a)$, $|W(h_a)| = 1$. From Lemma 3.1, $(h_i, h_a) \in E(H)$, therefore $W(h_i)$ and $W(h_a)$ must be adjacent in G'. But $a \notin N(\{b_i, s^\star_{Y_i}\})$, hence $W(h_i)$ and $W(h_a)$ are not adjacent in G', contradicting that $\mathcal{W}$ is an *H-witness structure* of G'.

Since $|W(h_i)| \geq 3$ and $G[W(h_i)]$ is connected, at least one of $s^\star_{Y_i}, b_i$ must be adjacent to an edge in T which is not $(s^\star_{Y_i}, b_i)$. □

Lemma 3.4. *Let (G', k') be a* YES *instance of* Split Contraction*. Then, for all $i \in [t]$, we have $|W(h_i)| \geq 2$ where $h_i = \varphi(c_i)$.*

Proof. Towards a contradiction assume that there exists $i \in [t]$, $h_i = \varphi(c_i)$, such that $|W(h_i)| < 2$. Let $A = \{c_j \mid j \in [t], j \neq i\} \cup \{x_i, y_i, z_i\}$. From Lemma 3.2 it follows that the edge $(b_j, s^\star_{Y_j}) \in T$, for each $j \in [t]$. By Lemma 3.3 it follows that there is an edge in T that is adjacent to exactly one of $\{b_j, s^\star_{Y_j}\}$ in T, for all $j \in [t]$. Therefore, at most t vertices in A can be incident to an edge in T, while $|A| = t + 2$. This implies that there exists $a \in A$, $h_a = \varphi(a)$ such that $|W(h_a)| = 1$. Observe that none of the vertices in A are adjacent to c_i in G'. Therefore, it follows that $W(h_i), W(h_a)$ are not adjacent in G'. But Lemma 3.1 implies that $(h_i, h_a) \in E(H)$, a contradiction to $\mathcal{W}$ being an *H-witness structure* of G'. $\square$

Lemma 3.5. *Let (G', k') be a* YES *instance of* Split Contraction *and $(b_i, s^\star_{Y_i}) \in T$ for $i \in [t]$. Then, for each $i \in [t]$, we have $|W(h_i)| = 3$ where $h_i = \varphi(s^\star_{Y_i})$.*

Proof. For $i \in [t]$, let $h_i = \varphi(s^\star_{Y_i})$. From Lemma 3.3 we know that $|W(h_i)| \geq 3$. Let $C = \{c_i \mid i \in [t]\}$ and $\mathcal{S} = \{\{b_i, s^\star_{Y_i}\} \mid i \in [t]\}$. From Lemma 3.3 and 3.4 it follows that each $c \in C$ must be incident to an edge in T and each $S \in \mathcal{S}$ must have a vertex which is incident to an edge in T with the other endpoint not in S. Since $|C| = |\mathcal{S}| = t$ and $(b_i, s^\star_{Y_i}) \in T$, for all $i \in [t]$, there are at most t edges in T that are incident to a vertex in C and a vertex in $S \in \mathcal{S}$. Therefore, each $c \in C$ is incident to exactly one edge in T. Similarly, each $S \in \mathcal{S}$ is incident to exactly one edge in T with one endpoint in S and the other not in S. This implies that exactly one vertex $c \in C$ belongs to $W(h_i)$ for $i \in [t]$, and c does not belong to $W(h_j)$, where $i \neq j$, $i, j \in [t]$. Also note that none of the vertices in $\{x_i, y_i, z_i \mid i \in [t]\}$ can be incident to an edge in T. Similarly, none of the vertices in $\{b'^i_j, c'^i_j, x'^i_j, y'^i_j, z'^i_j \mid i \in [t], j \in [4t+2]\}$ can be incident to an edge in T. Hence, we get that $|W(h_i)| = 3$, concluding the proof. $\square$

Lemma 3.6. *Let (G', k') be a* YES *instance of* Split Contraction *and $(b_i, s^\star_{Y_i}) \in T$ for $i \in [t]$. Then, for all $i \in [t]$, we have $c_i \in W(h_i)$ where $h_i = \varphi(s^\star_{Y_i})$.*

Proof. Suppose for some $i \in [t]$, $c_i \notin W(h_i)$ where $h_i = \varphi(s^\star_{Y_i})$. From Lemma 3.3 and 3.4, and $k' = 2t$, it follows that there exists some $j \in [t]$ such that $c_i \in W(h_j)$, where $h_j = \varphi(s^\star_{Y_j})$. By our assumption, $j \neq i$. From Lemma 3.5 we know that $|W(h_j)| = 3$, therefore $W(h_j) = \{b_j, s^\star_{Y_j}, c_i\}$. Moreover, by Lemma 3.4 and 3.5 and since $k' = 2t$, $|W(x_i)| = 1$. However, we then get that $W(h_j), W(x_i)$ are not adjacent in G'. By Lemma 3.1, we obtain a contradiction to the assumption that $\mathcal{W}$ is an *H-witness structure* of G'. This completes the proof. $\square$

We are now ready to prove the main equivalence lemma of this section.

Lemma 3.7. *$(G, \mathcal{P} = \{C_1, C_2, \ldots, C_t\}, k_1, \ldots, k_t)$ is a* YES *instance of* Sub-Cubic PVC *if and only if (G', k') is a* YES *instance of* Split Contraction.

Proof. In the forward direction, let Y be a vertex cover in G such that for each $i \in [t]$, $|Y \cap C_i| \leq k_i$. For $i \in [t]$, we let $Y_i = Y \cap C_i$. Let $T = \{(b_i, s_{Y_i}), (c_i, s_{Y_i}) \mid i \in [t]\}$. Let $H = G'/T$, $\varphi : V(G') \to V(H)$ be the underlying surjective map and $\mathcal{W}$ be the *H-witness structure* of G'. To show that T is a solution to Split Contraction in (G', k'), it is enough to show that H is a split graph. Let $I = \cup_{i \in [t]} (S_i \setminus \{s_{Y_i}\}) \cup \{b'^i_j, c'^i_j, x'^i_j, y'^i_j, z'^i_j \mid i \in [t], j \in [4t+2]\}$. Recall that for each $v \in I$, $|W(\varphi(v))| = 1$. Furthermore, for $v, v' \in I$, $(v, v') \notin E(G')$. Hence, it follows that $\hat{I} = \{\varphi(v) \mid v \in I\}$ induces an independent set in H. Let $C_1 = \{x_i, y_i, z_i \mid i \in [t]\}$. Recall that $G'[C_1]$ is a clique and from the construction of T, $|W(\varphi(c))| = 1$ for all $c \in C_1$. Therefore, $\hat{C}_1 = \{\varphi(c) \mid c \in C_1\}$ induces a clique in H. Let $C_2 = \{s_{Y_i} \mid i \in [t]\}$, $h_i = \varphi(s_{Y_i})$ for $i \in [t]$, and $\hat{C}_2 = \{h_i \mid i \in [t]\}$. From the construction of T, we have $W(h_i) = \{b_i, c_i, s_{Y_i}\}$ for all $i \in [t]$. Observe that for $c_1 \in \hat{C}_1$ and $c_2 \in \hat{C}_2$, $W(c_1), W(c_2)$ are adjacent in G', therefore, $(c_1, c_2) \in E(H)$. Consider $h_i, h_j \in \hat{C}_2$, where $i, j \in [t], i \neq j$. Recall $W(h_i) = \{b_i, s_{Y_i}, c_i\}$ and $W(h_j) = \{b_j, s_{Y_j}, c_j\}$. Since Y is a vertex cover, at least one of Y_i or Y_j covers the unique edge between C_i and C_j in G, say Y_i covers the edge between C_i and C_j. But then $(s_{Y_i}, c_j) \in E(G')$, therefore $(h_i, h_j) \in E(H)$. The above argument implies that $\hat{C} = \hat{C}_1 \cup \hat{C}_2$ induces a clique in H. Furthermore, $V(H) = \hat{I} \cup \hat{C}$. This implies that H is a split graph.

In the reverse direction, let T be a solution to Split Contraction in (G', k'). Let $H = G'/T$, $\varphi : V(G') \to V(H)$ be the underlying surjective map and $\mathcal{W}$ be the *H-witness structure* of G'.

From Lemma 3.2, it follows that for all $i \in [t]$, there exists $s_{Y_i} \in S_i$ such that $(b_i, s_{Y_i}) \in T$. For $i \in [t]$, let Y_i be the set such that $(b_i, s_{Y_i}) \in T$. We let $Y = \cup_{i \in [t]} Y_i$. For $i \in [t]$, from the definition of the vertices in S_i, it follows that $|Y \cap C_i| \leq k_i$. We will show that Y is a vertex cover in G. Towards a contradiction assume that there exists $i, j \in [t], i \neq j$, such that Y does not cover the unique edge between C_i and C_j. From Lemmas 3.2 and 3.6 it follows that $W(h_i) = \{b_i, s_{Y_i}, c_i\}$ and $W(h_j) = \{b_j, s_{Y_j}, c_j\}$, where $h_i = \varphi(s_{Y_i})$ and $h_j = \varphi(s_{Y_j})$. From Lemma 3.1 it follows that $(h_i, h_j) \in E(H)$. Therefore, $W(h_i)$ and $W(h_j)$ are adjacent in G'. Recall that $N_{G'}(b_i) \cap W(h_j) = \emptyset$, $N_{G'}(b_j) \cap W(h_i) = \emptyset$, $(c_i, c_j), (s_{Y_i}, s_{Y_j}) \notin E(G')$. Therefore, at least one of $(c_i, s_{Y_j}), (c_j, s_{Y_i})$ must belong to $E(G')$, say $(c_i, s_{Y_j}) \in E(G')$. But then by construction it follows that $Y_j \subseteq Y$ covers the unique edge between C_i and C_j in G, a contradiction. This completes the proof. □

Finally, we restate Theorem 1.1 and prove its correctness.

Theorem 3.8. *Unless the* ETH *fails,* Split Contraction *parameterized by ℓ, the size of a minimum vertex cover of the input graph, does not have an algorithm running in time $2^{o(\ell^2)} \cdot n^{O(1)}$. Here, n denotes the number of vertices in the input graph.*

Proof. Towards a contradiction assume that there is an algorithm $\mathcal{A}$ for Split Contraction, parameterized by ℓ, the size of a minimum vertex cover, running in time $2^{o(\ell^2)} n^{O(1)}$. Let $(G, \mathcal{P} = \{C_1, C_2, \ldots, C_t\}, k_1, \ldots, k_t)$ be an instance of Sub-Cubic PVC. We create an instance (G', k') of Split Contraction as described in the **Construction**, running in time $2^{o(n)} \cdot n^{O(1)}$, where $n = |V(G)|$. Recall that in the instance created, the size of a minimum vertex cover is $\ell = 5t = O(\sqrt{n})$. Then we use algorithm $\mathcal{A}$ for deciding if (G', k') is a YES instance of Split Contraction and return the same answer for Sub-Cubic PVC on $(G, \mathcal{P}, k_1, \ldots, k_t)$. The correctness of the answer returned follows from Lemma 3.7. But then we can decide whether $(G, \mathcal{P}, k_1, \ldots, k_t)$ is a YES instance of Sub-Cubic PVC in time $2^{o(n)} \cdot n^{O(1)}$, which contradicts ETH assuming Theorem 3.9. This concludes the proof. □

3.2 Reduction from Sub-Cubic VC to Sub-Cubic PVC

Finally, to complete our proof we show that Sub-Cubic PVC on graphs with n vertices can not be solved in time $2^{o(n)} n^{O(1)}$ unless the ETH fails. In this section we give a Turing reduction from Sub-Cubic VC to Sub-Cubic PVC that will imply our desired assertion.

Let (G, k) be an instance of Sub-Cubic VC and $n = |V(G)|$. We first create a new instance (G', k') of Sub-Cubic VC satisfying certain properties. We start by computing a harmonious coloring of G using $t \in O(\sqrt{n})$ color classes such that each color class contains at most $O(\sqrt{n})$ vertices. A harmonious coloring on bounded degree graphs can be computed in polynomial time using at most $O(\sqrt{n})$ colors with each color class having at most $O(\sqrt{n})$ vertices [21, 36, 40]. Let $C_1, \ldots, C_t$ be the color classes. Recall that between each pair of the color classes, C_i, C_j for $i, j \in [t]$, $i \neq j$, we have at most one edge. If for some $i, j \in [t], i \neq j$, there is no edge between a vertex in C_i and a vertex in C_j, then we add a new vertex x_{ij} in C_i and a new vertex x_{ji} in C_j and add the edge (x_{ij}, x_{ji}). Observe that we add a matching corresponding to a missing edge between a pair of color classes. In this process we can add at most $t - 1$ new vertices to a color class C_i, for $i \in [t]$. Therefore, the number of vertices in C_i for $i \in [t]$ after addition of new vertices is also bounded by $O(\sqrt{n})$. We denote the resulting graph by G' with partition of vertices $C_1, \ldots, C_t$ (including the newly added vertices, if any). Observe that the number of vertices n' in G' is at most $O(n)$. Let m be the number of matching edges added in G to obtain G' and let $k' = k + m$. It is easy to see that (G, k) is a YES instance of Sub-Cubic VC if and only if (G', k') is a yes instance of Sub-Cubic VC.

We will now be working with the instance (G', k') of Sub-Cubic VC with the partition of vertices $C_1, \ldots, C_t$ obtained by extending the color classes of the harmonious coloring of G we started with. We guess the size of the intersection of the vertex cover in G' with each C_i, for $i \in [t]$. That is, for $i \in [t]$, we guess an integer $0 \leq k'_i \leq \min(|C_i|, k')$, such that $\sum_{i \in [t]} k'_i = k'$. Finally, we let $(G', \mathcal{P} = \{C_1, \ldots, C_t\}, k'_1, \ldots, k'_t)$ be an instance of Sub-Cubic PVC. Notice that G' and $\mathcal{P}$ satisfies all the requirements for $(G', \mathcal{P} = \{C_1, \ldots, C_t\}, k'_1, \ldots, k'_t)$ to be an instance of Sub-Cubic PVC. It is easy to see that (G', k') is a YES instance of Sub-Cubic VC if and only if for some guess of k_i, for $i \in [t]$, $(G', \mathcal{P} = \{C_1, \ldots, C_t\}, k'_1, \ldots, k'_t)$ is a YES instance of Sub-Cubic PVC. This finishes the reduction from Sub-Cubic VC to Sub-Cubic PVC.

Theorem 3.9. *Unless the* ETH *fails, Sub-Cubic PVC does not admit an algorithm running in time* $2^{o(n)} \cdot n^{O(1)}$*, for a graph with n vertices.*

Proof. Towards a contradiction assume that there is an algorithm $\mathcal{A}$ for Sub-Cubic PVC running in time $2^{o(n)} \cdot n^{O(1)}$. Let (G, k) be an instance of Sub-Cubic VC. We apply the above mentioned reduction to create an instance (G', k') of Sub-Cubic VC with vertex partitions $C_1, \ldots, C_t$ such that $t \in O(\sqrt{n})$ and $|C_i| \in O(\sqrt{n})$, for all $i \in [t]$. Furthermore, there is exactly one edge between C_i, C_j, for $i, j \in [t]$, $i \neq j$, and C_i induces an independent set in G'. For each guess $0 \leq k'_i \leq \min(|C_i|, k')$ of the size of intersection of vertex cover with C_i, for $i \in [t]$, we solve the instance $(G', \mathcal{P}, k'_1, \ldots, k'_t)$. By the exhaustiveness of the guesses of the size of intersection for each partition, (G', k') is a YES instance of Sub-Cubic VC if and only if for some guess $k'_1, \ldots, k'_t$, $(G', \mathcal{P}, k'_1, \ldots, k'_t)$ is a YES instance of Sub-Cubic PVC. We emphasize the fact that the number of guesses we make is bounded by $\sqrt{n}^{O(\sqrt{n})} = 2^{o(n)}$, since $|C_i| \in O(\sqrt{n})$ and $t \in O(\sqrt{n})$. But then we have an algorithm for Sub-Cubic VC running in time $2^{o(n)} \cdot n^{O(1)}$, contradicting the ETH. This concludes the proof. $\square$

4 W[1]-HARDNESS OF SPLIT CONTRACTION

In this section we show that Split Contraction parameterized by the solution size is W[1]-hard. Towards this we first define an intermediate problem from which we give the desired reduction.

> Special Red-Blue Perfect Code (SRBPC) **Parameter:** k
> **Input:** A bipartite graph G with vertex set $V(G)$ partitioned into red set $\mathcal{R}$ and blue set $\mathcal{B}$. Furthermore, $\mathcal{R}$ is partitioned (disjoint) into $R_1 \uplus R_2 \uplus \ldots \uplus R_k$ and for all $r, r' \in \mathcal{R}$, $d_G(r) = d_G(r')$. That is, every vertex in $\mathcal{R}$ has the same degree, say d.
> **Question:** Does there exist $X \subseteq \mathcal{R}$, such that for all $b \in \mathcal{B}$, $|N(b) \cap X| = 1$ and for all $i \in [k]$, $|R_i \cap X| = 1$?

SRBPC is a variant of Perfect Code which is known to be W[1]-hard [16]. We postpone the W[1]-hardness proof of SRBPC to Section 4.2 and first give a parameterized reduction from SRBPC to Split Contraction, showing that Split Contraction is W[1]-hard.

4.1 Reduction from SRBPC to Split Contraction

Let $(G, \mathcal{R} = R_1 \uplus, R_2 \uplus \ldots \uplus R_k, \mathcal{B})$ be an instance of SRBPC. We will assume that $|\mathcal{B}| = dk$, otherwise, the instance is a trivial NO instance of SRBPC. For technical reasons we assume that $|\mathcal{B}| = \ell > 4k$ (and hence $d > 4$). Such an assumption is valid because otherwise, the problem is FPT. Indeed, if $|\mathcal{B}| = \ell \leq 4k$ then for every partition $P_1, \ldots, P_k$ of $\mathcal{B}$ into k parts such that each part is non-empty, we first guess a permutation π on k elements and then for every $i \in [k]$, we check whether there exists a vertex $r_{\pi(i)} \in R_{\pi(i)}$ that dominates exactly all the vertices in P_i (and none in other parts P_j, $j \neq i$). Clearly, all this can be done in time $2^{O(k \log k)} n^{O(1)}$. Furthermore, we also assume that $\boxed{k \geq 2}$, else the problem is solvable in polynomial time. Now we give the desired reduction. We construct an instance (G', k') of Split Contraction as follows. Initially, $V(G') = \mathcal{R} \cup \mathcal{B}$ and $E(G') = E(G)$. For all $b, b' \in \mathcal{B}$, $b \neq b'$, we add the edge (b, b') to $E(G')$. That is, we transform $\mathcal{B}$ into a clique.

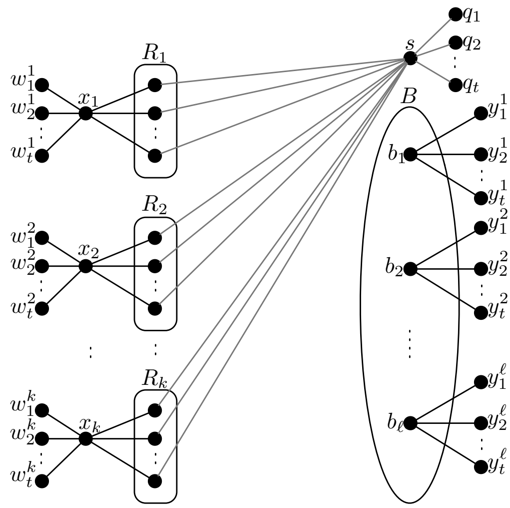

Fig. 2. W[1]-Hardness of Split Contraction.

Let $\boxed{t = 2k + 2}$. For each $b_i \in \mathcal{B}$, we add a set of t vertices $y_1^i, \ldots, y_t^i$ each adjacent to b_i in G'. We add a vertex s adjacent to every vertex $r \in \mathcal{R}$ in G'. Also, we add a set of t vertices $q_1, \ldots, q_t$ each adjacent to s in G'. For each $i \in [k]$, we add a vertex x_i adjacent to each vertex $r \in R_i$. Finally, for all $i \in [k]$, we add a set of t vertices $w_1^i, \ldots, w_t^i$ adjacent to x_i in G'. We set the new parameter k' to be $2k$. This completes the description of the reduction. We refer the reader to Figure 2 for an illustration of the reduction.

In the next four lemmas (Lemma 4.1 to 4.4) we prove certain structural properties of the instance (G', k') of Split Contraction. These will later be used in showing that $(G, \mathcal{R} = R_1 \uplus R_2 \uplus \ldots \uplus R_k, \mathcal{B})$ is a YES instance of SRBPC if and only if (G', k') is a YES instance of Split Contraction. For the next four lemmas, we let S be a solution to Split Contraction in (G', k') and $H = G'/S$ with $\hat{C}$, $\hat{I}$ being a partition of $V(H)$ inducing a clique and an independent set, respectively, in H. Let $\varphi : V(G) \to V(H)$ denote the function defining the contractibility of G to H, and $\mathcal{W}$ be the *H-witness structure* of G.

In the following lemma, we show that the vertices in $\mathcal{B}$, s and x_i, for $i \in [k]$ (or their contracted counterparts) always belong to the clique side (which is $\hat{C}$, for our case).

Lemma 4.1. *Let (G', k') be a* YES *instance of Split Contraction. Then, for all $v \in (\{s\} \cup \mathcal{B} \cup \{x_i \mid i \in [k]\})$, we have $\varphi(v) \in \hat{C}$.*

Proof. We only give an argument for the vertex s. The argument for vertices in $\mathcal{B} \cup \{x_i \mid i \in [k]\}$ is analogous and thus omitted. Recall that there are t pendant vertices $q_1, \ldots, q_t$ adjacent to s, where $t = 2k+2$. At most $2k < t$ edges in $\{(q_i, s) \mid i \in [t]\}$ can belong to S. Therefore, there exist $j_1, j_2 \in [t]$, $j_1 \neq j_2$ such that no edge incident to q_{j_1} or q_{j_2} is in S. In other words, for $h_1 = \varphi(q_{j_1})$ and $h_2 = \varphi(q_{j_2})$, $W(h_1)$ and $W(h_2)$ are singleton sets. Since $\mathcal{W}$ is a H-witness structure of G', $(h_1, h_2) \notin E(H)$. Therefore, at least one of h_1, h_2 belongs to $\hat{I}$, say $h_1 \in \hat{I}$. This implies that $\varphi(s) \in \hat{C}$. $\square$

Next lemma shows that for each $i \in [k]$, there is a vertex $r_i \in R_i$, such that the edge $(x_i, r_i) \in S$. This will be helpful in selecting a vertex from R_i (while constructing a desired type of dominating set in the graph G).

Lemma 4.2. *Let (G', k') be a* YES *instance of* Split Contraction*. Then, for all $i \in [k]$, there exists $r_i \in R_i$ such that $(x_i, r_i) \in S$.*

Proof. Towards a contradiction assume that there exists an index $i \in [k]$ such that for all $r \in R_i$, $(x_i, r) \notin S$. Let $h = \varphi(x_i)$. Observe that the edges in S can only be incident to at most $4k$ vertices and thus there exists $j \in [\ell]$ ($\ell = |\mathcal{B}| > 4k$) such that for $h' = \varphi(b_j)$, $W(h')$ is a singleton set. From Lemma 4.1, we know that $h, h' \in \hat{C}$. Hence, $W(h)$ and $W(h')$ are adjacent in G'. Thus there is a vertex $v \in W(h)$ and $v' \in W(h')$ such that $(v, v') \in E(G')$. Since $|W(h')| = 1$, we have that $v' = b_j$. But $(x_i, b_j) \notin E(G')$, hence $v \neq x_i$. Observe that v is a vertex of degree at least 2 in G' and all the neighbors of x_i with degree at least 2 are in R_i. Hence it follows that there exists $r \in R_i$ such that $r \in W(h)$. The solution S must contain all the edges of a spanning tree of $G[W(h)]$. Any spanning tree of $G[W(h)]$ must contain an edge (x_i, r') where $r' \in R_i$ (possibly $r' = r$) since all the paths between x_i and r in G must contain a vertex in R_i. This is contrary to our assumption that for all $r \in R_i$, $(x_i, r) \notin S$. This completes the claim. □

For each $i \in [k]$ we arbitrarily choose a vertex $r_i^\star \in R_i$ such that $e_i^\star = (x_i, r_i^\star) \in S$. The existence of such a vertex is guaranteed by Lemma 4.2. Note that we have already used a budget of (at least) k edges, which are precisely the edges $e_i^\star$, for $i \in [k]$. In our next lemma (Lemma 4.3), we show that for each $i \in [k]$, there is an edge $e_i \neq e_i^\star$ in S. We note that the existence of such k edges (for each $i \in [k]$) will guarantee that we have used all our budget. This will be helpful in ensuring that the vertex we select from R_i (for the desired type of dominating set for G) is $r_i^\star$, and there is exactly one such selected vertex, for each $i \in [k]$.

Lemma 4.3. *Let (G', k') be a* YES *instance of* Split Contraction*. Then, for all $i \in [k]$ and $h_i = \varphi(r_i^\star)$, we have $|W(h_i)| \geq 3$. Furthermore, there is an edge $e_i \neq e_i^\star$ in S incident to exactly one of $x_i, r_i^\star$ and not incident to the vertices in $\{w_1^i, \ldots, w_t^i\}$.*

Proof. Towards a contradiction assume that for some $i \in [k]$ and $h_i = \varphi(r_i^\star)$, $|W(h_i)| < 3$. From our assumption that $(x_i, r_i^\star) \in S$ we have that $x_i \in W(h_i)$. Also, note that there is a set $\mathcal{B}' \subseteq \mathcal{B}$ of at least $\ell - 2k$ vertices such that for $h_b = \varphi(b)$, $|W(h_b)| = 1$. This follows from the fact that at most $2k$ vertices in $\mathcal{B}$ can be incident to an edge in S. Let $\hat{\mathcal{B}} = \mathcal{B}' \setminus N(r_i^\star)$. We claim that $|\hat{\mathcal{B}}| \geq \ell - 2k - d > 0$. Towards the claim observe that if $(G, \mathcal{R}, \mathcal{B})$ is a YES instance of SRBPC then $\ell = dk$. The last assertion follows from the fact that every vertex in $\mathcal{R}$ has degree exactly d and we are seeking a solution $X \subseteq \mathcal{R}$, such that for all $b \in \mathcal{B}$, $|N(b) \cap X| = 1$ and for all $i \in [k]$, $|R_i \cap X| = 1$. That is, the set X is of size k and it partitions $\mathcal{B}$. This implies that $d > 4$, since $\ell = dk > 4k$. Thus, combining this with the fact that $k \geq 2$ we have that $|\hat{\mathcal{B}}| \geq \ell - 2k - d = (d-2)k - d > 0$. This completes the claim. Since the size of $|W(h_i)| < 3$ and it contains x_i and $r_i^\star$ we have that $W(h_i) = \{x_i, r_i^\star\}$. Now, consider $\hat{b} \in \hat{\mathcal{B}}$ with $\hat{h} = \varphi(\hat{b})$. Observe that $W(h_i)$ and $W(\hat{h})$ are not adjacent in G, however since $x_i \in W(h_i)$ Lemma 4.1 implies that $h_i \in \hat{C}$. But then $(\hat{h}, h_i) \in E(H)$, a contradiction. This implies that for all $i \in [k]$ and $h_i = \varphi(r_i^\star)$ we have $|W(h_i)| \geq 3$. However, since $h_i, \hat{h} \in \hat{C}$ there must be a vertex in $W(h_i)$ that is adjacent to a vertex in $W(\hat{h})$. But since $W(\hat{h}) = \{\hat{b}\}$, $W(h_i)$ must contain a vertex that is adjacent to $\hat{b}$. But, none of the vertices in $\{w_1^i, \cdots, w_t^i\}$ are adjacent to $\hat{b}$. Thus, $W(h_i)$ must contain a vertex that is adjacent to either x_i or $r_i^\star$ but not to any of the vertices in $\{w_1^i, \cdots, w_t^i\}$. Let such a vertex be z_i and let it be adjacent to $r_i^\star$ (or x_i). Since a solution to (G', k') can be formed by taking spanning trees of each of the witness sets, we can assume that S contains a spanning tree of $W(h_i)$ that contains the edge $e_i = (z_i, r_i^\star)$ (or $e_i = (z_i, x_i)$) and $e_i^\star$. This completes the proof of the lemma. □

From Lemma 4.2 we know that for each $i \in [k]$, we have $r_i^\star \in R_i$ such that $(x_i, r_i^\star) \in S$. Similarly, from Lemma 4.3 we know that, for each $i \in [k]$, there is an edge incident to one of x_i, r_i other than $e_i^\star = (x_i, r_i^\star)$ in every solution. Recall that for $i, j \in [k]$, $i \neq j$ none of x_i, r_i is adjacent to x_j, r_j. Hence, it follows that we have already used up our budget of $k' = 2k$ by forcing certain types of edges to be in S. Finally, we prove Lemma 4.4, where we prove that s and $r_i^\star$, for $i \in [k]$, belong to the same witness set. This will be crucial in proving that the set $\{r_i^\star \mid i \in [k]\}$ forms a solution for the instance $(G, \mathcal{R} = R_1 \uplus R_2 \uplus \ldots \uplus R_k, \mathcal{B})$ of SRBPC.

Lemma 4.4. *Let (G', k') be a* YES *instance of Split Contraction. Then, for all $i \in [k]$, $r_i^\star \in W(\varphi(s))$.*

Proof. Let $h_s = \varphi(s)$ and $\hat{R} = \{r_i^\star \mid i \in [k], r_i^\star \in W(h_s)\}$. For a contradiction assume that $|\hat{R}| < k$, otherwise the claim trivially holds. By Lemma 4.2, for each $i \in [k]$, $e_i^\star = (x_i, r_i^\star) \in S$. This implies that for all $r_i^\star \in \hat{R}$, $x_i \in W(h_s)$ and hence $|W(h_s)| \geq 2|\mathcal{R}| + 1$. From Lemma 4.3 we know that there exists an edge $e_i \neq e_i^\star \in S$ incident to either x_i or $r_i^\star$ and not incident to any vertex in $\{w_1^i, \ldots, w_t^i\}$. Thus, every edge in S is incident to either x_i or $r_i^\star$. This implies that for every vertex $z \in \{q_1, \ldots, q_t\} \cup \{y_1^j, \ldots, y_t^j \mid j \in [\ell]\}$, $|W(\varphi(z))| = 1$. Now we show that there exists a vertex in $\mathcal{B}$ that is not adjacent to any vertex in $W(h_s)$. We start proving the claim that S does not contain an edge of the form $(r_i^\star, b_j)$, where $i \in [k]$ and $b_j \in \mathcal{B}$. Suppose not. Consider the sets $\hat{R}_b = \{r_i^\star \in \hat{R} \mid (r_i^\star, b) \in S, b \in \mathcal{B}\}$ and $\hat{\mathcal{B}} = \{b \in \mathcal{B} \mid (r_i^\star, b) \in S, i \in [k]\}$. By our assumption we have $|\hat{R}_b| = q > 0$. Moreover, for each $b \in \hat{\mathcal{B}}$, we have $\varphi(s)$ and $\varphi(b)$ are adjacent in H and $|\hat{\mathcal{B}}| \leq q$. Observe that $|W(\varphi(s)) \cap \mathcal{R}| \leq k - q$, and $W(\varphi(s)) \cap \hat{R}_b = \emptyset$. From Lemma 4.1, $\varphi(s)$ must be adjacent in H to each $\varphi(b)$, where $b \in \mathcal{B}$. Since degree of each vertex in $\mathcal{R}$ is d therefore, $\varphi(s)$ can be adjacent in H to at most $d(k - q)$ vertices $\varphi(b)$, where $b \in \mathcal{B} \setminus \hat{\mathcal{B}}$. As $d > 4$, there is a vertex $b \in \mathcal{B} \setminus \hat{\mathcal{B}}$ such that $\varphi(s)$ and $\varphi(b)$ are non-adjacent in H, which is not possible. This concludes the proof of the claim. The claim allows us to assume that the only vertices in $W(h_s)$ that can be adjacent to a vertex in $\mathcal{B}$ are in $\hat{R}$. However, every vertex in $\hat{R}$ has exactly d neighbors in $\mathcal{B}$. This together with the fact that $|\mathcal{B}| = \ell = dk > d|\hat{R}|$ implies that there exists a subset $\mathcal{B}'$ of size $d(k - |\hat{R}|)$ such that none of these vertices are adjacent to any vertex in $\hat{R}$. However, at most $(k - |\hat{R}|)$ vertices in $\mathcal{B}'$ can be incident to an edge in S. This implies that there exists a vertex $b \in \mathcal{B}'$ with $h = \varphi(b)$ such that it is not incident to any edge in S and thus $|W(h)| = 1$. But then we can conclude that $W(h)$ and $W(h_s)$ are not adjacent in G'. However, by Lemma 4.1 we know that $h_s, h \in \hat{C}$ and thus there is an edge $(h = \varphi(b), h_s) \in E(H')$, a contradiction. This contradicts our assumption that $|\hat{R}| < k$ and gives us the desired result. ◻

We are now ready to prove the equivalence between the instance $(G, \mathcal{R}, \mathcal{B})$ of SRBPC and the instance (G', k') of Split Contraction.

Lemma 4.5. *$(G, \mathcal{R} = R_1 \uplus \ldots \uplus R_k, \mathcal{B})$ is a* YES *instance of SRBPC if and only if (G', k') is a* YES *instance of Split Contraction.*

Proof. In the forward direction, let $Z = \{r_i \mid r_i \in R_i, i \in [k]\} \subseteq \mathcal{R}$ be a solution to $(G, \mathcal{R}, \mathcal{B})$ of SRBPC. Let $Z' = \{(r_i, x_i), (r_i, s) \mid i \in [k]\}$. Observe that $|Z'| = 2k$. Let $T = \{r_i, x_i \mid i \in [k]\}$. We define the following surjective function $\varphi : V(G') \to V(G') \setminus T$. If $v \in T \cup \{s\}$ then $\varphi(v) = s$, else $\varphi(v) = v$. Observe that $G'[W(s)]$ is connected and for all $v \in V(G') \setminus (T \cup \{s\})$, $W(v)$ is a singleton set. Consider the graph H with $V(H) = V(G') \setminus T$ and $(v, u) \in E(H)$ if and only if $\varphi^{-1}(v), \varphi^{-1}(u)$ are adjacent in G'. Note that the graphs G'/Z' and H are isomorphic, therefore we prove that H is a split graph. Let $\hat{C} = \{\varphi(v) \mid \mathcal{B} \cup \{s\}\}$ and $\hat{I} = V(H) \setminus \hat{C}$. For $v, u \in \hat{I}$, $\varphi^{-1}(v) = \{v\}$ and $\varphi^{-1}(u) = \{u\}$ and $\{v\}, \{u\}$ are non-adjacent in G'. Therefore, $(v, u) \notin E(H)$. This proves that $\hat{I}$ is an independent set in H. For $b, b' \in \mathcal{B} \subset \hat{C}$, $(b, b') \in E(G')$, therefore $(\varphi(v), \varphi(u)) \in E(H)$. Since Z is a solution to SRBPC in $(G, \mathcal{R}, \mathcal{B})$, for $b \in \mathcal{B}$, there exists $r_i \in Z$ such that $(b, r_i) \in E(G')$, therefore, $W(s)$ and b

are adjacent in G'. Hence, $(\varphi(s), \varphi(b)) \in E(H')$. This finishes the proof that $\hat{C}$ induces a clique in H and that H is a split graph.

In the reverse direction, let S be a solution to (G', k') of Split Contraction, and denote $H = G'/S$. Let $\mathcal{W}$ be the H-*witness structure* of G, φ be the associated surjective function and $h_s = \varphi(s)$. From Lemmas 4.2 and 4.4 it follows that for all $i \in [k]$, there exists $r_i^\star \in R_i$ such that $(x_i, r_i^\star) \in S$ and $r_i^\star, x_i \in W(h_s)$. Let $Z = \{r_i^\star \mid i \in [k]\}$. We will show that Z is a solution to SRBPC in $(G, \mathcal{R}, \mathcal{B})$. Since $|W(h_s)| \geq k' + 1 = 2k + 1$, it holds that for all $v \in V(H) \setminus \{h_s\}$, $|W(v)| = 1$. This implies that for all $b \in \mathcal{B}$, $b \notin W(h_s)$. Also observe that since $x_i \in W(h_s)$ for all $i \in [k]$ and $|W(h_s)| = k' + 1 = 2k + 1$, we have that $|W(h_s) \cap R_i| = 1$. This implies that $|Z| = k$ and $|Z \cap R_i| = 1$, for all $i \in [k]$. To show that Z is indeed a solution, it is enough to show that for all $b_j \in \mathcal{B}$, $|Z \cap N(b_j)| = 1$. Towards a contradiction, assume there exists $b_j \in \mathcal{B}$ such that $|Z \cap N(b_j)| \neq 1$. Let $h_{b_j} = \varphi(b_j)$. We consider the following two cases.

- If $|Z \cap N_{G'}(b_j)| < 1$. Recall that $W(h_{b_j}) = \{b_j\}$. Furthermore, $N_{G'}(b_j) \subseteq \mathcal{R} \cup \{y_1^j, \ldots, y_t^j\}$, $Z = W(h_s) \cap \mathcal{R}$ and by our assumption $Z \cap N_{G'}(b_j) = \emptyset$. But then $W(h_s)$ and $W(h_{b_j})$ are not adjacent in G'. However, Lemma 4.1 implies that $(h_s, h_{b_j}) \in E(H)$, contradicting our assumption that $|Z \cap N(b_j)| < 1$.
- If $|Z \cap N_{G'}(b_j)| > 1$, then there exist $j, j' \in [k]$, $j \neq j'$ such that $r_j^\star, r_{j'}^\star \in N_{G'}(b)$. Then it follows that $| \cup_{i \in [k]} N(r_i^\star)| < \ell = dk$. But then there exists $b' \in \mathcal{B}$ such that $W(\varphi(b'))$ and $W(h_s)$ are non-adjacent in G', contradicting that $(\varphi(b'), h_s) \in E(H)$ from Lemma 4.1.

This completes the proof. □

By Lemma 4.5 and the W[1]-hardness of SRBPC (Theorem 4.10), we clearly have the proof of Theorem 1.2: "Split Contraction is W[1]-hard when parmeterized by the size of a solution".

4.2 W[1]-Hardness of Special Red-Blue Perfect Code

In this section we show that SRBPC is W[1]-hard parameterized by the solution size. We give a reduction from Multi-Colored Clique to SRBPC. The problem Multi-Colored Clique is known to be W[1]-hard [22], and is formally defined below.

> Multi-Colored Clique (MCC) **Parameter:** k
> **Input:** A k-partite graph G with vertex partition $V_1, \ldots, V_k$ of $V(G)$.
> **Question:** Does there exist $X \subseteq V(G)$ such that for all $i \in [k]$, $|X \cap V_i| = 1$ and $G[X]$ is a clique?

The intuitive description of the reduction we are going to construct below is as follows. Let $(G, V_1, \ldots, V_k)$ of be an instance of MCC. We will often refer to the sets V_i as color classes. For each color class we create a vertex selection gadget. Then we have edge selection gadgets which ensure that between every pair of color classes an edge is selected. The vertex selection gadget ensures that the vertex chosen is the same as the one incident to the edge chosen by the edge selection gadget. Finally, we have a coherence gadget which ensures that all the edges that are incident to a color class are incident to the same vertex in this color class.

For technical reasons we will assume that the number of vertices in G is 2^t, for some $t \in \mathbb{N}$. Note that this can be easily achieved by adding dummy vertices to an arbitrary color class with no edge incident to them. This results in at most doubling of the number of vertices in the graph. For

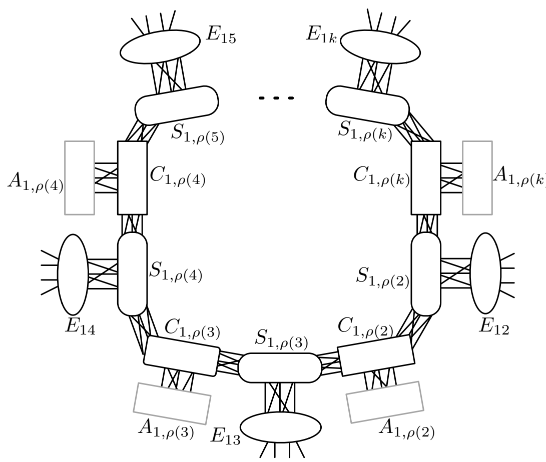

Fig. 3. Illustration of edges between Vertex Selection Gadget, Coherence Gadget for $i = 1$, and Edge Selection Gadget.

our purposes, we also assign a unique t-bit-string to each vertex $v \in V(G)$. Next, we move to the description of the instance $(G', \mathcal{R}, \mathcal{B})$ of SRBPC that we create.

Edge Selection Gadget. For $i, j \in [k]$, $i \neq j$, we create an edge selection gadget E_{ij} as follows. For each edge $(u, v) \in E(G)$, such that $u \in V_i$ and $v \in V_j$, we add a vertex e_{uv} to E_{ij}. We emphasize the fact that E_{ij} and E_{ji} denote the same set. Similarly, for an edge $(u, v) \in E(G)$, the vertices e_{vu} and e_{uv} are the same vertex. The symmetry in the indices/subscripts holds only for the *edge selection* gadgets.

For the description of the *vertex selection* and *coherence* gadgets we will need the following notation. For $i \in [k]$, the set $T_i = \{j \in [k] \mid j \neq i\}$ has a natural total ordering ρ_i, specifically the order given by the relation $<$ defined on $\mathbb{N}$. Therefore, by $\rho_i(j)$ we denote the position of j in the total ordering of T_i (1st position is denoted by 1). We will slightly abuse the notation and drop the subscript i from ρ_i whenever it is clear from the context.

Vertex Selection Gadget. For each color class $i \in [k]$ we have a vertex selection gadget $\mathcal{S}_i$. For $i \in [k]$, $\mathcal{S}_i$ consists of $k - 1$ sets of vertices $S_{i,\rho(j)}$, where $j \in [k] \setminus \{i\}$. Here, $S_{i,\rho(j)}$ is a set of $2t$ vertices denoted by $x_0^{i,\rho(j)}, x_1^{i,\rho(j)}, \ldots, x_{t-1}^{i,\rho(j)}, y_0^{i,\rho(j)}, y_1^{i,\rho(j)}, \ldots, y_{t-1}^{i,\rho(j)}$. The intuition behind the construction of the set $S_{i,\rho(j)}$ is to encode the bit representation of the vertices in V_i. The size of $S_{i,\rho(j)}$ is twice the size of the bit-representation for achieving the degree constraints of the vertices in the instance of SRBPC to be created.

Coherence Gadget. Consider $i \in [k]$ and $j \in [k] \setminus \{i\}$. We have a set $C_{i,\rho(j)}$ containing copies of vertices in V_i, *i.e.* $|C_{i,\rho(j)}| = |V_i|$. For a vertex $v \in V_i$, its copy in $C_{i,\rho(j)}$ is denoted by $c_v^{i,\rho(j)}$. Also, we have a set $A_{i,\rho(j)}$ containing a vertex $a_\ell^{i,\rho(j)}$, for each $\ell \in [t]$. The set $A_{i,\rho(j)}$ is added only to ensure some degree constraints in the construction. For each $u \in A_{i,\rho(j)}$ and $v \in C_{i,\rho(j)}$, we add the edge (u, v) to $E(G')$, i.e., $G'[A_{i,\rho(j)} \cup C_{i,\rho(j)}]$ is a complete bipartite graph. By $\mathcal{A}_i$ we denote the set $\cup_{j \in [k] \setminus \{i\}} A_{i,\rho(j)}$.

We now move to the description of the edges between vertex selection, edge selection and coherence gadgets. We refer the reader to Figure 3 for an illustration of the reduction.

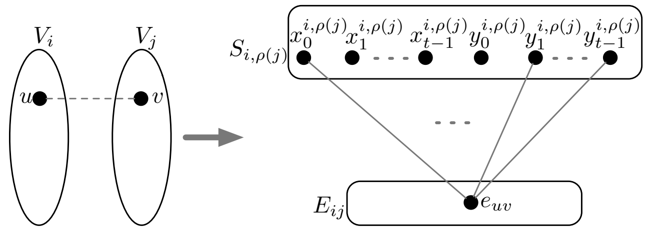

Fig. 4. Edges between E_{ij} and S_{ij}, assuming the bit-string associated with v has $b_0 = 1$ and $b_\ell = 0$ for all $\ell \in [t-1]$.

Edges between gadgets. Let $i, j \in [k], i \neq j$, and $u \in V_i, v \in V_j$ such that $(u, v) \in E(G)$. Recall that corresponding to the edge (u, v), we have a vertex e_{uv} in E_{ij} (which is the same as E_{ji}). Let $b_0 b_1 \dots b_{t-1}$ be the unique bit-string assigned to u. We add an edge between $x_\ell^{i,\rho(j)} \in S_{i,\rho(j)}$ and e_{uv} in G' if and only if $b_\ell = 1$, here $\ell \in \{0, \dots, t-1\}$. Similarly, we add an edge between $y_\ell^{i,\rho(j)} \in S_{i,\rho(j)}$ and e_{uv} in G' if and only if $b_\ell = 0$; here, $\ell \in \{0, \dots, t-1\}$. Refer to Figure 4 for a pictorial illustration.

We now describe the edges between $C_{i,\rho(j)}$ and $S_{i,\rho(j)}$. We will assume modulo k-arithmetics for the computation of indices. We note that the notation ρ is used only for ease in specification and modulo index computation to work properly. For $i, j \in [k]$, $i \neq j$ and $v \in V_i$, there is a vertex $c_v^{i,\rho(j)} \in C_{i,\rho(j)}$. Let $b_0 b_1 \dots b_{t-1}$ be the unique bit-string assigned to v. We add an edge between $x_\ell^{i,\rho(j)} \in S_{i,\rho(j)}$ and $c_v^{i,\rho(j)}$ in G' if and only if $b_\ell = 0$, here $\ell \in \{0, \dots, t-1\}$. Similarly, we add an edge between $y_\ell^{i,\rho(j)+1} \in S_{i,\rho(j)+1}$ and $c_v^{i,\rho(j)}$ in G' if and only if $b_\ell = 1$, here $\ell \in \{0, \dots, t-1\}$. This finishes the description of the graph G'.

Now we move on to partitioning the vertices in $V(G')$ into two sets $\mathcal{R}$ and $\mathcal{B}$. Then we further partition $\mathcal{R}$. For $i, j \in [k]$, $i \neq j$ we add all the vertices in $C_{i,\rho(j)}$ and E_{ij} to $\mathcal{R}$. All the remaining vertices are added to the set $\mathcal{B}$. The set $\mathcal{R}$ is partitioned into E_{ij} and $C_{i,\rho(j)}$, where $i \neq j$. Observe that since $E_{ij} = E_{ji}$ for all $i \neq j$ we have $k(k-1) + \binom{k}{2}$ parts of $\mathcal{R}$ and the degree of each vertex in $\mathcal{R}$ is $2t$. This completes the description of the instance $(G', \mathcal{R}, \mathcal{B})$ of SRBPC.

We will prove some lemmas (Lemma 4.6 to 4.8) that will help us in establishing the equivalence between the two instances. In Lemma 4.6 we consider the case when $(G', \mathcal{R}, \mathcal{B})$ is a YES instance of SRBPC and has a solution R. Consider the case when $e_{uv} \in R$ (which corresponds to the selection of edge (u, v) for the MCC instance, to establish the reverse direction), where $u \in V_i$ and $v \in V_i$. Then we show that the vertices corresponding to u and v in the sets $C_{i,\rho(j)}$ and $C_{i,\rho(j)-1}$, and $C_{j,\rho(i)}$ and $C_{j,\rho(i)-1}$, respectively also belong to the set R. The above will be helpful in showing that the edges selected are incident to the vertices selected for their respective color classes.

Lemma 4.6. *Let $(G', \mathcal{R}, \mathcal{B})$ be a* YES *instance of SRBPC and R be one of its solution. If for some $i, j \in [k], i \neq j, u \in V_i, v \in V_j$ we have $e_{uv} \in R$ then the following holds.*

- $c_u^{i,\rho(j)}, c_u^{i,\rho(j)-1} \in R$.
- $c_v^{j,\rho(i)}, c_v^{j,\rho(i)-1} \in R$.

Proof. We give proof only for the first part of the lemma. The second one follows from an analogous argument. Consider $i, j \in [k], i \neq j$, $u \in V_i$, $v \in V_j$, such that $e_{uv} \in R$. Let $\bar{b}_u = b_0 b_1 \dots b_{t-1}$ be the unique bit-string assigned to u. Observe that all the vertices $x_\ell^{i,\rho(j)}$ with $b_\ell = 1$, for $\ell \in \{0, \dots, t-1\}$ are adjacent to e_{uv}. Since R is a solution, it must contain a vertex from $C_{i,\rho(j)}$.

Let the unique vertex in $R \cap C_{i,\rho(j)}$ be $c_w^{i,\rho(j)}$. Suppose $w \neq u$. Consider the difference in the bit-string representation $\bar{b}_w$, of w and $\bar{b}_u$. Since $w \neq u$, $\bar{b}_w$ and $\bar{b}_u$ differs in at least one position, let the first such position be q. If $b_q = 1$ (q^{th} bit in $\bar{b}_u$) then q^{th} bit in $\bar{b}_w$ is 0. But then, $x_q^{i,\rho(j)}$ is adjacent to two vertices, namely e_{uv} and $c_w^{i,\rho(j)}$, contradicting that R is a solution. If $b_q = 0$, then $x_q^{i,\rho(j)}$ is not adjacent to e_{uv} and $c_u^{i,\rho(j)}$. Recall that $N(x_q^{i,\rho(j)}) \subseteq E_{ij} \cup C_{i,\rho(j)}$. Hence, $x_q^{i,\rho(j)}$ is non-adjacent to any vertex in R, a contradiction. Therefore, $u = w$ and $c_u^{i,\rho(j)} \in R$. A similar argument can be given for proving $c_u^{i,\rho(j)-1} \in R$. This completes the proof. □

In our next lemma (Lemma 4.7), we consider the case when a vertex say $c_u^{i,\rho(j)}$ is selected from the set $C_{i,\rho(j)}$ in a solution R for the instance $(G', \mathcal{R}, \mathcal{B})$ of SRBPC. We then show that edges incident to u must be selected for each color class V_j ($j \neq i$). That is, for each $j \in [k] \setminus \{i\}$, there is some $v \in V_j$ such that $e_{uv} \in R$.

Lemma 4.7. *Let $(G', \mathcal{R}, \mathcal{B})$ be a* YES *instance of SRBPC and R be a solution. If for some $i, j \in [k], i \neq j$ and $u \in V_i$ we have $c_u^{i,\rho(j)} \in R$ then there exists some $v \in V_j$ such that $e_{uv} \in R$.*

Proof. Towards a contradiction assume that for some $i, j \in [k], i \neq j$ and $u \in V_i$ we have $c_u^{i,\rho(j)} \in R$ and for all $v \in V_j$, $e_{uv} \notin R$. Let $\bar{b}_u = b_0 b_1 \dots b_{t-1}$ be the unique bit-string assigned to u. For all $\ell \in \{0, \dots, t-1\}$ such that $b_\ell = 0$, $x_\ell^{i,\rho(j)}$ is adjacent to $c_u^{i,\rho(j)}$. Since R is a solution it must contain a vertex $e_{wz} \in E_{ij}$, where $w \in V_i$ and $z \in V_j$. By assumption $w \neq u$. But by Lemma 4.6, $c_w^{i,\rho(j)} \in R$, contradicting that $|R \cap C_{i,\rho(j)}| = 1$. This implies that $w = u$. □

In the following lemma (Lemma 4.8), we consider the case when for $i, j \in [k]$, where $i \neq j$, a vertex say $c_u^{i,\rho(j)} \in C_{i,\rho(j)}$ is selected in a solution R to the SRBPC instance $(G', \mathcal{R}, \mathcal{B})$. Then we show that for each for each $\ell \in [k] \setminus \{i\}$ we have $c_u^{i,\rho(\ell)} \in R$. The above will be useful to argue that for each i, the vertices selected from the sets $C_{i,\rho(\ell)}$, for $\ell \in [k] \setminus \{i\}$, correspond to the same vertex in V_i. This will be useful in the selection of exactly one vertex for each V_i, for the MCC instance $(G, V_1, \dots, V_k)$.

Lemma 4.8. *Let $(G', \mathcal{R}, \mathcal{B})$ be a* YES *instance of SRBPC and R be a solution. If for some $i, j \in [k], i \neq j$ and $u \in V_i$ we have $c_u^{i,\rho(j)} \in R$ then for all $\ell \in [k] \setminus \{i\}$ we have $c_u^{i,\rho(\ell)} \in R$.*

Proof. Follows from Lemma 4.6 and 4.7. □

We are now ready to establish the equivalence of the instance $(G, V_1, \dots, V_k)$ of MCC and the instance $(G', \mathcal{R}, \mathcal{B})$ of SRBPC.

Lemma 4.9. *$(G, V_1, \dots, V_k)$ is a* YES *instance of MCC if and only if $(G', \mathcal{R}, \mathcal{B})$ is a* YES *instance of SRBPC.*

Proof. In the forward direction, let $V = \{v_i \mid i \in [k]\}$ be a solution to MCC for $(G, V_1, \dots, V_k)$. Let $\bar{b}_i$ be the unique bit-string assigned to v_i, for $i \in [k]$. Also, we let $R = \{c_{v_i}^{i,\rho(j)} \mid i, j \in [k], i \neq j\} \cup \{e_{v_i v_j} \mid i, j \in [k], i \neq j\}$. Observe that $|R \cap C_{ij}| = 1$, for all $i, j \in [k], i \neq j$. Similarly, $|R \cap E_{ij}| = 1$, for all $i, j \in [k], i \neq j$. Recall that $\mathcal{B} = V(G') \setminus \mathcal{R} = (\cup_{i \in [k]} \mathcal{S}_i) \cup (\cup_{i \in [k]} \mathcal{A}_i)$. Here, for $i \in [k]$, we have $\mathcal{S}_i = \cup_{j \in [k] \setminus \{i\}} S_{i,\rho(j)}$ and $\mathcal{A}_i = \cup_{j \in [k] \setminus \{i\}} A_{i,\rho(j)}$. Observe that for each $i \in [k]$, each vertex in $\mathcal{A}_i$ is adjacent to exactly one vertex in R. Next, we show that for $i, j \in [k]$, $i \neq j$, each vertex in $S_{i,\rho(j)}$ is adjacent to exactly one vertex in R. Recall that $S_{i,\rho(j)}$ is adjacent only to vertices in $C_{i,\rho(j)}, C_{i,\rho(j)-1}$ and E_{ij}. Consider a vertex $x_\ell^{i,\rho(j)} \in S_{i,\rho(j)}$, for $\ell \in \{0, \dots, t-1\}$. Assume that ℓ^{th} bit of $\bar{b}_i$ is 1. This implies that $x_\ell^{i,\rho(j)}$ is adjacent to $e_{v_i v_j}$ and not adjacent to $c_{v_i}^{i,\rho(j)}$. Also, $x_\ell^{i,\rho(j)}$ is non-adjacent to any other vertex in R. Hence it follows that $|R \cap N(x_{v_i}^{i,\rho(j)})| = 1$. An analogous argument can be given for the case when ℓ^{th} bit of $\bar{b}_i$ is 0. Furthermore, we can give a symmetric argument for a vertex $y_\ell^{i,\rho(j)} \in S_{i,\rho(j)}$, where $\ell \in \{0, \ldots, t-1\}$. This finishes the proof of the forward direction.

In the reverse direction, let R be a solution to SRBPC for $(G', \mathcal{R}, \mathcal{B})$. Note that for $i, j \in [k]$, $i \neq j$, $|R \cap E_{ij}| = 1$ and $|R \cap C_{i,\rho(j)}| = 1$. Let $X = \{v \in V(G) \mid c_v^{i,\rho(j)} \in R\}$. It follows from Lemma 4.8 that for all $i \in [k]$, $|X \cap V_i| = 1$. Consider $u, v \in X$, where $u \in V_i$, $v \in V_j$ and $i \neq j$. From Lemma 4.8 for all $\ell \in [k]$, $i \neq \ell$ we have $c_u^{i,\rho(\ell)} \in R$ and for all $\ell' \in [k]$, $j \neq \ell'$ we have $c_u^{j,\rho(\ell')} \in R$. This together with Lemma 4.7 imply that $e_{uv} \in R$. Hence $(u, v) \in E(G)$. Since choice of u, v was arbitrary, it implies that $G[X]$ is a clique. □

We are now ready to prove the main theorem of this section.

Theorem 4.10. *SRBPC when parameterized by the number of parts in $\mathcal{R}$ is* W[1]-hard.

Proof. Follows from construction of the instance $(G', \mathcal{R}, \mathcal{B})$ of SRBPC for the given instance (G, k) of MCC, Lemma 4.9, and W[1]-hardness of MCC. □

5 FPT ALGORITHM FOR SPLIT CONTRACTION PARAMETERIZED BY VERTEX COVER

In this section we give an FPT algorithm for Split Contraction when parameterized by the size of a minimum vertex cover. In Section 5.1 we give an algorithm running in time $2^{O(\ell^2)} \cdot n^{O(1)}$ for Split Contraction parameterized by ℓ, the size of minimum vertex cover, when the input graph is connected. In this section we use the algorithm for solving Split Contraction parameterized by the size of a minimum vertex cover on connected graphs to solve Split Contraction on general graphs.

Let (G, k) be an instance of Split Contraction and $C_1, \ldots, C_t$ be the set of connected components of G. Observe that except for one connected component in G, every other component must be contracted to a single vertex, since all the vertices in these components must be part of the independent set. Also, note that for contracting a component to a single vertex we need to contract a spanning tree in it. Therefore, for each $i \in [t]$ let $k_i = k - \sum_{j \in [t] \setminus \{i\}} |V(C_j) - 1|$ and solve the instance (C_i, k_i). If for any $i \in [t]$ the algorithm returns a YES instance then we return that (G, k) is a YES instance, otherwise return that (G, k) is a NO instance. The correctness of the above algorithm relies on the correctness of the algorithm for connected graphs and thus results in the following theorem.

Theorem 5.1. *Split Contraction admits an algorithm running in time $2^{O(\ell^2)} \cdot n^{O(1)}$, where ℓ is the size of the minimum vertex cover of the input graph.*

5.1 Algorithm for Split Contraction on Connected Graphs

In this section we give an FPT algorithm for Split Contraction parameterized by the size of a minimum vertex cover when the input graph is a connected. Let (G, k) be an instance of Split Contraction, where G is a connected graph. We start by computing a minimum sized vertex cover S in G. We prove the following lemma which will be useful for the algorithm.

Lemma 5.2. *Let G be a connected graph, S be a minimum vertex cover in G and $K \subseteq E(G)$ be a set of minimum size such that G/K is a split graph, then $|K| < 2|S|$.*

Proof. Let T be a *dfs*-tree of G and L_T denote the set of leaves in T. It is well known that $V(T) \setminus L_T$ is a connected vertex cover of G and $|V(T) \setminus L_T| \leq 2|S|$ [42]. Let E_T be the edges in T

that are non-adjacent to vertices in L_T. Observe that G/E_T is a split graph. Thus, $|K| \leq |E_T| < |V(T) \setminus L_T| \leq 2|S|$. ◻

Let $I = V(G) \setminus S$. Since S is a vertex cover, I is an independent set in G. We define an equivalence relation $\mathcal{R}$ among the vertices in I based on their neighborhood in S. Basically, $u, v \in I$ belong to the same equivalence class if and only if $N(u) = N(v)$. Let $I_1, \ldots, I_t$ be the equivalence classes of $\mathcal{R}$. Note that $t \leq 2^{|S|}$. We apply the following reduction rules exhaustively.

Reduction Rule 1. *If $k \geq 2|S|$, then return that (G, k) is a YES instance.*

Lemma 5.3. *Reduction Rule 1 is safe.*

Proof. The proof follows from Lemma 5.2. ◻

Reduction Rule 2. *If there is an equivalence class I_j, for $j \in [t]$ such that $|I_j| > 2k + 2$, then delete an arbitrary vertex $v \in I_j$ from G. That is, the resulting instance is $(G - \{v\}, k)$.*

Lemma 5.4. *Reduction Rule 2 is safe.*

Proof. Let (G, k) be an instance of Split Contraction. Consider $j \in [t]$, such that $|I_j| > 2k + 2$. Let $v \in I_j$ be an arbitrarily chosen vertex from I_j, and let $G' = G - \{v\}$. We will show that (G, k) is a YES instance of Split Contraction if and only if (G', k) is a YES instance of Split Contraction.

In the forward direction let X be a solution to (G, k), $\mathcal{W}$ be the $H = G/X$-*witness structure* of G with φ being the underlying surjective function. If no edge in X is incident to v, then X is also a solution in (G', k) as G'/X is an induced subgraph of G/X. Let $X_v \subseteq X$ be those edges which are incident to v. There is a vertex $v' \in I_j$ that is not adjacent to any edge in X since $|I_j| > 2k + 2$. Let $X_{v'} = \{(u, v') \mid (u, v) \in X_v\}$, *i.e.*, $X_{v'}$ is the set of edges obtained by replacing v by v' in X_v. Note that such a replacement is possible because $N(v) = N(v')$. Let $X' = (X \setminus X_v) \cup X_{v'}$. Clearly, the size of $|X'| \leq |X| \leq k$. We define the surjective function $\varphi' : V(G') \to V(H) \setminus \{\varphi(v')\}$ as follows. For $u \in V(G'), u \neq v'$, $\varphi'(u) = \varphi(u)$ and $\varphi'(v') = \varphi(v)$ (recall, $\varphi(v) \neq \varphi(v')$). For $h \in V(H) \setminus \{\varphi(v')\}$ we let $W'(h) = \varphi^{-1}(h)$. Let H' to be the graph with $V(H') = V(H) \setminus \{\varphi(v')\}$ and $(h_1, h_2) \in E(H')$ if and only if $W'(h_1)$ and $W'(h_2)$ are adjacent in G'. Since, $|W(\varphi(v'))| = 1$ we have that for any vertex $h \in V(H') \setminus \{\varphi'(v')\}$, $W'(h) = W(h)$ and $W'(\varphi'(v')) = (W(\varphi(v)) \setminus \{v\}) \cup \{v'\}$. Observe that since $N_G(v) = N_G(v')$, we have that for all $h \in V(H')$, $G'[W(h)]$ is connected, and hence it follows that G' is contractible to H'. Furthermore, to show that G'/X' is a split graph, it is enough to show that H' is a split graph. Since $N_G(v) = N_G(v')$, the graphs H, H' differs only in the vertex $\varphi(v') \in V(H)$ $(\varphi(v') \notin V(H'))$. But any induced subgraph of a split graph, is a split graph, hence it follows that H' is a split graph.

In the reverse direction, let X be a solution to Split Contraction in (G', k), $H = G'/X$ and $\varphi, \mathcal{W}$ be the underlying surjective function and H-*witness structure* of G', respectively. Observe that X can be incident to at most $2k$ vertices in I_j, therefore there are vertices $u, u' \in V(G') \cap I_j$, $u \neq u'$ which are not incident to any edge in X *i.e.* $|W(\varphi(u))| = |W(\varphi(u')| = 1$. Let C' and $\mathcal{I}'$ be the clique and independent set respectively in H. Note that at least one of $\varphi(u), \varphi(u')$ belongs to $\mathcal{I}'$, say $\varphi(u) \in \mathcal{I}'$. We define the surjective function $\varphi_v : V(G) \to V(H) \cup \{v\}$ as follows. For $x \in V(G) \setminus \{v\}$, $\varphi_v(x) = \varphi(x)$ and $\varphi(v) = v$. Let H_v be the graph with vertex set $V(H) \cup \{v\}$ and $(h, h') \in E(H_v)$ if and only if $W_v(h)$ and $W_v(h')$ are adjacent in G. Notice that φ_v satisfies all the properties for it to define the contractibility of G to H_v. Recall that $N(v) = N(u)$. But then $\mathcal{I}' \cup \{v\}$ is an independent set and C' is a clique, partitioning the vertices of H_v, therefore H_v is a split graph. But notice that indeed $H_v = G/X$, hence the claim follows. ◻

Given an instance (G, k) to Split Contraction, we apply Reduction Rule 1 and 2 until no longer applicable. For simplicity we denote the resulting instance where none of the reduction rules are applicable by (G, k) itself. Observe that the number of vertices in G is upper bounded by $(2k+2) \cdot 2^{\ell} + \ell \leq (4\ell + 2) \cdot 2^{\ell} + \ell = 2^{O(\ell)}$, where $\ell = |S|$. This follows from the fact that none of the reduction rules are applicable and Lemma 5.2.

Observe that the number of vertices in G that are incident to an edge of the solution is bounded by $2k$. We guess $X \subseteq V(G)$ of size at most $2k$, which is incident to at least one edge in the solution. Note that the number of such guesses is upper bounded by $\binom{2^{O(\ell)}}{2\ell} = 2^{O(\ell^2)}$. The number of edges in $G[X]$ is bounded by $O(\ell^2)$. For each $E' \subseteq E(G[X])$ of size at most k, we check if G/E' is a split graph. If for all $X \subseteq V(G)$ and $E' \subseteq E(G[X])$, G/E' is not a split graph then we return that (G, k) is a NO instance, otherwise we return that (G, k) is a YES instance of Split Contraction.

Correctness and running time analysis. Given an instance (G, k), where G is a connected graph on n vertices, the algorithm starts by computing a minimum sized vertex cover S in G and an equivalence relation based on the neighborhood in G. The time required for this step of the algorithm is bounded by $O(1.2738^{\ell} \cdot n^{O(1)})$, where $\ell = |S|$ [11]. The algorithm then applies one of the reduction rules, if applicable. The reduction rules can be applied in polynomial time and their safeness follows from Lemma 5.3 and 5.4. When none of the reduction rules are applicable then the algorithm solves the instance in a brute force way and here its correctness is immediate. In the brute force step the algorithm guess a subset $X \subseteq V(G)$ of size at most $2k$ which are incident to an edge in the solution. The number of such subsets is bounded by $2^{O(\ell \cdot k)}$, which in turn is bounded by $2^{O(\ell^2)}$. For the guessed subset X, the algorithm tries for all possible sets of edges E' of size at most k in $E(G[X])$. The number of such edge subsets is upper bounded by $2^{O(k \log k)}$ which is bounded by $2^{O(\ell^2)}$. Checking if G/E' is a split graph takes linear time [28]. Hence, the total running time is bounded by $1.2738^{\ell} \cdot n^{O(1)} + 2^{O(\ell^2)} \cdot 2^{O(\ell^2)} \cdot n^{O(1)} = 2^{O(\ell^2)} \cdot n^{O(1)}$.

Theorem 5.5. *Split Contraction on connected graphs admits an algorithm running in time $2^{O(\ell^2)} \cdot n^{O(1)}$, where ℓ is the size of a minimum vertex cover of the input graph.*

6 CONCLUSION

In this paper, we have established two important results regarding the complexity of Split Contraction. First, we have shown that under the ETH, this problem cannot be solved in time $2^{o(\ell^2)} \cdot n^{O(1)}$ where ℓ is the vertex cover number of the input graph, and this lower bound is tight. To the best of our knowledge, this is the first tight lower bound of the form $2^{o(\ell^2)} \cdot n^{O(1)}$ for problems parameterized by the vertex cover number of the input graph. Second, we have proved that Split Contraction, despite its deceptive simplicity, is actually W[1]-hard with respect to the solution size. We believe that techniques integrated in our constructions can be used to derive conditional lower bounds and W[1]-hardness results in the context of other edge contraction problems.

We would like to conclude our paper with the following intriguing question. In the exact setting, it is easy to see that Split Contraction can be solved in time $2^{O(n \log n)}$. Can it be solved in time $2^{o(n \log n)}$? A negative answer would imply, for instance, that it is neither possible to find a topological clique minor in a given graph in time $2^{o(n \log n)}$, which is an interesting open problem [12]. It might be possible that tools developed in our paper, such as the usage of harmonious coloring, can be utilized to shed light on such problems.

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REFERENCES

[1] Takao Asano and Tomio Hirata. 1983. Edge-contraction problems. *J. Comput. System Sci.* 26, 2 (1983), 197–208.

[2] Ivan Bliznets, Marek Cygan, Pawel Komosa, Lukás Mach, and Michal Pilipczuk. 2016. Lower bounds for the parameterized complexity of Minimum Fill-In and other completion problems. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA)*. ACM-SIAM, Arlington, VA, USA, 1132–1151.

[3] Ivan Bliznets, Fedor V. Fomin, Marcin Pilipczuk, and Michal Pilipczuk. 2015. A Subexponential Parameterized Algorithm for Proper Interval Completion. *SIAM Journal on Discrete Mathematics* 29, 4 (2015), 1961–1987.

[4] Ivan Bliznets, Fedor V. Fomin, Marcin Pilipczuk, and Michal Pilipczuk. 2018. Subexponential Parameterized Algorithm for Interval Completion. *ACM Transactions on Algorithms* 14, 3 (2018), 35:1–35:62.

[5] Leizhen Cai. 1996. Fixed-parameter tractability of graph modification problems for hereditary properties. *Inform. Process. Lett.* 58, 4 (1996), 171–176.

[6] Leizhen Cai and Chengwei Guo. 2013. Contracting Few Edges to Remove Forbidden Induced Subgraphs. In *Parameterized and Exact Computation - 8th International Symposium (IPEC)*. Springer, Cham, Sophia Antipolis, France, 97–109.

[7] Yixin Cao. 2016. Linear Recognition of Almost Interval Graphs. In *Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. ACM-SIAM, Arlington, Virginia, 1096–1115.

[8] Yixin Cao. 2017. Unit interval editing is fixed-parameter tractable. *Information and Computation* 253 (2017), 109–126.

[9] Yixin Cao and Dániel Marx. 2015. Interval Deletion Is Fixed-Parameter Tractable. *ACM Transactions on Algorithms* 11, 3 (2015), 21:1–21:35.

[10] Yixin Cao and Dániel Marx. 2016. Chordal Editing is Fixed-Parameter Tractable. *Algorithmica* 75, 1 (2016), 118–137.

[11] Jianer Chen, Iyad A Kanj, and Ge Xia. 2010. Improved upper bounds for vertex cover. *Theoretical Computer Science* 411, 40 (2010), 3736–3756.

[12] Marek Cygan, Fedor V. Fomin, Alexander Golovnev, Alexander S. Kulikov, Ivan Mihajlin, Jakub Pachocki, and Arkadiusz Socala. 2017. Tight Lower Bounds on Graph Embedding Problems. *J. ACM* 64, 3 (2017), 18:1–18:22.

[13] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. 2015. *Parameterized Algorithms*. Springer International Publishing, Switzerland.

[14] Marek Cygan, Marcin Pilipczuk, and Michal Pilipczuk. 2016. Known Algorithms for Edge Clique Cover are Probably Optimal. *SIAM J. Comput.* 45, 1 (2016), 67–83.

[15] Reinhard Diestel. 2012. *Graph Theory, 4th Edition*. Graduate texts in mathematics, Vol. 173. Springer-Verlag Berlin Heidelberg, Germany.

[16] Rodney G. Downey and Michael R. Fellows. 1995. Fixed-Parameter Tractability and Completeness II: On Completeness for W[1]. *Theoretical Computer Science* 141, 1&2 (1995), 109–131.

[17] Rod G. Downey and Michael R. Fellows. 2013. *Fundamentals of Parameterized complexity*. Springer-Verlag, London.

[18] Pål Grønås Drange, Markus Sortland Dregi, Daniel Lokshtanov, and Blair D. Sullivan. 2015. On the Threshold of Intractability. In *Algorithms - 23rd Annual European Symposium (ESA)*. Springer-Verlag Berlin Heidelberg, Germany, 411–423.

[19] Pål Grønås Drange, Fedor V. Fomin, Michal Pilipczuk, and Yngve Villanger. 2015. Exploring the Subexponential Complexity of Completion Problems. *ACM Transactions on Computation Theory* 7, 4 (2015), 14:1–14:38.

[20] Pål Grønås Drange and Michal Pilipczuk. 2018. A Polynomial Kernel for Trivially Perfect Editing. *Algorithmica* 80 (2018), 3481–3524.

[21] Keith Edwards. 1997. The Harmonious Chromatic Number and the Achromatic Number. In *Surveys in Combinatorics*. Cambridge University Press, Cambridge, 13–48.

[22] Michael R. Fellows, Danny Hermelin, Frances A. Rosamond, and Stéphane Vialette. 2009. On the parameterized complexity of multiple-interval graph problems. *Theoretical computer science* 410, 1 (2009), 53–61.

[23] Jörg Flum and Martin Grohe. 2006. *Parameterized Complexity Theory*. Springer-Verlag Berlin Heidelberg, Germany.

[24] Fedor V Fomin, Stefan Kratsch, Marcin Pilipczuk, Michał Pilipczuk, and Yngve Villanger. 2014. Tight bounds for parameterized complexity of cluster editing with a small number of clusters. *J. Comput. System Sci.* 80, 7 (2014), 1430–1447.

[25] Fedor V Fomin and Yngve Villanger. 2013. Subexponential parameterized algorithm for minimum fill-in. *SIAM J. Comput.* 42, 6 (2013), 2197–2216.

[26] Esha Ghosh, Sudeshna Kolay, Mrinal Kumar, Pranabendu Misra, Fahad Panolan, Ashutosh Rai, and M. S. Ramanujan. 2015. Faster parameterized algorithms for deletion to split graphs. *Algorithmica* 71, 4 (2015), 989–1006.

[27] Petr A. Golovach, Pim van 't Hof, and Daniel Paulusma. 2013. Obtaining planarity by contracting few edges. *Theoretical Computer Science* 476 (2013), 38–46.

[28] Martin Charles Golumbic. 2004. *Algorithmic graph theory and perfect graphs*. Vol. 57. Elsevier, Academic Press.

[29] Sylvain Guillemot and Dániel Marx. 2013. A faster FPT algorithm for Bipartite Contraction. *Inform. Process. Lett.* 113, 22–24 (2013), 906–912.

[30] Chengwei Guo and Leizhen Cai. 2015. Obtaining split graphs by edge contraction. *Theoretical Computer Science* 607 (2015), 60–67.

[31] Pinar Heggernes, Pim van 't Hof, Benjamin Lévêque, Daniel Lokshtanov, and Christophe Paul. 2014. Contracting Graphs to Paths and Trees. *Algorithmica* 68, 1 (2014), 109–132.

[32] Pinar Heggernes, Pim van 't Hof, Daniel Lokshtanov, and Christophe Paul. 2013. Obtaining a Bipartite Graph by Contracting Few Edges. *SIAM Journal on Discrete Mathematics* 27, 4 (2013), 2143–2156.

[33] Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. 2001. Which Problems Have Strongly Exponential Complexity? *J. Comput. System Sci.* 63, 4 (2001), 512–530.

[34] Haim Kaplan, Ron Shamir, and Robert Endre Tarjan. 1999. Tractability of Parameterized Completion Problems on Chordal, Strongly Chordal, and Proper Interval Graphs. *SIAM J. Comput.* 28, 5 (1999), 1906–1922.

[35] Christian Komusiewicz. 2018. Tight Running Time Lower Bounds for Vertex Deletion Problems. *ACM Transactions on Computation Theory* 10, 2, Article 6 (2018), 18 pages.

[36] Sin-Min Lee and John Mitchem. 1987. An Upper Bound for the Harmonious Chromatic Number. *Journal of Graph Theory* 11, 4 (1987), 565–567.

[37] Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. 2018. Slightly Superexponential Parameterized Problems. *SIAM J. Comput.* 47, 3 (2018), 675–702.

[38] Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. 2013. On the hardness of eliminating small induced subgraphs by contracting edges. In *Parameterized and Exact Computation - 8th International Symposium (IPEC)*. Springer, Cham, Sophia Antipolis, France, 243–254.

[39] Dániel Marx and Valia Mitsou. 2016. Double-Exponential and Triple-Exponential Bounds for Choosability Problems Parameterized by Treewidth. In *43rd International Colloquium on Automata, Languages, and Programming, (ICALP)*. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 28:1–28:15.

[40] Colin McDiarmid and Luo Xinhua. 1991. Upper Bounds for Harmonious Coloring. *Journal of Graph Theory* 15, 6 (1991), 629–636.

[41] Rolf Niedermeier. 2006. *Invitation to fixed-parameter algorithms.* Oxford University Press, Oxford.

[42] Carla D. Savage. 1982. Depth-First Search and the Vertex Cover Problem. *Inform. Process. Lett.* 14, 5 (1982), 233–237.

[43] Toshimasa Watanabe, Tadashi Ae, and Akira Nakamura. 1981. On the removal of forbidden graphs by edge-deletion or by edge-contraction. *Discrete Applied Mathematics* 3, 2 (1981), 151–153.

[44] Toshimasa Watanabe, Tadashi Ae, and Akira Nakamura. 1983. On the NP-hardness of edge-deletion and-contraction problems. *Discrete Applied Mathematics* 6, 1 (1983), 63–78.

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