CS 111 Sample Final -- December 2010

You may use your textbook and notes, but no computers or other electronic devices.

Problem 1. Let **`x`**, **`p`**, and **`A`** be defined by the following Matlab statements:

```
x = [27 18 28 18];
p = [3 4 2 1];
A = [1 2 3 ; 4 5 6; 7 8 9; 10 11 12];
```

What is the value of ...

(a) **`B`** after: **`B = A(p,:)`**

(b) **`y`** after: **`y = x(p)`**

(c) **`y`** after: **`y(p) = x`**

(d) **`z`** after: **`z = A * ones(3,1)`**

(e) **`A`** after: **`A(3:4, 2:3) = [x(1:2) ; x(3:4)]`**

2. (a) Let `A = [-1 0 3 ; 1 6 8 ; 2 4 4]`
Show the result of LU factorization of A with partial pivoting, so `A(p,:) = L * U`.

`L =`

`U =`

`p =`

(b) Let `A = [2 2 6 ; 3 3 3 ; 2 4 0]`
Show the result of LU factorization of A with complete pivoting, so `A(p,q) = L * U` .

```
L =

U =

p =

q =
```

3. Norman, Evan, and Hans work in the AA grocery shop. They are all paid the same amount per hour. Sometimes they take home groceries instead of pay.

Last week, Norman worked 18 hours and took home 2 loaves of bread and one box of fruit. Evan worked 12 hours and took home 3 loaves of bread and 4 boxes of fruit. Hans worked 20 hours and took home one box of fruit. At the end of the week, Norman was paid \$344, Evan \$196, and Hans \$392.

We want to find out three things: the pay per hour, the cost of a loaf of bread, and the cost of a box of fruit. Write down the equations you would solve, and the Matlab code to solve them. (This should be at most a few lines of Matlab.) You don't have to actually solve the equations, but explain clearly which outputs correspond to which unknowns.

4. How many times does each of the following loops iterate? (Full credit if you're within 20% of the right answer.)

(a) `x = 1; while x-1 < x; x = 100*x; end;`

(b) `x = 1; while x+x > x; x = 100*x; end;`

(c) `x = 1; while x+x > 0; x = 100*x; end;`

Problem 5. (This exercise requires you to run Matlab, but the problems on the real final exam will not require you to actually run Matlab.)

The singular value decomposition of matrix **`A`** is **`A = X*D*Y';`**

Here is an example:

```
>> A =
     [ 1     0     5    -2
       5    -2     1     0
       0    -1     2    -5
       2    -5     0    -1 ];

>> [X, D, Y] = svd(A)

X =
   -0.5       0.5       0.5       0.5
   -0.5      -0.5       0.5      -0.5
   -0.5       0.5      -0.5      -0.5
   -0.5      -0.5      -0.5       0.5

D =
     8         0         0         0
     0         6         0         0
     0         0         4         0
     0         0         0         2

Y =
   -0.5      -0.5       0.5      -0.5
    0.5       0.5       0.5      -0.5
   -0.5       0.5       0.5       0.5
    0.5      -0.5       0.5       0.5
```

5a. Type matrices **`X`**, **`D`**, and **`Y`** into Matlab and multiply them to verify that the product is really **`A`**.

5b. Fill in the blanks below to give four properties that the matrices **`X`**, **`D`**, and **`Y`** in the SVD must satisfy. For each one, use Matlab to verify that the relevant matrix does indeed satisfy the property.

Each column of **`X`** or **`Y`** is a vector of length _____ .

Any two different columns of **`X`** are _______________, as are any two columns of **`Y`**.

Matrix D is ____________________, and all its diagonal elements are ___________.

The product **`X'*X`** is equal to _______________.

Problem 6. (This exercise requires you to run Matlab, but the problems on the real final exam will not require you to actually run Matlab.)

Consider the singular value decomposition `A = X*D*Y'`, and suppose the diagonal elements of `D` are $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$, **...**, $\boldsymbol{\sigma_n}$. Recall that we can also write the SVD as

$$\mathbf{A} = \boldsymbol{\sigma_1 x_1 y_1}^{\mathbf{T}} + \boldsymbol{\sigma_2 x_2 y_2}^{\mathbf{T}} + \ldots + \boldsymbol{\sigma_n x_n y_n}^{\mathbf{T}}.$$

Define $\mathbf{A}^{(\mathbf{k})} = \boldsymbol{\sigma_1 x_1 y_1}^{\mathbf{T}} + \boldsymbol{\sigma_2 x_2 y_2}^{\mathbf{T}} + \ldots + \boldsymbol{\sigma_k x_k y_k}^{\mathbf{T}}$ for each $\mathbf{k}$.

For $\mathbf{k}$ from 1 to 4, compute $\mathbf{A}^{(\mathbf{k})}$. Then use Matlab to compute the norm $\|\mathbf{A} - \mathbf{A}^{(\mathbf{k})}\|$. (Use the Matlab "norm" function.) Compare your results to the singular values in **D**. What do you observe? What is the theorem (from class) that describes this pattern?

7. You are given the following data:

time	*Temp*
0.1	1
2.2	14
4.3	20
6.3	32
8.3	45
10.5	45
12.1	60

This is the temperature T at time t inside a pipe that is being heated from outside, and also being cooled and heated periodically by liquid flowing through it.

We want to fit a curve of the form:

$$T = \alpha t + \beta sin(t)$$

Write down the matrices involved, and write down Matlab code to solve this problem and find α and β.

8. The standard form of an ODE initial value problem is:

$$\dot{y} = f(t, y), y(t_0) = y_0$$

Express this ODE problem in the standard form.

$$\ddot{y} = t^2 - \dot{y} - z^2 + 1$$

$$\ddot{z} = t + \dot{z} + y^3 - 2$$

where all function values and derivatives are zero at $t = 0$.

9. This is a simpler version of exercise 7.15 in the online version of the book (exercise 7.18 in the SIAM paperback).

Consider a 2-dimensional coordinate system that has a horizontal x-*axis* and a vertical y-*axis*. A cannon standing on a 100-meter cliff, at position $x = 0$ and $y = 100$, fires a cannonball horizontally (in the $+x$ direction, parallel to the x axis) at an initial velociy of $v_0 = 100$ meters per second.

Let θ be the angle between the ball's direction and the x axis (initially zero), let (x, y) be its coordinates, and let v be its velocity. If we pick the units to make all the various constants cancel out, the equations describing the motion of the ball are

$$\dot{x} = v \cos \theta$$

$$\dot{y} = v \sin \theta$$

$$\dot{\theta} = -\frac{\cos \theta}{v}$$

$$\dot{v} = -D - \sin \theta$$

where $D(t) = (\dot{x} + 10)^2 + \dot{y}^2$ is the aerodynamic drag on the ball in a 10 meter per second headwind.

Write Matlab code to plot the path of the ball from time $t = 0$ to time $t = 10$. Your code should include a (short) function to compute the derivatives, and a (short) main program that calls ode45 to solve the equations and then calls "plot" appropriately on the output of ode45.

We won't take off points for minor Matlab errors, but your code should be substantially correct for full credit.

10. Using pencil and paper, take one step of the forward Euler algorithm for the ODE problem

$$\dot{y} = 2y + 1$$

$$y(0) = 1$$

Express your results in terms of an arbitrary step size h.