Integrating Differential Equations

$$\frac{dy(t)}{dt} = f(t, y(t))$$

$$y(t_0) = y_0$$

$$y_n \approx y(t_n),\ \ n = 0, 1, \ldots$$

$$h_n = t_{n+1} - t_n$$

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))ds$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s)ds$$

$$\dot{y} = \frac{dy(t)}{dt}$$

$$\ddot{y} = \frac{d^2y(t)}{dt^2}$$

Systems of Equations

$$\ddot{x}(t) = -x(t)$$

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

$$\begin{aligned} \dot{y}(t) &= \begin{bmatrix} \dot{x}(t) \\ -x(t) \end{bmatrix} \\ &= \begin{bmatrix} y_2(t) \\ -y_1(t) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \ddot{u}(t) &= -u(t)/r(t)^3 \\ \ddot{v}(t) &= -v(t)/r(t)^3 \end{aligned}$$

$$r(t) = \sqrt{u(t)^2 + v(t)^2}$$

$$y(t) = \begin{bmatrix} u(t) \\ v(t) \\ \dot{u}(t) \\ \dot{v}(t) \end{bmatrix}$$

$$\dot{y}(t) = \begin{bmatrix} \dot{u}(t) \\ \dot{v}(t) \\ -u(t)/r(t)^3 \\ -v(t)/r(t)^3 \end{bmatrix}$$

Linearized Differential Equations

$$f(t, y) = f(t_c, y_c) + \alpha(t - t_c) + J(y - y_c) + \ldots$$

$$\alpha = \frac{\partial f}{\partial t}(t_c, y_c)$$

$$J = \frac{\partial f}{\partial y}(t_c, y_c)$$

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} f_1(t, y_1, \ldots, y_n) \\ f_2(t, y_1, \ldots, y_n) \\ \vdots \\ f_n(t, y_1, \ldots, y_n) \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}$$

$$\dot{y} = Jy$$

$$\lambda_k = \mu_k + i\nu_k = \mathrm{eig}(J)$$

$$\Lambda = \mathrm{diag}(\lambda_k)$$

$$J = V\Lambda V^{-1}$$

$$Vx = y$$

$$\dot{x}_k = \lambda_k x_k$$

$$x_k(t) = e^{\lambda_k (t - t_c)} x(t_c)$$

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of J are $\pm i$ and the solutions are purely oscillatory linear combinations of e^{it} and e^{-it}.

$$\dot{y}(t) = \begin{bmatrix} y_3(t) \\ y_4(t) \\ -y_1(t)/r(t)^3 \\ -y_2(t)/r(t)^3 \end{bmatrix}$$

$$r(t) = \sqrt{y_1(t)^2 + y_2(t)^2}$$

$$J = \frac{1}{r^5} \begin{bmatrix} 0 & 0 & r^5 & 0 \\ 0 & 0 & 0 & r^5 \\ 2y_1^2 - y_2^2 & 3y_1y_2 & 0 & 0 \\ 3y_1y_2 & 2y_2^2 - y_1^2 & 0 & 0 \end{bmatrix}$$

$$\lambda = \frac{1}{r^{3/2}} \begin{bmatrix} \sqrt{2} \\ i \\ -\sqrt{2} \\ -i \end{bmatrix}$$

Single Step Methods

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n, y_n) \\ t_{n+1} &= t_n + h \end{aligned}$$

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_1) \\
y_{n+1} &= y_n + h s_2 \\
t_{n+1} &= t_n + h
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + h, y_n + h s_1) \\
y_{n+1} &= y_n + h \frac{s_1 + s_2}{2} \\
t_{n+1} &= t_n + h
\end{aligned}
$$

Classical Runge-Kutta

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_1) \\
s_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_2) \\
s_4 &= f(t_n + h, y_n + h s_3) \\
y_{n+1} &= y_n + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \\
t_{n+1} &= t_n + h
\end{aligned}
$$

$$s_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{i,j} s_j)$$

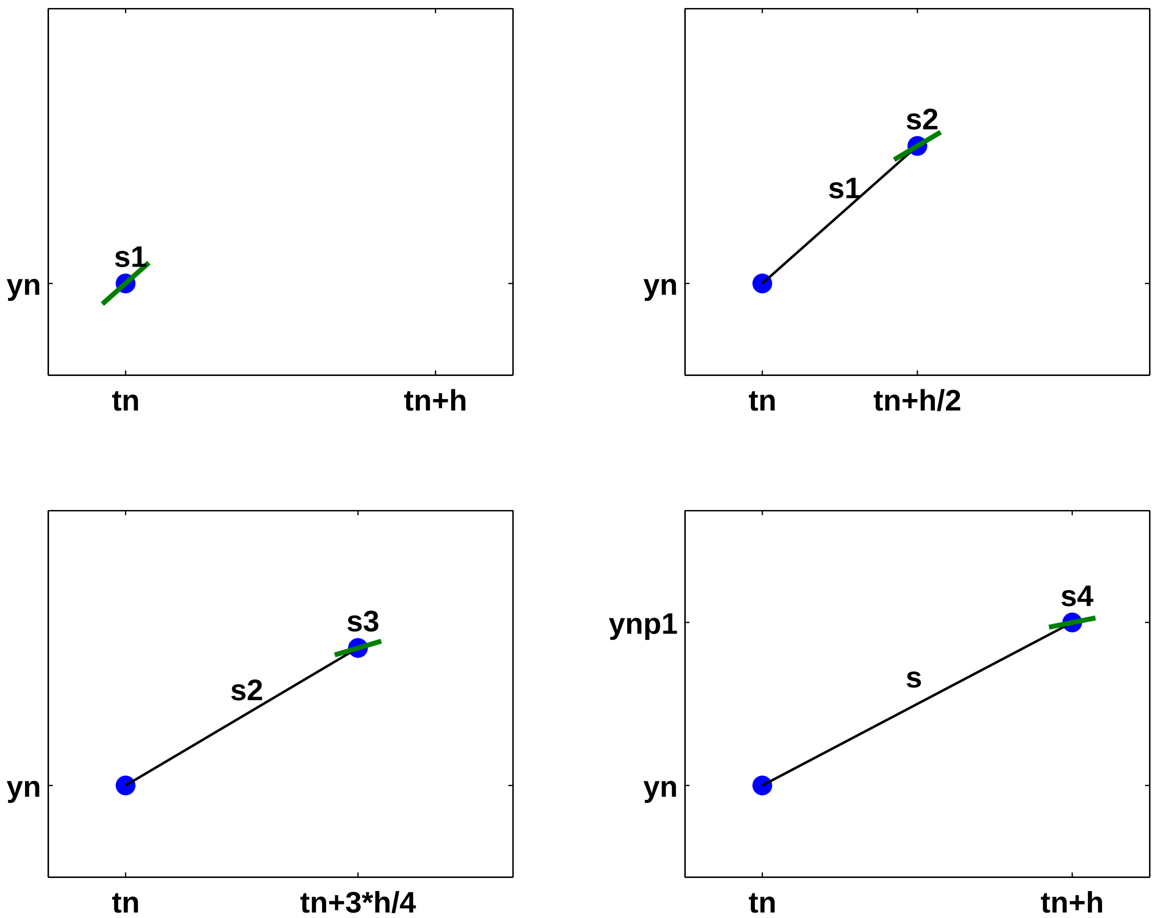
$$i = 1, \ldots, k$$

$$y_{n+1} = y_n + h \sum_{i=1}^{k} \gamma_i s_i$$

$$e_{n+1} = h \sum_{i=1}^{k} \delta_i s_i$$

The BS23 algorithm

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}s_1) \\
s_3 &= f(t_n + \frac{3}{4}h, y_n + \frac{3}{4}hs_2) \\
t_{n+1} &= t_n + h \\
y_{n+1} &= y_n + \frac{h}{9}(2s_1 + 3s_2 + 4s_3) \\
s_4 &= f(t_{n+1}, y_{n+1}) \\
e_{n+1} &= \frac{h}{72}(-5s_1 + 6s_2 + 8s_3 - 9s_4)
\end{aligned}
$$


s1
yn
tn
tn+h
s2
s1
yn
tn
tn+h/2
s3
s2
yn
tn
tn+3*h/4
s4
ynp1
s
yn
tn
tn+h

Lorenz Attractor

$$\dot{y} = Ay$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$A = \begin{bmatrix} -\beta & 0 & y_2 \\ 0 & -\sigma & \sigma \\ -y_2 & \rho & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -\beta & 0 & \eta \\ 0 & -\sigma & \sigma \\ -\eta & \rho & -1 \end{bmatrix}$$

$$\eta = \pm\sqrt{\beta(\rho - 1)}$$

$$y(t_0) = \begin{pmatrix} \rho - 1 \\ \eta \\ \eta \end{pmatrix}$$

$$\dot{y}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Stiffness

A problem is stiff if the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results.

$$\begin{aligned} \dot{y} &= y^2 - y^3 \\ y(0) &= \eta \\ 0 \le t &\le 2/\eta \end{aligned}$$

Events

$$\dot{y} = f(t, y)$$

$$y(t_0) = y_0$$

$$g(t_*, y(t_*)) = 0$$

$$\ddot{y} = -1 + \dot{y}^2$$

$$y(0) = 1,\ \dot{y}(0) = 0.$$

$$g(t, y) = \dot{d}(t)^T d(t)$$

$$d = (y_1(t) - y_1(0), y_2(t) - y_2(0))^T$$

Local discretization error

$$\dot{u}_n = f(t, u_n)$$
$$u_n(t_n) = y_n$$

$$d_n = y_{n+1} - u_n(t_{n+1})$$

Global discretization error

$$e_n = y_n - y(t_n)$$

$$\int_{t_0}^{t_N} f(\tau)d\tau \approx \sum_{0}^{N-1} h_n f(t_n)$$

$$d_n = h_n f(t_n) - \int_{t_n}^{t_{n+1}} f(\tau)d\tau$$

$$e_N = \sum_{n=0}^{N-1} h_n f(t_n) - \int_{t_0}^{t_N} f(\tau)d\tau$$

$$e_N = \sum_{n=0}^{N-1} d_n$$

order

$$|d_n| \le Ch_n^{p+1}$$

$$d_n = O(h_n^{p+1})$$

$$y_{n+1} = y_n + h_n f(t_n, y_n)$$

$$u_n(t) = u_n(t_n) + (t - t_n)u_n'(t_n) + O((t - t_n)^2)$$

$$u_n(t_{n+1}) = y_n + h_n f(t_n, y_n) + O(h_n^2)$$

$$d_n = y_{n+1} - u_n(t_{n+1}) = O(h_n^2)$$

$$N = \frac{t_f - t_0}{h}$$

$$N\epsilon = \frac{L\epsilon}{h}$$

$$Ch^p + \frac{L\epsilon}{h}$$

$$h \approx \left(\frac{L\epsilon}{C}\right)^{\frac{1}{p+1}}$$

$$N \approx L\left(\frac{C}{L\epsilon}\right)^{\frac{1}{p+1}}$$