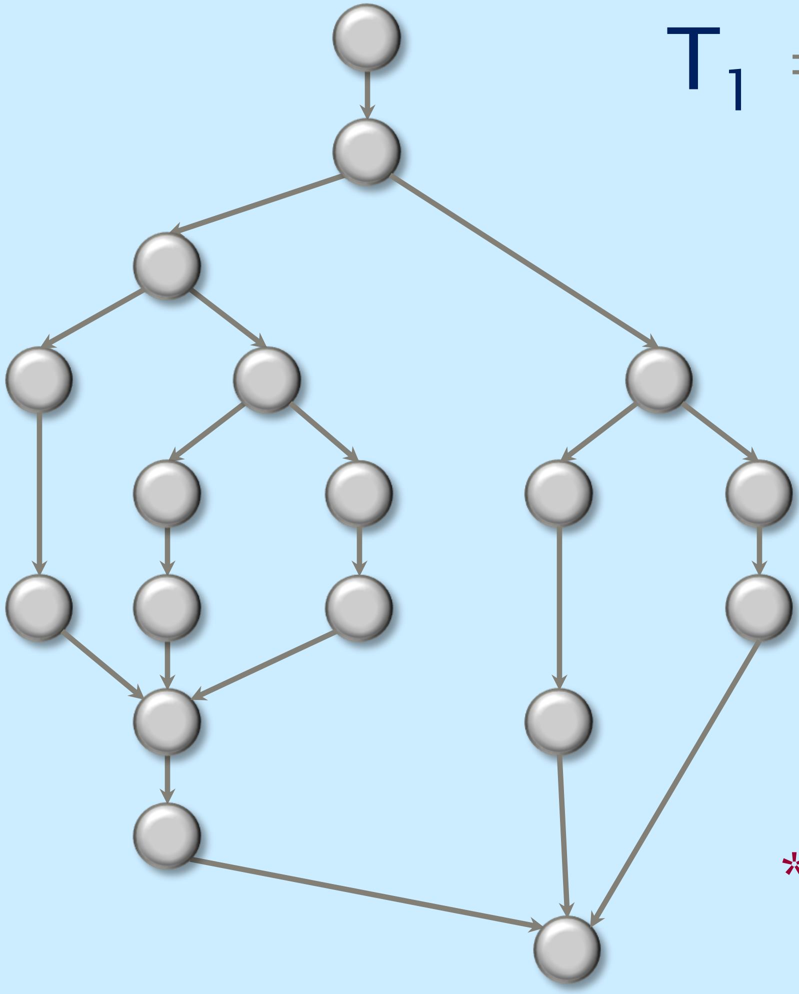
CS 140 : Non-numerical Examples with Cilk++

- Divide and conquer paradigm for Cilk++
- Quicksort
- Mergesort

Thanks to Charles E. Leiserson for some of these slides

Work and Span (Recap)

T_P = execution time on P processors



T_1 = *work* T_∞ = *span**

Speedup on p processors

- T_1/T_p

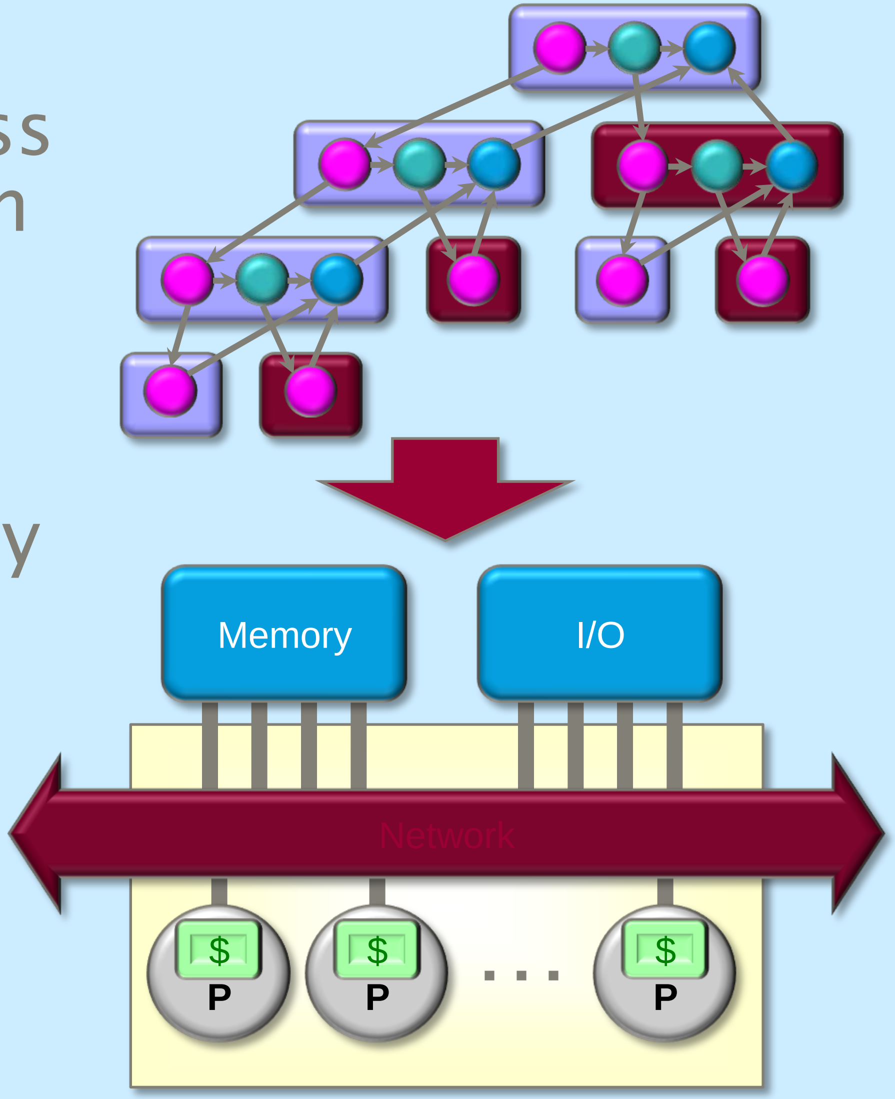
Parallelism

- T_1/T_∞

*Also called *critical-path length* or *computational depth*.

Scheduling

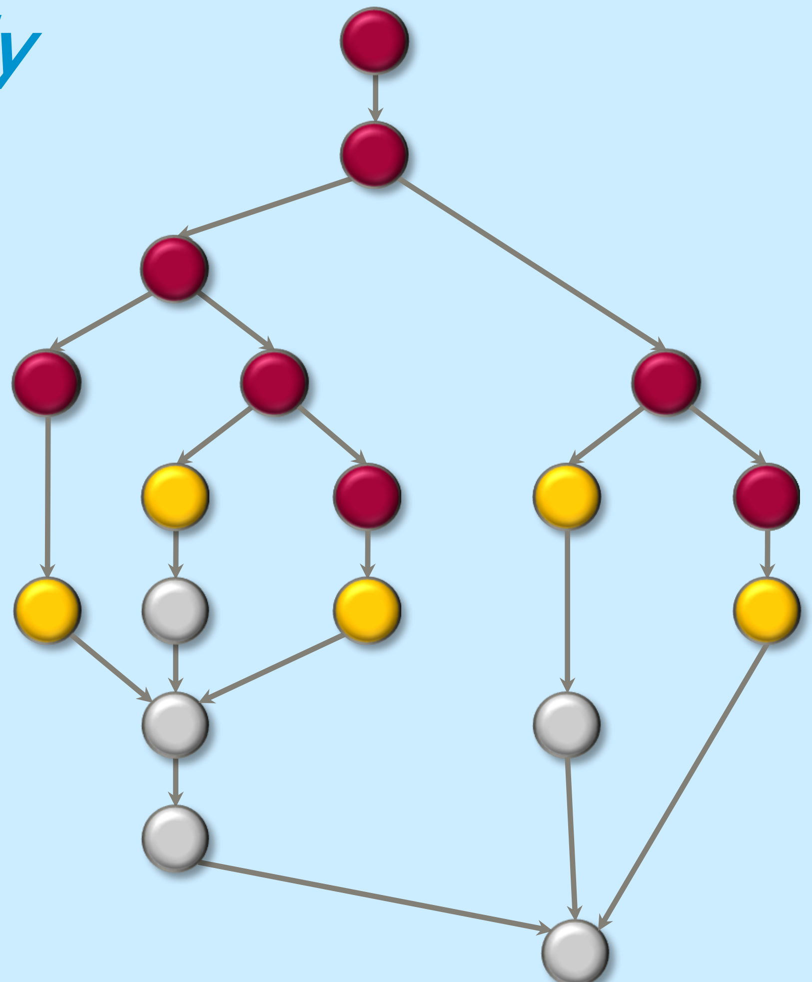
- Cilk++ allows the programmer to express *potential* parallelism in an application.
- The Cilk++ ***scheduler*** maps strands onto processors dynamically at runtime.
- Since ***on-line*** schedulers are complicated, we'll explore the ideas with an ***off-line*** scheduler.



Greedy Scheduling

IDEA: Do as much as possible on every step.

Definition: A strand is ***ready*** if all its predecessors have executed.



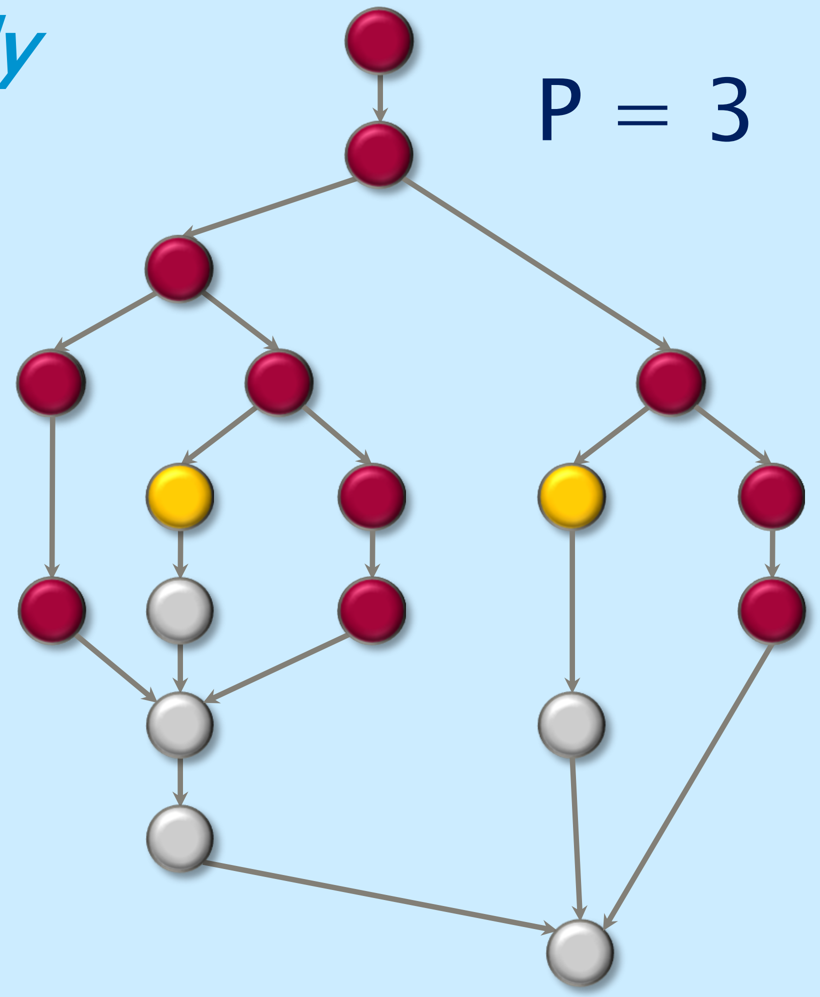
Greedy Scheduling

IDEA: Do as much as possible on every step.

Definition: A strand is ***ready*** if all its predecessors have executed.

Complete step

- $\geq$ P strands ready.
- Run any P.



Greedy Scheduling

IDEA: Do as much as possible on every step.

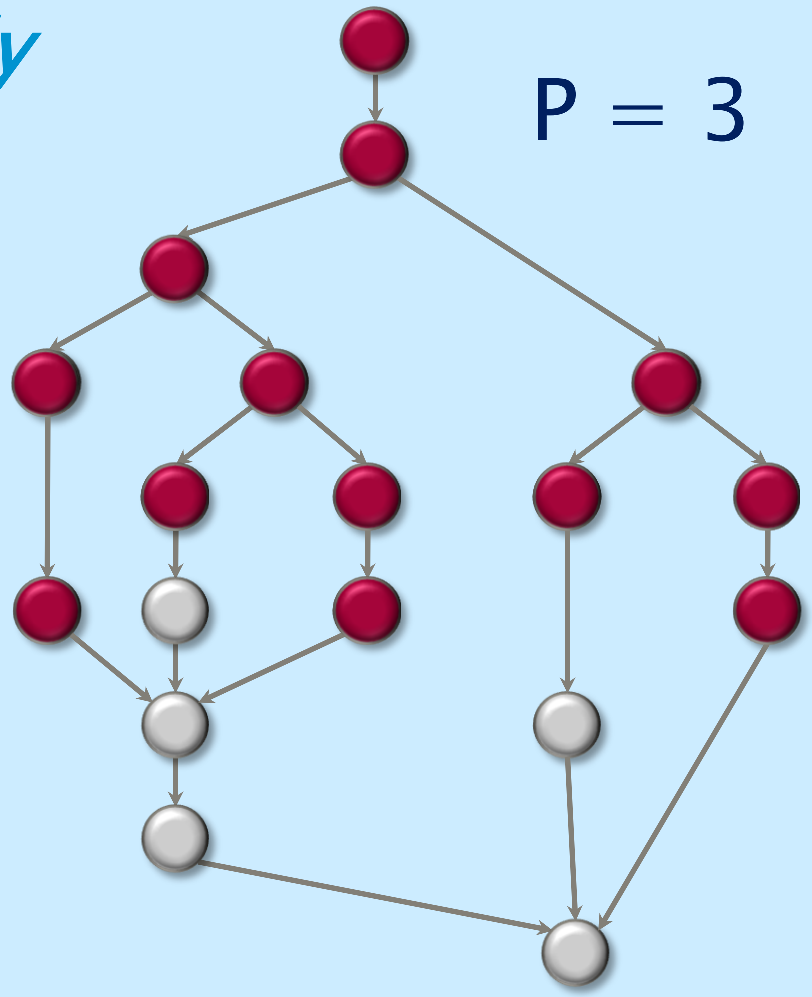
Definition: A strand is ***ready*** if all its predecessors have executed.

Complete step

- $\geq$ P strands ready.
- Run any P.

Incomplete step

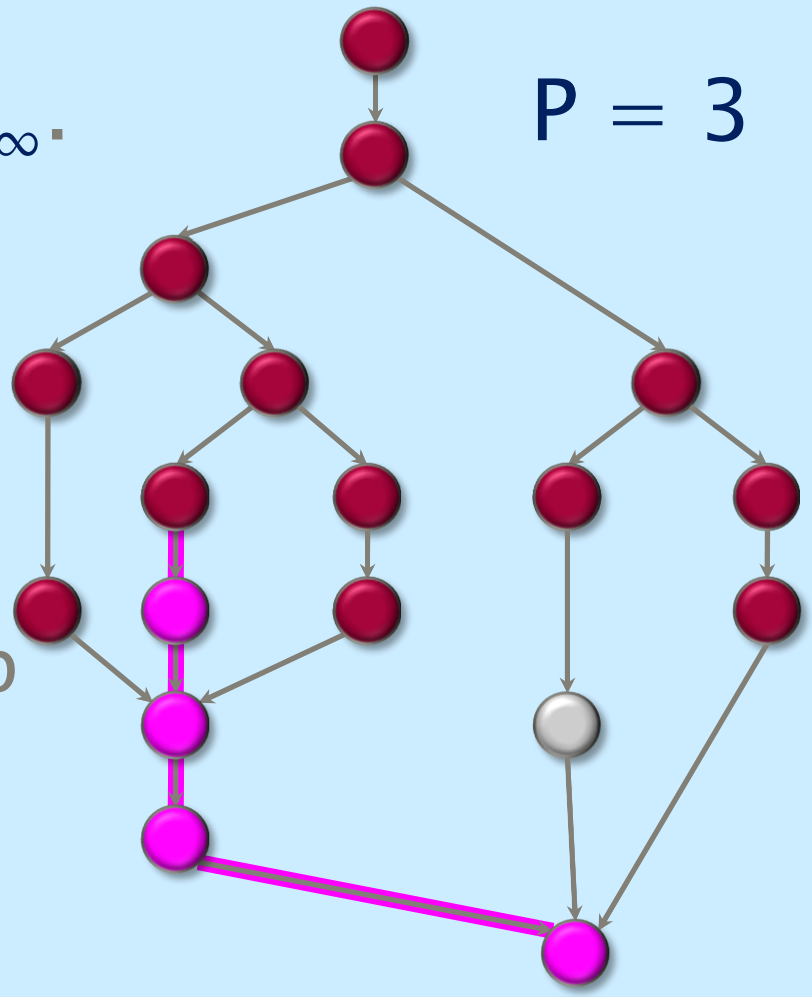
- $<$ P strands ready.
- Run all of them.



Analysis of Greedy

Theorem [G68, B75, BL93]. Any greedy scheduler achieves

$$T_P \leq T_1/P + T_\infty.$$



Proof.

- # complete steps $\leq T_1/P$, since each complete step performs P work.
- # incomplete steps $\leq T_\infty$, since each incomplete step reduces the span of the unexecuted dag by 1. ■

Optimality of Greedy

Corollary. Any greedy scheduler achieves within a factor of 2 of optimal.

Proof. Let T_P^* be the execution time produced by the optimal scheduler. Since $T_P^* \geq \max\{T_1/P, T_\infty\}$ by the Work and Span Laws, we have

$$\begin{aligned} T_P &\leq T_1/P + T_\infty \\ &\leq 2\cdot\max\{T_1/P, T_\infty\} \\ &\leq 2T_P^* . \quad \blacksquare \end{aligned}$$

Linear Speedup

Corollary. Any greedy scheduler achieves near-perfect linear speedup whenever $P \ll T_1/T_\infty$.

Proof. Since $P \ll T_1/T_\infty$ is equivalent to $T_\infty \ll T_1/P$, the Greedy Scheduling Theorem gives us

$$T_P \le T_1/P + T_\infty \approx T_1/P \,.$$

Thus, the speedup is $T_1/T_P \approx P$. ■

Definition. The quantity T_1/PT_∞ is called the ***parallel slackness***.

Sorting

- Sorting is possibly the most frequently executed operation in computing!
- **Quicksort** is the fastest sorting algorithm in practice with an average running time of O(N log N), (but $O(N^2)$ worst case performance)
- **Mergesort** has worst case performance of O(N log N) for sorting N elements
- Both based on the recursive **divide-and-conquer** paradigm

Parallelizing Quicksort

- Serial Quicksort sorts an array S as follows:
  - If the number of elements in S is 0 or 1, then return.
  - Pick any element *v* in S. Call this **pivot**.
  - Partition the set S–{v} into two disjoint groups:
    - $S_1 = \{x \in S-\{v\} \mid x \leq v\}$
    - $S_2 = \{x \in S-\{v\} \mid x \geq v\}$
  - **Return quicksort(S_1) followed by v followed by quicksort(S_2)**

Not necessarily so !

Parallel Quicksort (Basic)

- The second recursive call to *qsort* does not depend on the results of the first recursive call
- We have an opportunity to speed up the call by making both calls in parallel.

```
template <typename T>
void qsort(T begin, T end) {
  if (begin != end) {
     T middle = partition(
                  begin,
                  end,
                  bind2nd( less<typename iterator_traits<T>::value_type>(),
                           *begin )
                );
     cilk_spawn qsort(begin, middle);
     qsort(max(begin + 1, middle), end);
     cilk_sync;
  }
}
```

Performance

- ./qsort 500000 -cilk_set_worker_count 1
  >> 0.122 seconds
- ./qsort 500000 -cilk_set_worker_count 4
  >> 0.034 seconds
- Speedup = T_1/T_4 = 0.122/0.034 = **3.58**

- ./qsort 50000000 -cilk_set_worker_count 1
  >> 14.54 seconds
- ./qsort 50000000 -cilk_set_worker_count 4
  >> 3.84 seconds
- Speedup = T_1/T_4 = 14.54/3.84 = **3.78**

Measure Work/Span Empirically

- cilkview ./qsort

  Work : 6,008,068,218 instructions

  Span : 1,635,913,102 instructions

  Burdened span : 1,636,331,960 instructions

  Parallelism : 3.67

  …

- Only the qsort function, exclude data generation (page 112 of the cilk++ programmer's guide)

  qsort only ws 1 1.000000

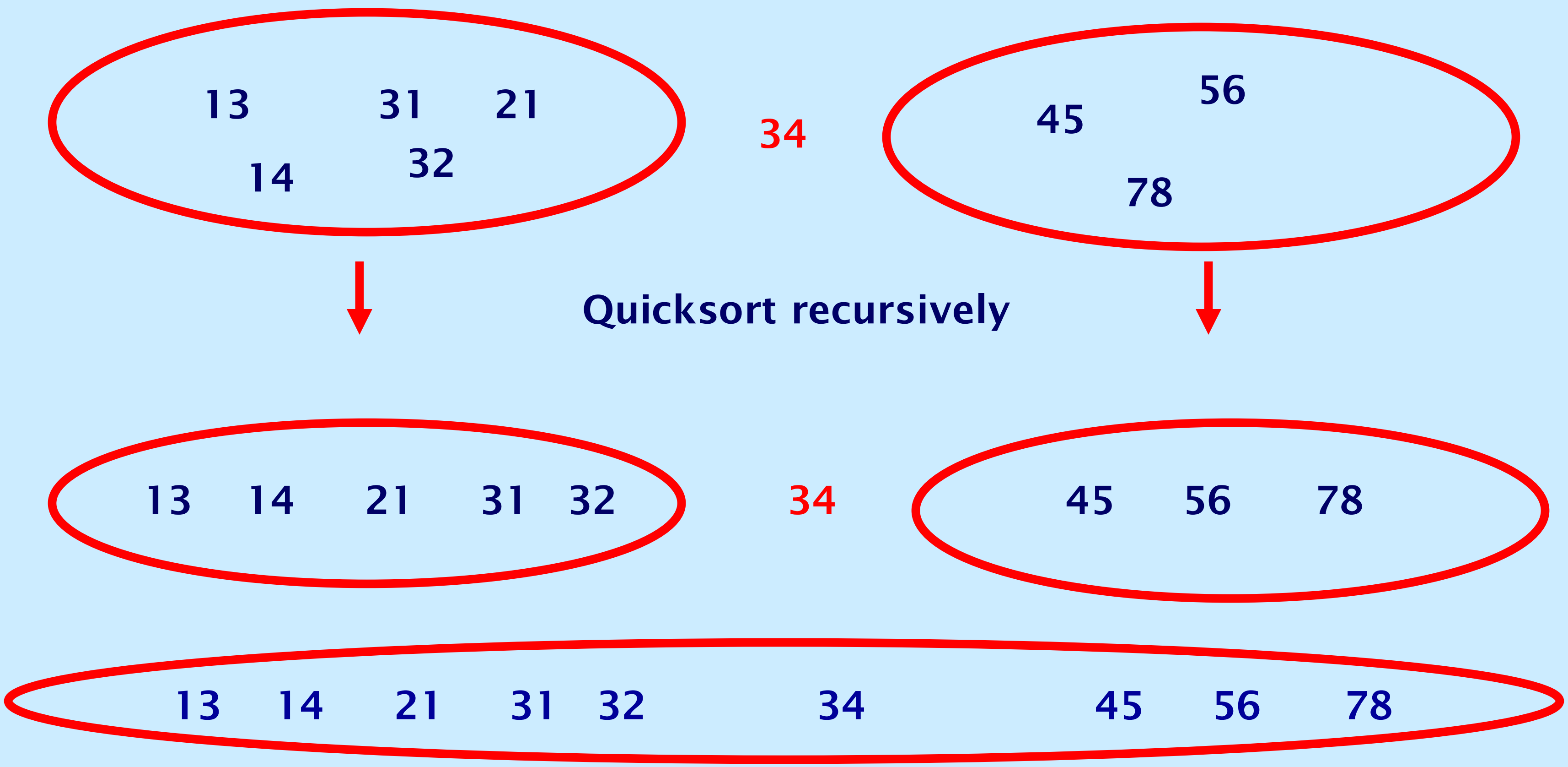
  qsort only ws 2 1.764697

  …

  qsort only ws 16 5.333179

  qsort only ws infinity 12.749447

Analyzing Quicksort



Assume we have a “great” partitioner that always generates two balanced sets

Analyzing Quicksort

- Work:

$$T_1(n) = 2T_1(n/2) + \Theta(n)$$
$$2T_1(n/2) = 4T_1(n/4) + 2\,\Theta(n/2)$$
....
....
$$+ \quad n/2\; T_1(2) = n\, T_1(1) + n/2\; \Theta(2)$$

$$T_1(n) = \Theta(n \lg n)$$

•Partitioning
•not parallel !

- Span recurrence: $T_\infty(n) = T_\infty(n/2) + \Theta(n)$

Solves to $T_\infty(n) = \Theta(n)$

Analyzing Quicksort

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\lg n)$$ Not much !

- Indeed, partitioning (i.e., constructing the array $S_1 = \{x \in S - \{v\} \mid x \le v\}$) can be accomplished in parallel in time $\Theta(\lg n)$
- Which gives a span $T_\infty(n) = \Theta(\lg^2 n)$
- And parallelism $\Theta(n/\lg n)$

Way better !

- Basic parallel qsort can be found under `$cilkpath/examples/qsort`
- Parallel partitioning might be a final project

The Master Method (Optional)

The ***Master Method*** for solving recurrences applies to recurrences of the form

$$T(n) = a\, T(n/b) + f(n)\ ,^{*}$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

IDEA: Compare $n^{\log_b a}$ with $f(n)$.

* The unstated base case is $T(n) = \Theta(1)$ for sufficiently small n.

Master Method — Case 1

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \gg f(n)$$

Specifically, $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.

Solution: $T(n) = \Theta(n^{\log_b a})$.

Master Method — Case 2

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \approx f(n)$$

Specifically, $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.

Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n))$.

Ex(qsort): $a = 2$, $b=2$, $k=0$ ➔ $T_1(n)=\Theta(n \lg n)$

Master Method — CASE 3

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \ll f(n)$$

Specifically, $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and $f(n)$ satisfies the ***regularity condition*** that $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.

Example: Span of qsort

Master Method Summary

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \epsilon})$, constant $\epsilon > 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$, constant $\epsilon > 0$, and regularity condition

$\Rightarrow T(n) = \Theta(f(n))$.

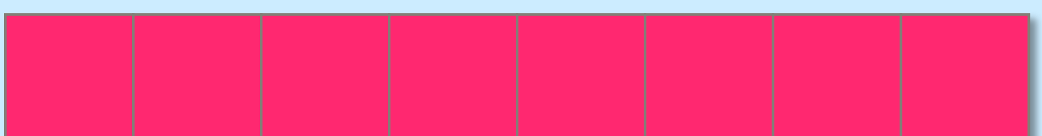
MERGESORT

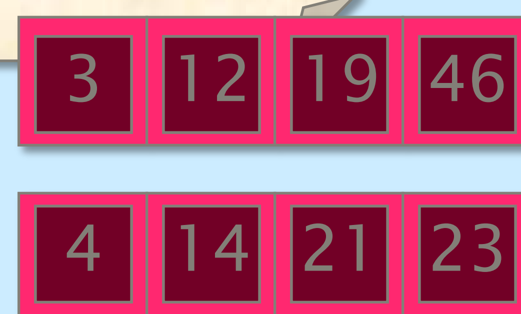
- Mergesort is an example of a recursive sorting algorithm.
- It is based on the divide-and-conquer paradigm
- It uses the merge operation as its fundamental component (which takes in two sorted sequences and produces a single sorted sequence)
- Simulation of Mergesort
- Drawback of mergesort: Not in-place (uses an extra temporary array)

Merging Two Sorted Arrays

```
template <typename T>
void Merge(T *C, T *A, T *B, int na, int nb) {
  while (na>0 && nb>0) {
    if (*A <= *B) {
      *C++ = *A++; na--;
    } else {
      *C++ = *B++; nb--;
    }
  }
  while (na>0) {
    *C++ = *A++; na--;
  }
  while (nb>0) {
    *C++ = *B++; nb--;
  }
}
```

Time to merge n elements = $\Theta(n)$.

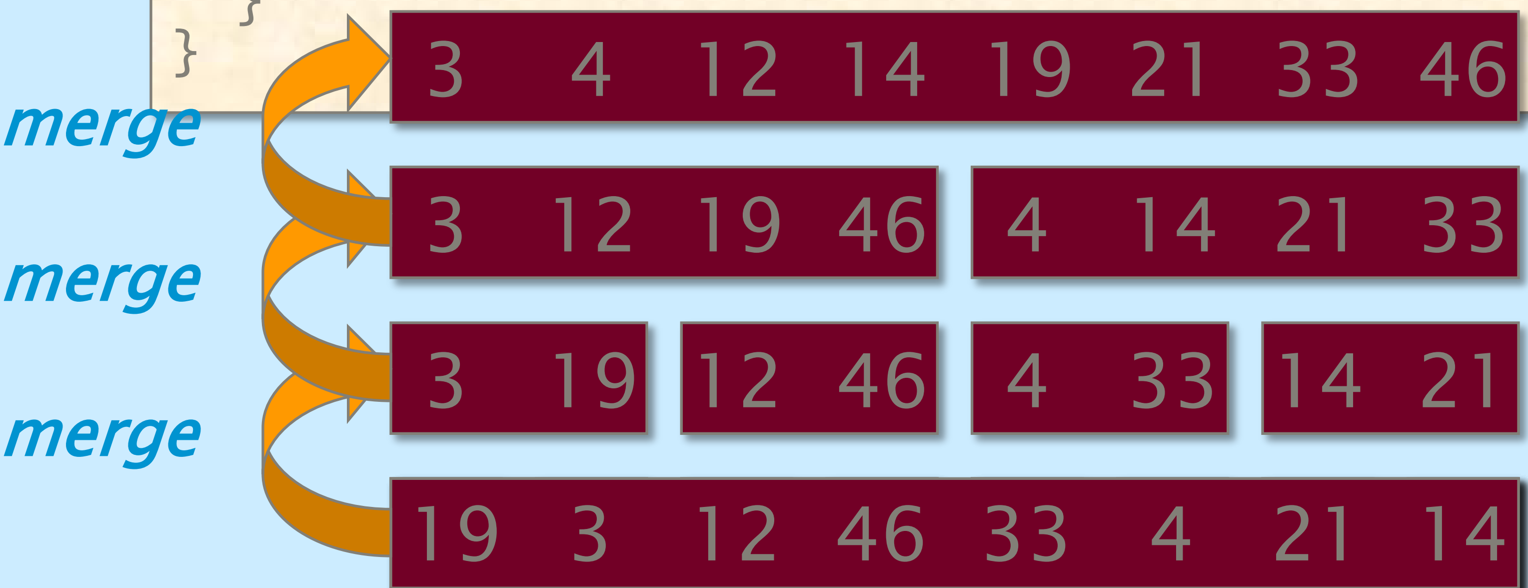




Parallel Merge Sort

A: input (unsorted)
B: output (sorted)
C: temporary

```
template <typename T>
void MergeSort(T *B, T *A, in
   if (n==1) {
      B[0] = A[0];
   } else {
      T* C = new T[n];
      cilk_spawn MergeSort(C, A, n/2);
                 MergeSort(C+n/2, A+n/2, n-n/2);
      cilk_sync;
      Merge(B, C, C+n/2, n/2, n-n/2);
      delete[] C;
   }
}
```



Work of Merge Sort

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C = new T[n];
    cilk_spawn MergeSort(C,
               MergeSort(C+
    cilk_sync;
    Merge(B, C, C+n/2, n/2,
    delete[] C;
  }
}
```

CASE 2:

$n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = \Theta(n^{\log_b a} \lg^0 n)$

Work: $T_1(n) = 2T_1(n/2) + \Theta(n)$

$= \Theta(n \lg n)$

Span of Merge Sort

```
template <typename T>
void MergeSort(T *B, T *A, i
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C = new T[n];
    cilk_spawn MergeSort(C,
               MergeSort(C+n/2,  , n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-  );
    delete[] C;
  }
}
```

CASE 3:

$n^{\log_b a} = n^{\log_2 1} = 1$

$f(n) = \Theta(n)$

Span: $T_\infty(n) = T_\infty(n/2) + \Theta(n)$

$= \Theta(n)$

Parallelism of Merge Sort

Work: $T_1(n) = \Theta(n \lg n)$

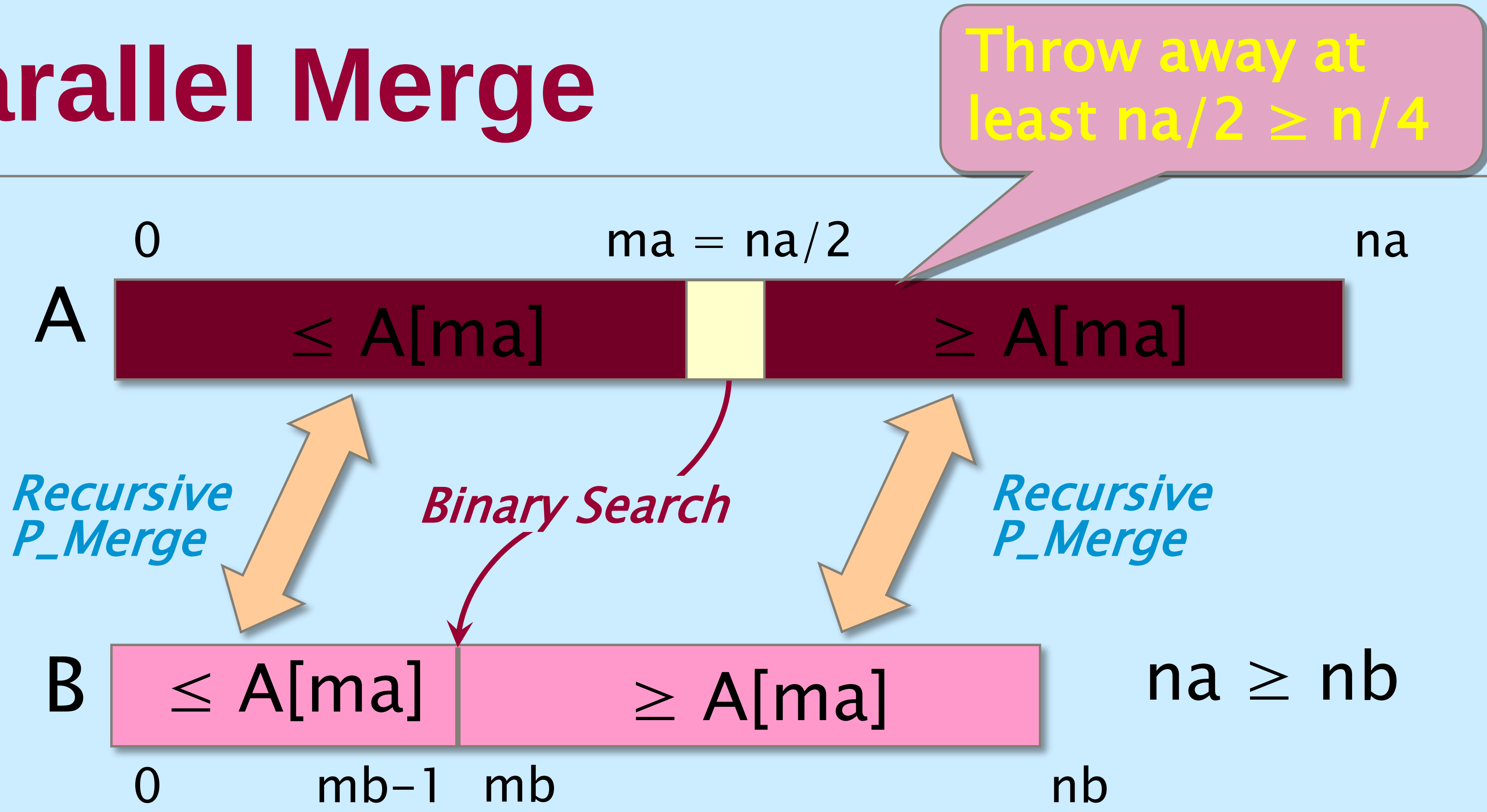
Span: $T_\infty(n) = \Theta(n)$

PUNY!

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\lg n)$$

We need to parallelize the merge!

Parallel Merge



KEY IDEA: If the total number of elements to be merged in the two arrays is $n = na + nb$, the total number of elements in the larger of the two recursive merges is at most $(3/4)n$.

Parallel Merge

```
template <typename T>
void P_Merge(T *C, T *A, T *B, int na, int nb) {
  if (na < nb) {
    P_Merge(C, B, A, nb, na);
  } else if (na==0) {
      return;
  } else {
    int ma = na/2;
    int mb = BinarySearch(A[ma], B, nb);
    C[ma+mb] = A[ma];
    cilk_spawn P_Merge(C, A, B, ma, mb);
    P_Merge(C+ma+mb+1, A+ma+1, B+mb, na-ma-1, nb-mb);
    cilk_sync;
  }
}
```

Coarsen base cases for efficiency.

Span of Parallel Merge

```
template <typename T>
void P_Merge(T *C, T *A, T *B, int na, int nb) {
  if (na < nb) {
    ⋮
    int mb = BinarySearch(A[ma]
    C[ma+mb] = A[ma];
    cilk_spawn P_Merge(C, A, B
    P_Merge(C+ma+mb+1, A+ma+1,
    cilk_sync;
  }
}
```

CASE 2:

$n^{\log_b a} = n^{\log_{4/3} 1} = 1$

$f(n) = \Theta(n^{\log_b a} \lg^1 n)$

Span: $T_\infty(n) = T_\infty(3n/4) + \Theta(\lg n)$

$= \Theta(\lg^2 n)$

Work of Parallel Merge

```
template <typename T>
void P_Merge(T *C, T *A, T *B, int na, int nb) {
  if (na < nb) {
     ⋮
    int mb = BinarySearch(A[ma], B, nb);
    C[ma+mb] = A[ma];
    cilk_spawn P_Merge(C, A, B, ma, mb)
    P_Merge(C+ma+mb+1, A+ma+1, B+mb, na-        );
    cilk_sync;
  }
}
```

HAIRY!

Work: $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$,
where $1/4 \le \alpha \le 3/4$.

Claim: $T_1(n) = \Theta(n)$.

Analysis of Work Recurrence

Work: $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$, where $1/4 \le \alpha \le 3/4$.

Substitution method: Inductive hypothesis is $T_1(k) \le c_1 k - c_2 \lg k$, where $c_1, c_2 > 0$. Prove that the relation holds, and solve for c_1 and c_2.

$$T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$$
$$\le c_1(\alpha n) - c_2 \lg(\alpha n)$$
$$+ c_1(1-\alpha)n - c_2 \lg((1-\alpha)n) + \Theta(\lg n)$$

Analysis of Work Recurrence

Work: $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$,
where $1/4 \le \alpha \le 3/4$.

$$\begin{aligned} T_1(n) &= T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n) \\ &\le c_1(\alpha n) - c_2\lg(\alpha n) \\ &\quad + c_1(1-\alpha)n - c_2\lg((1-\alpha)n) + \Theta(\lg n) \end{aligned}$$

Analysis of Work Recurrence

Work: $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$, where $1/4 \le \alpha \le 3/4$.

$$
\begin{aligned}
T_1(n) &= T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n) \\
&\le c_1(\alpha n) - c_2\lg(\alpha n) \\
&\quad + c_1(1-\alpha)n - c_2\lg((1-\alpha)n) + \Theta(\lg n) \\
&\le c_1 n - c_2\lg(\alpha n) - c_2\lg((1-\alpha)n) + \Theta(\lg n) \\
&\le c_1 n - c_2(\lg(\alpha(1-\alpha)) + 2\lg n) + \Theta(\lg n) \\
&\le c_1 n - c_2 \lg n \\
&\quad - (c_2(\lg n + \lg(\alpha(1-\alpha))) - \Theta(\lg n)) \\
&\le c_1 n - c_2 \lg n
\end{aligned}
$$

by choosing c_2 large enough. Choose c_1 large enough to handle the base case.

Parallelism of P_Merge

Work: $T_1(n) = \Theta(n)$

Span: $T_\infty(n) = \Theta(\lg^2 n)$

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(n/\lg^2 n)$$

Parallel Merge Sort

```
template <typename T>
void P_MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T C[n];
    cilk_spawn P_MergeSo
    P_MergeSort(C+n/2, A+n/2            2);
    cilk_sync;
    P_Merge(B, C, C+n/2, n/      n-n/2);
  }
}
```

CASE 2:

$n^{\log_b a} = n^{\log_2 2} = n$

$f(n) = \Theta(n^{\log_b a} \lg^0 n)$

Work: $T_1(n) = 2T_1(n/2) + \Theta(n)$

$= \Theta(n \lg n)$

Parallel Merge Sort

```
template <typename T>
void P_MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T C[n];
    cilk_spawn P_MergeSor
    P_MergeSort(C+n/2, A+n/2            );
    cilk_sync;
    P_Merge(B, C, C+n/2, n/      -n/2);
  }
}
```

CASE 2:

$n^{\log_b a} = n^{\log_2 1} = 1$

$f(n) = \Theta(n^{\log_b a} \lg^2 n)$

Span: $T_\infty(n) = T_\infty(n/2) + \Theta(\lg^2 n)$

$= \Theta(\lg^3 n)$

Parallelism of P_MergeSort

Work: $T_1(n) = \Theta(n \lg n)$

Span: $T_\infty(n) = \Theta(\lg^3 n)$

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(n/\lg^2 n)$$