CS 290H: Preconditioning iterative methods // Homework 3

Assigned November 8, 2004

Due by class time Monday, November 21

1. [**40 points**] You may use Matlab for this problem or do it by hand; I'm not sure which is easier.

(a) Show the 9-by-9 matrix of the two-dimensional model problem on a 3-by-3 grid, factored as $A = UU^T$, where U is a matrix that has (i) 9 rows, (ii) no more than 2 nonzeros per column, and (iii) if a column has two nonzeros a and b, then $a = -b$. Notice that U is not square.

(b) Show the graph of matrix A, with weights on the edges for nonzero values. Show a maximum-weight spanning tree for that graph. (There are a lot of choices.) Show the matrix B that is the support-graph preconditioner corresponding to the spanning tree you chose.

(c) Show B factored as $B = VV^T$, where V is a matrix that satisfies the same three conditions as U above.

(d) Find a matrix W with $U = VW$. (Note that W is not square.)

(e) What are the 2-norm, the 1-norm, the infinity norm, and the Frobenius norm of W? What is $||W^T W||_1$? What is the condition number of A? What is the condition number of $B^{-1}A$? (Answers computed by Matlab are ok.)

2. [**60 points**] Let $Au = f$ be the one-dimensional model problem, so that (following the notation of Briggs et al.) A is an $n-1$-by-$n-1$ tridiagonal matrix with all diagonal elements equal to 2 and all super- and sub-diagonal elements equal to -1; the right-hand side f is an $n-1$-vector of "forces"; and u is an $n-1$-vector of unknowns.

(a) Prove that for each k from 1 to $n-1$,

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2n}\right)$$

is an eigenvalue of A, with an eigenvector w_k whose j-th element is $\sin(jk\pi/n)$. Hint: you may want to use the trigonometric identity $\sin((j+1)\theta) + \sin((j-1)\theta) = 2\sin(j\theta)\cos\theta$.

(b) What is the condition number of A as a function of n? Express your answer in the form $O(n^c)$ for some constant c. Expressed in the same form $O(n^c)$, how many iterations would conjugate gradient with no preconditioning require to converge to a fixed tolerance on this problem? (Answer by analysis, not experiment, for this and the following two parts.)

(c) Now consider the unweighted Jacobi iteration for the system $Au = f$. Writing $A = D - L - U$ where D is diagonal, L is strictly lower triangular, and U is strictly upper triangular, recall that the iteration is

$$v^{(new)} = Rv^{(old)} + D^{-1}f,$$

where $R = D^{-1}(L + U)$ is the iteration matrix. Prove that the eigenvectors of R are the same as the eigenvectors of A. What are the eigenvalues of R?

(d) How many iterations does unweighted Jacobi require to reduce the error in the initial guess by a fixed factor, say 10^{-8}? Express your answer in the form $O(n^c)$.

(e) [**optional, extra credit**] Implement an unweighted Jacobi iteration in Matlab, and use it to solve the 3-dimensional model problem for the same sizes you used in Homework 2. Make semilog plots of the relative residual reduction versus iteration number for Jacobi and for unpreconditioned CG. Also, make a log-log plot of number of iterations to convergence versus n. Does your experiment support the analysis of part (d) above?