Applied Math Review for Deep Learning

UCSB CS165B W22 Section 1

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1. Linear Algebra

2. Calculus

3. Probability

Vectors, Matrices and Tensors

Scalar	Vector	Matrix	Tensor
1	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 7 \end{bmatrix} & \begin{bmatrix} 5 & 4 \end{bmatrix} \end{bmatrix}$

Vectors, Matrices and Tensors

Notations

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}, \text{ or } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We often denote the set of all possible real value vectors with d elements as $\mathbb{R}^d$. The shape of such vectors is $d \times 1$, i.e. they are column vectors.

Similarly, the set of real value matrices of shape $m \times n$ is denoted as $\mathbb{R}^{m \times n}$.

Matrix Transpose

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \rightarrow \boldsymbol{A}^\top = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Formally, the **transpose** of a matrix $\boldsymbol{A}$ is denoted as $\boldsymbol{A}^\top$. It is defined such that

$$(\boldsymbol{A}^\top)_{i,j} = \boldsymbol{A}_{j,i}$$

The transpose of a vector $\boldsymbol{x}$ therefore becomes a row vector.

Matrix Multiplication

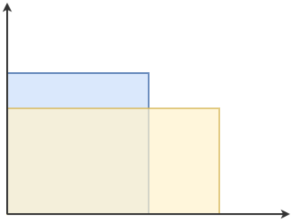
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

For matrix $\boldsymbol{A}$ of shape $m \times n$ and matrix $\boldsymbol{B}$ of shape $n \times p$, the **matrix product** of the two is another matrix $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ of shape $m \times p$ where

$$\boldsymbol{C}_{i,j} = \sum_k \boldsymbol{A}_{i,k}\boldsymbol{B}_{k,j}$$

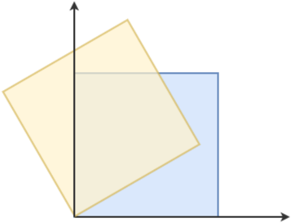
The dot product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with the same dimensions can be written as $\boldsymbol{x}^\top \boldsymbol{y}$.

Matrix Multiplication as Linear Transformation



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.75 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Scaling



$$\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Rotation

Identity and Inverse Matrices

An n-dimensional **identity matrix** is denoted as $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$. All its diagonal elements are 1's and all other elements are 0's. For example,

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is called identity matrix because for any n-dimensional vector $\boldsymbol{x}$, $\boldsymbol{I}_n\boldsymbol{x} = \boldsymbol{x}$.

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The **matrix inverse** of $\boldsymbol{A}$ is denoted as $\boldsymbol{A}^{-1}$, and it is defined as the matrix such that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

Finding the inverse of a matrix $\boldsymbol{A}$ helps us to solve linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. i.e. $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

Vector Norms

Norms are functions to measure the size of a vector. The L^p norm is given by

$$\|\boldsymbol{x}\|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

L^2 norm, or **Euclidean norm**, is frequently used in machine learning and simply represents the Euclidean distance from point $\boldsymbol{x}$ to the origin.

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Derivatives and Gradients

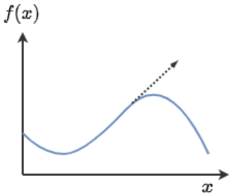
For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the **derivative** of f is defined as

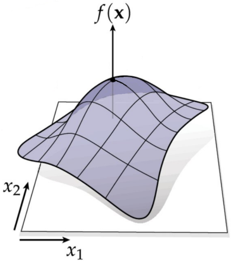
$$f'(x) = \frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The derivative gives the slope of the function at x.

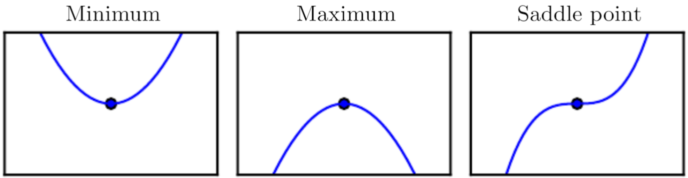
For a general function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the **gradient** of f with respect to the input $\boldsymbol{x}$ is defined as the vector of all partial derivatives

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right]^\top$$





Stationary Points



Points where $f'(x) = 0$ are called **stationary points**. Local minimum, local maximum, and saddle points are all stationary.

Derivative Calculation

Common Functions:
$$\frac{d}{dx}x^n = n \cdot x^{n-1},$$
$$\frac{d}{dx}e^x = e^x,$$
$$\frac{d}{dx}\log x = \frac{1}{x}$$

Product Rule:
$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

Chain Rule:
$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Chapter 2 of the Matrix Cookbook[1] has all formula needed to compute derivatives with respect to vectors and matrices.

[1] Kaare Brandt Petersen, Michael Syskind Pedersen, et al. "The matrix cookbook". In: *Technical University of Denmark* 7.15 (2008), p. 510.

Derivative Calculation

Exercise

Consider vector $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{R}^n$ and scalar b, find $\frac{\partial}{\partial \boldsymbol{x}} f(\boldsymbol{x})$ with the function f defined as

$$f(\boldsymbol{x}) = \frac{1}{1 + e^{-(\boldsymbol{w}^\top \boldsymbol{x} + b)}}$$

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f can be seen as a composite of $f_1(x) = \frac{1}{x}$, $f_2(x) = 1 + e^{-x}$, and $f_3(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{x} + b$:

$$f(\boldsymbol{x}) = f_1(f_2(f_3(\boldsymbol{x})))$$

Now let's denote $y = f_3(\boldsymbol{x})$ and $z = f_2(y)$. Using chain rule:

$$\frac{\partial}{\partial \boldsymbol{x}} f_1(f_2(f_3(\boldsymbol{x}))) = \frac{\partial f_1(z)}{\partial z} \cdot \frac{\partial f_2(y)}{\partial y} \cdot \frac{\partial f_3(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

$$= -z^{-2} \cdot (-e^{-y}) \cdot \boldsymbol{w} = \frac{e^{-(\boldsymbol{w}^\top \boldsymbol{x} + b)}}{(1 + e^{-(\boldsymbol{w}^\top \boldsymbol{x} + b)})^2} \cdot \boldsymbol{w}$$

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Key Concepts

Conditional Probability: $$p(y|x) = \frac{p(x, y)}{p(x)}$$

Marginal Probability: $$p(x) = \int p(x, y)\, dy$$

Independence: $$p(x, y) = p(x)p(y)$$

Expectation: $$\mathbb{E}_{x \sim p}\left[f(x)\right] = \int f(x)p(x)\, dx$$

Bayes' Rule

When we are interested in the value of $P(x|y)$, but only have access to $P(x)$ and $P(y|x)$, we can apply the Bayes' rule to compute it.

$$P(x|y) = \frac{P(x)P(y|x)}{P(y)} = \frac{P(x)P(y|x)}{\sum_x P(x)P(y|x)}$$

$P(x)$ if often referred to as the **prior distribution**, and $P(x|y)$ is known as the **posterior distribution** of x.

Bayes' Rule Application

The distribution of Mark's body temperature is $\mathcal{N}(98, 0.5)$ under healthy conditions. When sick, the distribution is $\mathcal{N}(99, 0.7)$. We know Mark is sick 10% of the time, and his body temperature right now is 98.5. What is the probability that Mark is sick at the moment?

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We are essentially looking for the posterior distribution of "Mark is sick" given the prior distribution and the conditionals. Denote x as the event "Mark is sick" and y as Mark's body temperature, we have

Prior: $P(x = \mathsf{T}) = 0.1$

Conditionals: $y \mid (x = \mathsf{T}) \sim \mathcal{N}(99, 0.7), \quad y \mid (x = \mathsf{F}) \sim \mathcal{N}(98, 0.5)$

Posterior: $$P(x = \mathsf{T} \mid y = 98.5) = \frac{P(x = \mathsf{T})P(y = 98.5 \mid x = \mathsf{T})}{P(x = \mathsf{T})P(y = 98.5 \mid x = \mathsf{T}) + P(x = \mathsf{F})P(y = 98.5 \mid x = \mathsf{F})}$$