Lecture 2:
Review of Linear Algebra
Last Lecture

• What is Computer Graphics?
• Why study Computer Graphics?
• Course Topics
• Course Logistics
Announcements

- Course slides and (pre)-reading materials

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<td>Overview of Computer Graphics [PDF]</td>
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Today

A Swift and Brutal
Introduction to Linear Algebra!

(in fact it’s relatively easy…)
Graphics’ Dependencies

• Basic mathematics
  - Linear algebra, calculus, statistics

• Basic physics
  - Optics, Mechanics

• Misc
  - signal processing
  - numerical analysis

• And a bit of aesthetics
This Course

• More dependent on Linear Algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)

• For example,
  - A point is a vector (?)
  - An operation like translating or rotating objects can be matrix-vector multiplication
An Example of Rotation

Rendering Glints on High-Resolution Normal-Mapped Specular Surfaces, Lingqi Yan, 2014
Vectors

- Usually written as $\vec{a}$ or in bold $\mathbf{a}$
- Or using start and end points $\overrightarrow{AB}$
- Direction and length
- No absolute starting position
Vector Normalization

- Magnitude (length) of a vector written as $||\vec{a}||$

- Unit vector
  - A vector with magnitude of 1
  - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a} / ||\vec{a}||$
  - Used to represent directions
Vector Addition

- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords
Cartesian Coordinates

- X and Y can be any (usually orthogonal unit) vectors

\[
A = \begin{pmatrix} x \\ y \end{pmatrix} \quad A^T = (x, y) \quad \|A\| = \sqrt{x^2 + y^2}
\]
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
Dot (scalar) Product

\[ \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta \]

- For unit vectors
  \[ \cos \theta = \hat{a} \cdot \hat{b} \]
Dot (scalar) Product

- Properties

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \]
\[ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]
\[ (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b}) \]
Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up
  - In 2D
    \[ \vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b \]
  - In 3D
    \[ \vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b \]
Dot Product in Graphics

- Find angle between two vectors (e.g. cosine of angle between light source and surface)
- Finding projection of one vector on another
Dot Product for Projection

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $\vec{b}_\perp = k\hat{a}$
  - What’s its magnitude $k$?
    - $k = ||\vec{b}_\perp|| = ||\vec{b}|| \cos \theta$
Dot Product in Graphics

- Measure how close two directions are
- Determine forward / backward
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
Cross (vector) Product

- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)
Cross product: Properties

\[
\begin{align*}
\vec{x} \times \vec{y} &= +\vec{z} \\
\vec{y} \times \vec{x} &= -\vec{z} \\
\vec{y} \times \vec{z} &= +\vec{x} \\
\vec{z} \times \vec{y} &= -\vec{x} \\
\vec{z} \times \vec{x} &= +\vec{y} \\
\vec{x} \times \vec{z} &= -\vec{y}
\end{align*}
\]

\[
\begin{align*}
\vec{a} \times \vec{b} &= -\vec{b} \times \vec{a} \\
\vec{a} \times \vec{a} &= \vec{0} \\
\vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\
\vec{a} \times (k\vec{b}) &= k(\vec{a} \times \vec{b})
\end{align*}
\]
Cross Product: Cartesian Formula?

\[ \vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix} \]

- Later in this lecture

\[ \vec{a} \times \vec{b} = A^*b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \]

dual matrix of vector a
Cross Product in Graphics

- Determine left / right
- Determine inside / outside
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
Orthonormal Bases / Coordinate Frames

• Important for representing points, positions, locations

• Often, many sets of coordinate systems
  - Global, local, world, model, parts of model (head, hands, …)

• Critical issue is transforming between these systems/bases
  - A topic for next week
Orthonormal Coordinate Frames

- Any set of 3 vectors (in 3D) that

\[
\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1
\]
\[
\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0
\]
\[
\vec{w} = \vec{u} \times \vec{v} \quad \text{ (right-handed)}
\]
\[
\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}
\]
\quad \text{(projection)}
Questions?
Matrices

- Magical 2D arrays that haunt in every CS course
- In Graphics, pervasively used to represent transformations
  - Translation, rotation, shear, scale
  (more details in the next lecture)
What is a matrix

- Array of numbers \((m \times n = m \text{ rows, } n \text{ columns})\)

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\]

- Addition and multiplication by a scalar are trivial: element by element
Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  \((M \times N) (N \times P) = (M \times P)\)

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
\]
Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  $(M \times N) (N \times P) = (M \times P)$

$$
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
= 
\begin{pmatrix}
9 & ? & 33 & 13 \\
19 & 44 & 61 & 26 \\
8 & 28 & 32 & ?
\end{pmatrix}
$$

- Element $(i, j)$ in the product is the dot product of row $i$ from A and column $j$ from B
Matrix-Matrix Multiplication

- Properties
  - Non-commutative
    (AB and BA are different in general)
  - Associative and distributive
    - A(B+C) = AB + AC
    - (A+B)C = AC + BC
Matrix-Vector Multiplication

- Treat vector as a column matrix \((m \times 1)\)
- Key for transforming points (next lecture)

- Official spoiler: 2D reflection about y-axis

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
-x \\
y
\end{pmatrix}
\]
Transpose of a Matrix

• Switch rows and columns (ij -> ji)

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}^T = \begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}
\]

• Property

\[
(AB)^T = B^T A^T
\]
Identity Matrix and Inverses

$$I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$
Vector multiplication in Matrix form

• Dot product?

\[ \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} \]

\[ = (x_a \ y_a \ z_a) \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = (x_a x_b + y_a y_b + z_a z_b) \]

• Cross product?

\[ \vec{a} \times \vec{b} = A^*b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \]

dual matrix of vector \( \vec{a} \)
An Example of General Transformation

The Sponza Scene, rendered by Lingqi Yan using Real-time Ray Tracing (RTRT)
Questions?
Next Week

• Transform!

Transformers: The Last Knight, 2017 movie
Thank you!