Lecture 2:
Review of Linear Algebra
Announcements

• Slides and recordings of Lecture 1 now available
  - Slides (not recording) also available on the course website

• (Pre)-reading materials will be out before lectures

<table>
<thead>
<tr>
<th>Week</th>
<th>Date</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Jan 5</td>
<td>Overview of Computer Graphics [PDF]</td>
</tr>
<tr>
<td></td>
<td>Jan 7</td>
<td>Vectors and Linear Algebra</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Reading: Chapter 2 (Miscellaneous Math) and Chapter 5 (Linear Algebra)</td>
</tr>
</tbody>
</table>

• If you are on the waitlist
  - Let any TA add you on GauchoSpace
  - Let me know if you have not been enrolled automatically early next week
Last Lecture

• What is Computer Graphics?

• Why study Computer Graphics?

• Course Topics

• Course Logistics
A Swift and Brutal Introduction to Linear Algebra!

(in fact it’s relatively easy…)
Graphics’ Dependencies

• Basic mathematics
  - Linear algebra, calculus, statistics

• Basic physics
  - Optics, Mechanics

• Misc
  - Numerical analysis, signal processing

• And a bit of aesthetics
This Course

• More dependent on Linear Algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)

• For example,
  - A point is a vector (?)
  - An operation like translating or rotating objects can be matrix-vector multiplication
An Example of Rotation

Rendering Glints on High-Resolution Normal-Mapped Specular Surfaces, Lingqi Yan, 2014
Vectors

• Usually written as \( \vec{a} \) or in bold \( a \)

• Or using start and end points \( \overrightarrow{AB} = B - A \)

• Direction and length

• No absolute starting position
Vector Normalization

- Magnitude (length) of a vector written as $||\vec{a}||$

- Unit vector
  - A vector with magnitude 1
  - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a}/||\vec{a}||$
  - Used to represent directions
Vector Addition

• Geometrically: Parallelogram law & Triangle law

• Algebraically: Simply add coordinates
Cartesian Coordinates

- X and Y can be any (usually orthogonal unit) vectors

\[
\begin{align*}
A &= \begin{pmatrix} x \\ y \end{pmatrix} \\
A^T &= (x, y) \\
\|A\| &= \sqrt{x^2 + y^2}
\end{align*}
\]
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
Dot (scalar) Product

\[ \vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta \]

- For unit vectors

\[ \cos \theta = \hat{\vec{a}} \cdot \hat{\vec{b}} \]
Dot (scalar) Product

- Properties

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \]

\[ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]

\[ (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b}) \]
Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up
  - In 2D
    \[
    \vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b
    \]
  - In 3D
    \[
    \vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b
    \]
Dot Product in Graphics

• Find angle between two vectors (e.g. cosine of angle between light source and surface)

• Finding projection of one vector on another
Dot Product for Projection

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $\vec{b}_\perp = k\hat{a}$
  - What’s its magnitude $k$?
    - $k = ||\vec{b}_\perp|| = ||\vec{b}|| \cos \theta$
Dot Product in Graphics

- Measure how close two directions are
- Decompose a vector
Dot Product in Graphics

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames
Cross (vector) Product

- Cross product is orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

\[ a \times b = -b \times a \]

\[ |a \times b| = |a| |b| \sin \phi \]
Cross product: Properties

\[ \vec{x} \times \vec{y} = +\vec{z} \]
\[ \vec{y} \times \vec{x} = -\vec{z} \]
\[ \vec{y} \times \vec{z} = +\vec{x} \]
\[ \vec{z} \times \vec{y} = -\vec{x} \]
\[ \vec{z} \times \vec{x} = +\vec{y} \]
\[ \vec{x} \times \vec{z} = -\vec{y} \]

\[ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \]
\[ \vec{a} \times \vec{a} = \vec{0} \]

\[ \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \]

\[ \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b}) \]
Cross Product: Cartesian Formula?

\[ \vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix} \]

• Later in this lecture

\[ \vec{a} \times \vec{b} = A^*b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \]

dual matrix of vector \( \vec{a} \)
Cross Product in Graphics

- Determine left / right
- Determine inside / outside
Cross Product in Graphics

\[ \vec{a} \times \vec{b} \]

\[ \vec{a} \]

\[ \vec{b} \]

\[ \mathbf{P} \]

\[ \mathbf{A} \]

\[ \mathbf{B} \]

\[ \mathbf{C} \]
Vector Multiplication

- Dot product
- Cross product
  - Orthonormal bases and coordinate frames
Orthonormal Bases / Coordinate Frames

• Important for representing points, positions, locations

• Often, many sets of coordinate systems
  - Global, local, world, model, parts of model
    (head, hands, …)

• Critical issue is transforming within these systems/bases
  - A topic for next week
Orthonormal Coordinate Frames

• Any set of 3 vectors (in 3D) that

\[ \| \vec{u} \| = \| \vec{v} \| = \| \vec{w} \| = 1 \]

\[ \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0 \]

\[ \vec{w} = \vec{u} \times \vec{v} \text{ (right-handed)} \]

\[ \vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w} \text{ (projection)} \]
Matrices

- Magical 2D arrays that haunt in almost every CS course
- In Graphics, pervasively used to represent transformations
  - Translation, rotation, shear, scale
    (more details in the next lecture)
What is a matrix

• Array of numbers \((m \times n = m \text{ rows, } n \text{ columns})\)

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\]

• Addition and multiplication by a scalar are trivial: element by element
Matrix-Matrix Multiplication

• # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
\]
Matrix-Matrix Multiplication

- Number of columns in A must equal number of rows in B
  \((M \times N) (N \times P) = (M \times P)\)

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
=
\begin{pmatrix}
9 & ? & 33 & 13 \\
19 & 44 & 61 & 26 \\
8 & 28 & 32 & ?
\end{pmatrix}
\]

- Element \((i, j)\) in the product is the dot product of row \(i\) from A and column \(j\) from B
Matrix-Matrix Multiplication

- Properties
  - **Non-commutative**
    (AB and BA are different in general)
  - Associative and distributive
    - (AB)C = A(BC)
    - A(B+C) = AB + AC
    - (A+B)C = AC + BC
Matrix-Vector Multiplication

• Treat vector as a column matrix (m×1)

• Key for transforming points (next lecture)

• Official spoiler: 2D reflection about y-axis

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
-x \\
y
\end{pmatrix}
\]
Transpose of a Matrix

- Switch rows and columns (ij -> ji)

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}^T = \begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}
\]

- Property

\[
(AB)^T = B^T A^T
\]
Identity Matrix and Inverses

\[ I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ AA^{-1} = A^{-1}A = I \]

\[ (AB)^{-1} = B^{-1}A^{-1} \]
Vector multiplication in Matrix form

- Dot product?

\[
\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}
\]

\[
= (x_a \ y_a \ z_a) \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = (x_a x_b + y_a y_b + z_a z_b)
\]

- Cross product?

\[
\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}
\]

dual matrix of vector a
An Example of General Transformation

The Sponza Scene, rendered by Lingqi Yan using Real-time Ray Tracing (RTRT)
Next

- Transform!

Transformers: The Last Knight, 2017 movie
Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)