Lecture 5:
Transformation 3 (Projection)
Announcement

• Important: attendance time!
  - Just enter your name and perm number from this link
  - [https://forms.gle/yLGASNh69w6sHg6i7](https://forms.gle/yLGASNh69w6sHg6i7)
  - Do it before 11:59 PM on Jan 25 AoE, otherwise 3.33% off

• Homework 2 will be released on Thursday
  - On looking at a rotating triangle

• My office hour will be canceled next week
  - Jan 26 only, any other office hours will not be affected

• Still, this lecture will be difficult
Last Lecture

• 3D transformations

• Viewing transformation
  - View / Camera transformation
  - Projection transformation
    - Orthographic projection
    - Perspective projection
Today

• Perspective Projection
  - Defining the frustum
  - Relationship between perspective and orthographic projections
  - The “squish”-ing operation
  - Derivation

• Viewport Transform after MVP
Projection Transformation

• Characteristics
  - Closer objects appear larger
  - Parallel lines can be no longer parallel

Orthographic projection

Perspective projection
Perspective Projection

• Euclid was wrong??!!

In geometry, parallel lines are lines in a plane which do not meet; that is, two lines in a plane that do not intersect or touch each other at any point are said to be parallel. By extension, a line and a plane, or two planes, in three-dimensional Euclidean space that do not share a point are said to be parallel. However, two lines in three-dimensional space which do not meet must be in a common plane to be considered parallel; otherwise they are called skew lines. Parallel planes are planes in the same three-dimensional space that never meet.

Parallel lines are the subject of Euclid's parallel postulate. Parallelism is primarily a property of affine geometries and Euclidean geometry is a special instance of this type of geometry. In some other geometries, such as hyperbolic geometry, lines can have analogous properties that are referred to as parallelism.

https://en.wikipedia.org/wiki/Parallel_(geometry)
Perspective Projection

• Vanishing point and horizon
  - Vanishing point: all parallel lines intersect at the same point
  - Horizon (line): consists of all vanishing points from different pairs of parallel lines on the same plane
Perspective Projection

- Perspective projection vs. orthographic projection

https://stackoverflow.com/questions/36573283/from-perspective-picture-to-orthographic-picture
Defining the Frustum

• Recall: we have left a question in view transform
  - Why do we need to move the camera to the “canonical position”?

• Answer: this simplifies the frustum
Perspective vs. Orthographic

• How to do perspective projection
  - First “squish” the frustum into a cuboid (n -> n, f -> f) ($M_{persp->ortho}$)
  - Do orthographic projection ($M_{ortho}$, already known!)

Fig. 7.13 from *Fundamentals of Computer Graphics, 4th Edition*
Perspective vs. Orthographic

• Since orthographic projection discards z anyway
  - It doesn’t matter if the “squishing” moves the z coord of points
  - As long as the relative order is not changed (why?)
Perspective vs. Orthographic

• Before we move on

• Recall: property of homogeneous coordinates
  - \((x, y, z, 1), (kx, ky, kz, k), (xz, yz, z^2, z)\)
    all represent the same point \((x, y, z)\) in 3D (for non-zero \(k\) and \(z\))
  - e.g. \((1, 0, 0, 1)\) and \((2, 0, 0, 2)\) both represent \((1, 0, 0)\)

• Simple, but useful immediately next
Perspective to Orthographic

- In order to find a transformation
  - Recall the key idea: Find the relationship between the transformed point \((x', y', z')\) and the original point \((x, y, z)\)

\[
y' = \frac{n}{z} y
\]
Perspective to Orthographic

• In order to find a transformation
  - Find the relationship between transformed points \((x', y', z')\)
    and the original points \((x, y, z)\)

  \[
  y' = \frac{n}{z}y \quad x' = \frac{n}{z}x \quad \text{(similar to } y')
  \]

• In homogeneous coordinates,

  \[
  \begin{pmatrix}
  x \\
  y \\
  z \\
  1
  \end{pmatrix}
  \Rightarrow
  \begin{pmatrix}
  nx/z \\
  ny/z \\
  \text{unknown} \\
  1
  \end{pmatrix}
  \quad \text{mult. by } z
  \Rightarrow
  \begin{pmatrix}
  nx \\
  ny \\
  \text{still unknown} \\
  z
  \end{pmatrix}
  \]
Perspective to Orthographic

- So the “squish” (persp to ortho) projection does this

\[
M_{\text{persp} \rightarrow \text{ortho}}^{(4 \times 4)} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ \text{unknown} \\ z \end{pmatrix}
\]

- Already good enough to figure out part of \( M_{\text{persp} \rightarrow \text{ortho}} \)

\[
M_{\text{persp} \rightarrow \text{ortho}} = \begin{pmatrix} n & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 \\ ? & ? & ? & ? & ? \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\textbf{WHY?}
Perspective to Orthographic

• How to figure out the third row of $M_{\text{persp->ortho}}$
  - Any information that we can use?
    
    $M_{\text{persp->ortho}} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ ? & ? & ? & ? \\ 0 & 0 & 1 & 0 \end{pmatrix}$

• Observation: the third row is responsible for $z'$
  - Any point on the near plane will not change
  - Any point’s $z$ on the far plane will not change
Perspective to Orthographic

- Any point on the near plane \((z = n)\) will not change

\[
M_{\text{persp} \rightarrow \text{ortho}}^{(4 \times 4)} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ \text{unknown} \\ z \end{pmatrix}
\]

replace \(z\) with \(n\)

\[
\begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ n^2 \\ n \end{pmatrix}
\]

- So the third row must be of the form \((0 \ 0 \ A \ B)\)

\[
\begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix}
\]

\(n^2\) has nothing to do with \(x\) and \(y\)
Perspective to Orthographic

- What do we have now?

\[
(0 \quad 0 \quad A \quad B) \begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} = n^2 \quad \rightarrow \quad An + B = n^2
\]

- Any point’s z on the far plane will not change

\[
\begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f^2 \\ f \end{pmatrix} \quad \rightarrow \quad Af + B = f^2
\]
Perspective Projection

• Solve for A and B

\[
An + B = n^2 \\
Af + B = f^2
\]

\[
A = n + f \\
B = -nf
\]

• Finally, every entry in \( M_{\text{persp}} \rightarrow \text{ortho} \) is known!

• What’s next?
  - Do orthographic projection (\( M_{\text{ortho}} \)) to finish
  - \( M_{\text{persp}} = M_{\text{ortho}} M_{\text{persp} \rightarrow \text{ortho}} \)
Defining the Frustum

- What orthographic projection to perform?
  - Recall: just need the cuboid
  - \([l, r] \times [b, t] \times [n, f]\)
  - \(n\) and \(f\) did not change
  - \(l, r, b, t\) are near plane’s \(l, r, b, t\)
Defining the Frustum

• What’s near plane’s l, r, b, t then?
  - If explicitly specified, good
  - Sometimes people prefer:
    - vertical field-of-view (fovY) and aspect ratio
      (assume symmetry i.e. l = -r, b = -t)
Defining the Frustum

• How to convert from fovY and aspect to l, r, b, t?
  - Trivial

\[ \tan \frac{\text{fovY}}{2} = \frac{t}{|n|} \]

\[ \text{aspect} = \frac{r}{t} \]
Thank you!