MANCHESTER 1824

The University of Manchester

Conflict Resolution

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Outline

Our Main Problem

Checking satisfiability of systems of linear constraints over the rationals.

Example (i)

$$\begin{array}{rcrcrcl} 3x_1 & + & 4x_2 & - & 1 & > & 0 \\ -x_1 & + & x_2 & + & 1 & \geq & 0 \\ & & 2x_2 & - & x_3 & = & 0 \end{array}$$

Satisfiable: $x_1 = 0, x_2 = 1, x_3 = 2$

Example (ii)

$$\begin{array}{rcrcrcl} -x_1 & + & x_2 & + & 1 & \geq & 0 \\ & - & x_2 & - & x_3 & \geq & 0 \\ x_1 & + & x_3 & - & 2 & \geq & 0 \end{array}$$

Unsatisfiable.

Applications of linear constraint solving

- Combinatorial and optimisation problems;
- Linear Programming;
- Satisfiability Modulo Theories (SMT): software and hardware verification, constraint satisfaction;
- Constraint Programming;
- Hybrid Systems.

Existing Methods

There is a handful of methods for solving systems of linear inequalities:

- Fourier-Motzkin elimination (1820's)
- Simplex method (1950's)
- Interior point method (1970's)
- Various modifications of these methods

Outline

Example (iii)

$$
\begin{array}{rcrcrcrcrcl}
x_4 & - & 2x_3 & & & + & x_1 & + & 5 & \geq & 0 \quad (1)\\
x_4 & + & 2x_3 & + & x_2 & & & + & 3 & \geq & 0 \quad (2)\\
-x_4 & - & x_3 & - & 3x_2 & - & 3x_1 & + & 1 & \geq & 0 \quad (3)\\
-x_4 & + & 2x_3 & + & 2x_2 & + & x_1 & + & 6 & \geq & 0 \quad (4)\\
 & & x_3 & & & + & 3x_1 & - & 1 & \geq & 0 \quad (5)\\
 & & -x_3 & + & x_2 & - & 2x_1 & + & 5 & \geq & 0 \quad (6)
\end{array}
$$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

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$$-8x_1 + 27 \geq 0$$

$$x_1 \in [-\tfrac{15}{11}; \tfrac{59}{26}]$$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

$$x_4 - 2x_3 + x_1 + 5 \geq 0 \quad (1)$$
$$x_4 + 2x_3 + x_2 + 3 \geq 0 \quad (2)$$
$$-x_4 - x_3 - 3x_2 - 3x_1 + 1 \geq 0 \quad (3)$$
$$-x_4 + 2x_3 + 2x_2 + x_1 + 6 \geq 0 \quad (4)$$

$$x_3 + 3x_1 - 1 \geq 0 \quad (5)$$
$$x_3 - 2x_2 - 3x_1 + 4 \geq 0$$
$$x_3 + 4x_2 + x_1 + 9 \geq 0$$
$$-x_3 + x_2 - 2x_1 + 5 \geq 0 \quad (6)$$
$$-3x_3 - 3x_2 - 2x_1 + 6 \geq 0$$

$$2x_2 + 2x_1 + 11 \geq 0$$
$$x_2 + x_1 + 4 \geq 0$$
$$5x_2 - x_1 + 14 \geq 0$$
$$9x_2 + x_1 + 21 \geq 0$$
$$-3x_2 + 7x_1 + 3 \geq 0$$
$$-x_2 - 5x_1 + 9 \geq 0$$
$$-9x_2 - 11x_1 + 18 \geq 0$$

$$5x_1 + 9 \geq 0$$
$$2x_1 + 3 \geq 0$$
$$33x_1 + 67 \geq 0$$
$$11x_1 + 15 \geq 0$$
$$-8x_1 + 29 \geq 0$$
$$-4x_1 + 13 \geq 0$$
$$-22x_1 + 51 \geq 0$$
$$-10x_1 + 39 \geq 0$$
$$-26x_1 + 59 \geq 0$$
$$-4x_1 + 135 \geq 0$$
$$-x_1 + 27 \geq 0$$
$$-8x_1 + 27 \geq 0$$

$x_1 \in [-\frac{15}{11}; \frac{59}{26}]$ $x_1 = 0$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

$$x_4 - 2x_3 + x_1 + 5 \geq 0 \quad (1) \qquad -x_4 - x_3 - 3x_2 - 3x_1 + 1 \geq 0 \quad (3)$$

$$x_4 + 2x_3 + x_2 + 3 \geq 0 \quad (2) \qquad -x_4 + 2x_3 + 2x_2 + x_1 + 6 \geq 0 \quad (4)$$

$$x_3 + 3x_1 - 1 \geq 0 \quad (5) \qquad -x_3 + x_2 - 2x_1 + 5 \geq 0 \quad (6)$$

$$x_3 - 2x_2 - 3x_1 + 4 \geq 0 \qquad -3x_3 - 3x_2 - 2x_1 + 6 \geq 0$$

$$x_3 + 4x_2 + x_1 + 9 \geq 0$$

$$2x_2 + 2x_1 + 11 \geq 0 \qquad -3x_2 + 7x_1 + 3 \geq 0$$

$$x_2 + x_1 + 4 \geq 0 \qquad -x_2 - 5x_1 + 9 \geq 0$$

$$5x_2 - x_1 + 14 \geq 0 \qquad -9x_2 - 11x_1 + 18 \geq 0$$

$$9x_2 + x_1 + 21 \geq 0$$

$x_2 \in [-\frac{21}{9}; 1]$ $x_2 = 0$

$$5x_1 + 9 \geq 0$$
$$2x_1 + 3 \geq 0$$
$$33x_1 + 67 \geq 0$$
$$11x_1 + 15 \geq 0$$

$$-8x_1 + 29 \geq 0 \qquad -26x_1 + 59 \geq 0$$

$$-4x_1 + 13 \geq 0 \qquad -4x_1 + 135 \geq 0$$

$$-22x_1 + 51 \geq 0 \qquad -x_1 + 27 \geq 0$$

$$-10x_1 + 39 \geq 0 \qquad -8x_1 + 27 \geq 0$$

$x_1 \in [-\frac{15}{11}; \frac{59}{26}]$ $x_1 = 0$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

$$x_4 - 2x_3 + x_1 + 5 \geq 0 \quad (1) \qquad -x_4 - x_3 - 3x_2 - 3x_1 + 1 \geq 0 \quad (3)$$
$$x_4 + 2x_3 + x_2 + 3 \geq 0 \quad (2) \qquad -x_4 + 2x_3 + 2x_2 + x_1 + 6 \geq 0 \quad (4)$$

$$x_3 + 3x_1 - 1 \geq 0 \quad (5) \qquad -x_3 + x_2 - 2x_1 + 5 \geq 0 \quad (6)$$
$$x_3 - 2x_2 - 3x_1 + 4 \geq 0 \qquad -3x_3 - 3x_2 - 2x_1 + 6 \geq 0$$
$$x_3 + 4x_2 + x_1 + 9 \geq 0$$

$x_3 \in [1; 2]$ $x_3 = 1$

$$2x_2 + 2x_1 + 11 \geq 0 \qquad -3x_2 + 7x_1 + 3 \geq 0$$
$$x_2 + x_1 + 4 \geq 0 \qquad -x_2 - 5x_1 + 9 \geq 0$$
$$5x_2 - x_1 + 14 \geq 0 \qquad -9x_2 - 11x_1 + 18 \geq 0$$
$$9x_2 + x_1 + 21 \geq 0$$

$x_2 \in [-\frac{21}{9}; 1]$ $x_2 = 0$

$$5x_1 + 9 \geq 0 \qquad -8x_1 + 29 \geq 0 \qquad -26x_1 + 59 \geq 0$$
$$2x_1 + 3 \geq 0 \qquad -4x_1 + 13 \geq 0 \qquad -4x_1 + 135 \geq 0$$
$$33x_1 + 67 \geq 0 \qquad -22x_1 + 51 \geq 0 \qquad -x_1 + 27 \geq 0$$
$$11x_1 + 15 \geq 0 \qquad -10x_1 + 39 \geq 0 \qquad -8x_1 + 27 \geq 0$$

$x_1 \in [-\frac{15}{11}; \frac{59}{26}]$ $x_1 = 0$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

$$x_4 - 2x_3 + x_1 + 5 \geq 0 \quad (1)$$
$$x_4 + 2x_3 + x_2 + 3 \geq 0 \quad (2)$$
$$-x_4 - x_3 - 3x_2 - 3x_1 + 1 \geq 0 \quad (3)$$
$$-x_4 + 2x_3 + 2x_2 + x_1 + 6 \geq 0 \quad (4)$$

$x_4 \in [-3; 0]$ $x_4 = 0$

$$x_3 + 3x_1 - 1 \geq 0 \quad (5)$$
$$x_3 - 2x_2 - 3x_1 + 4 \geq 0$$
$$x_3 + 4x_2 + x_1 + 9 \geq 0$$
$$-x_3 + x_2 - 2x_1 + 5 \geq 0 \quad (6)$$
$$-3x_3 - 3x_2 - 2x_1 + 6 \geq 0$$

$x_3 \in [1; 2]$ $x_3 = 1$

$$2x_2 + 2x_1 + 11 \geq 0$$
$$x_2 + x_1 + 4 \geq 0$$
$$5x_2 - x_1 + 14 \geq 0$$
$$9x_2 + x_1 + 21 \geq 0$$
$$-3x_2 + 7x_1 + 3 \geq 0$$
$$-x_2 - 5x_1 + 9 \geq 0$$
$$-9x_2 - 11x_1 + 18 \geq 0$$

$x_2 \in [-\frac{21}{9}; 1]$ $x_2 = 0$

$$5x_1 + 9 \geq 0$$
$$2x_1 + 3 \geq 0$$
$$33x_1 + 67 \geq 0$$
$$11x_1 + 15 \geq 0$$

$$-8x_1 + 29 \geq 0$$
$$-4x_1 + 13 \geq 0$$
$$-22x_1 + 51 \geq 0$$
$$-10x_1 + 39 \geq 0$$

$$-26x_1 + 59 \geq 0$$
$$-4x_1 + 135 \geq 0$$
$$-x_1 + 27 \geq 0$$
$$-8x_1 + 27 \geq 0$$

$x_1 \in [-\frac{15}{11}; \frac{59}{26}]$ $x_1 = 0$

Fourier-Motzkin elimination

$x_4 \succ x_3 \succ x_2 \succ x_1$

$$x_4 - 2x_3 + x_1 + 5 \geq 0 \quad (1)$$
$$x_4 + 2x_3 + x_2 + 3 \geq 0 \quad (2)$$
$$-x_4 - x_3 - 3x_2 - 3x_1 + 1 \geq 0 \quad (3)$$
$$-x_4 + 2x_3 + 2x_2 + x_1 + 6 \geq 0 \quad (4)$$

$x_4 \in [-3; 0]$ $x_4 = 0$

$$x_3 + 3x_1 - 1 \geq 0 \quad (5)$$
$$x_3 - 2x_2 - 3x_1 + 4 \geq 0$$
$$x_3 + 4x_2 + x_1 + 9 \geq 0$$
$$-x_3 + x_2 - 2x_1 + 5 \geq 0 \quad (6)$$
$$-3x_3 - 3x_2 - 2x_1 + 6 \geq 0$$

$x_3 \in [1; 2]$ $x_3 = 1$

$$2x_2 + 2x_1 + 11 \geq 0$$
$$x_2 + x_1 + 4 \geq 0$$
$$5x_2 - x_1 + 14 \geq 0$$
$$9x_2 + x_1 + 21 \geq 0$$
$$-3x_2 + 7x_1 + 3 \geq 0$$
$$-x_2 - 5x_1 + 9 \geq 0$$
$$-9x_2 - 11x_1 + 18 \geq 0$$

$x_2 \in [-\frac{21}{9}; 1]$ $x_2 = 0$

$$5x_1 + 9 \geq 0$$
$$2x_1 + 3 \geq 0$$
$$33x_1 + 67 \geq 0$$
$$11x_1 + 15 \geq 0$$
$$-8x_1 + 29 \geq 0$$
$$-4x_1 + 13 \geq 0$$
$$-22x_1 + 51 \geq 0$$
$$-10x_1 + 39 \geq 0$$
$$-26x_1 + 59 \geq 0$$
$$-4x_1 + 135 \geq 0$$
$$-x_1 + 27 \geq 0$$
$$-8x_1 + 27 \geq 0$$

$x_1 \in [-\frac{15}{11}; \frac{59}{26}]$ $x_1 = 0$

Satisfiable : $x_4 = 0; x_3 = 1; x_2 = 0; x_1 = 0$

Fourier-Motzkin elimination

Iteratively eliminates variables, one at each step.

$$\frac{x + \bar{a}\bar{y} \geq 0 \qquad -x + \bar{b}\bar{y} \geq 0}{\bar{a}\bar{x} + \bar{b}\bar{y} \geq 0}$$

The main issues

- Very quickly becomes large and hard to deal with
- Generates many redundant inequalities
- Even if a trivial solution exists, we need to make all possible inferences

Fourier-Motzkin elimination

Iteratively eliminates variables, one at each step.

$$\frac{x + \bar{a}\bar{y} \geq 0 \qquad -x + \bar{b}\bar{y} \geq 0}{\bar{a}\bar{x} + \bar{b}\bar{y} \geq 0}$$

The main issues

- Very quickly becomes large and hard to deal with
- Generates many redundant inequalities
- Even if a trivial solution exists, we need to make all possible inferences

Fourier-Motzkin elimination

Generates over 280 million linear inequalities!

Example (iv)

$x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1$

$$
\begin{array}{rcrcrcrcrcrcl}
2x_5 & - & 3x_4 & + & x_3 & - & 3x_2 & - & 2x_1 & + & 3 & \geq & 0 \\
2x_5 & + & x_4 & - & 2x_3 & & & - & 2x_1 & + & 2 & \geq & 0 \\
-x_5 & & & & & + & 3x_2 & + & x_1 & + & 2 & \geq & 0 \\
-3x_5 & & & + & 2x_3 & & & - & 3x_1 & - & 2 & \geq & 0 \\
x_5 & - & 2x_4 & & & - & 2x_2 & + & 3x_1 & - & 2 & \geq & 0 \\
-2x_5 & + & 2x_4 & - & 3x_3 & - & x_2 & + & 2x_1 & + & 3 & > & 0 \\
3x_5 & - & 2x_4 & + & 2x_3 & + & 3x_2 & + & 2x_1 & + & 1 & > & 0 \\
x_5 & & & & & & & + & 2x_1 & + & 2 & > & 0 \\
 & & 2x_4 & - & x_3 & - & 3x_2 & - & x_1 & + & 3 & = & 0
\end{array}
$$

Outline

Conflict Resolution

$$2x_4 + 4x_3 - x_1 + 5 \geq 0$$

Normalised:

$$x_4 + 2x_3 - \frac{1}{2}x_1 + \frac{5}{2} \geq 0$$

Assignment $\sigma : \quad \{x_4 \mapsto 1; x_3 \mapsto -3; x_2 \mapsto -\frac{1}{3}; x_1 \mapsto 4\}$

$$p\sigma = 1 + 2 * (-3) - \frac{1}{2} * 4 = -8$$

σ is a solution of $p \geq 0$ if $p\sigma \geq 0$ is true.

Conflict Resolution

$$2x_4 + 4x_3 - x_1 + 5 \geq 0$$

Normalised:

$$x_4 + 2x_3 - \frac{1}{2}x_1 + \frac{5}{2} \geq 0$$

Assignment $\sigma : \quad \{x_4 \mapsto 1; x_3 \mapsto -3; x_2 \mapsto -\frac{1}{3}; x_1 \mapsto 4\}$

$$p\sigma = 1 + 2 * (-3) - \frac{1}{2} * 4 = -8$$

σ is a solution of $p \geq 0$ if $p\sigma \geq 0$ is true.

Conflict Resolution

$$2x_4 + 4x_3 - x_1 + 5 \geq 0$$

Normalised:

$$\underbrace{x_4 + 2x_3 - \tfrac{1}{2}x_1 + \tfrac{5}{2}}_{p} \geq 0$$

Assignment $\sigma : \quad \{x_4 \mapsto 1; x_3 \mapsto -3; x_2 \mapsto -\tfrac{1}{3}; x_1 \mapsto 4\}$

$$p\sigma = 1 + 2 * (-3) - \frac{1}{2} * 4 = -8$$

σ is a solution of $p \geq 0$ if $p\sigma \geq 0$ is true.

Conflict Resolution

$$2x_4 + 4x_3 - x_1 + 5 \geq 0$$

Normalised:

$$\underbrace{x_4 + 2x_3 - \tfrac{1}{2}x_1 + \tfrac{5}{2}}_{p} \geq 0$$

Assignment $\sigma : \quad \{x_4 \mapsto 1; x_3 \mapsto -3; x_2 \mapsto -\frac{1}{3}; x_1 \mapsto 4\}$

$$p\sigma = 1 + 2 * (-3) - \frac{1}{2} * 4 = -8$$

σ is a solution of $p \geq 0$ if $p\sigma \geq 0$ is true.

Conflict Resolution

$$2x_4 + 4x_3 - x_1 + 5 \geq 0$$

Normalised:

$$\underbrace{x_4 + 2x_3 - \tfrac{1}{2}x_1 + \tfrac{5}{2}}_{p} \geq 0$$

Assignment $\sigma : \quad \{x_4 \mapsto 1; x_3 \mapsto -3; x_2 \mapsto -\tfrac{1}{3}; x_1 \mapsto 4\}$

$$p\sigma = 1 + 2 * (-3) - \frac{1}{2} * 4 = -8$$

σ is a solution of $p \geq 0$ if $p\sigma \geq 0$ is true.

Conflict Resolution

A state $\mathbb{S} = (S, \sigma)$:

S - a system of linear constraints

σ - an assignment

The main idea behind CRA

1. Start with an initial state (S,σ)
2. Repeatedly transform $\mathbb{S}$ applying the transformation rules:
   - Conflict Resolution rule (CR)
   - Assignment Refinement rule (AR)
3. Until either σ is a solution or a contradiction is derived.

Conflict Resolution

Example (iii)

$$
\begin{array}{rcrcrcrcrcl}
x_4 & - & 2x_3 & & & + & x_1 & + & 5 & \geq & 0 \quad (1)\\
x_4 & + & 2x_3 & + & x_2 & & & + & 3 & \geq & 0 \quad (2)\\
-x_4 & - & x_3 & - & 3x_2 & - & 3x_1 & + & 1 & \geq & 0 \quad (3)\\
-x_4 & + & 2x_3 & + & 2x_2 & + & x_1 & + & 6 & \geq & 0 \quad (4)\\
 & & x_3 & & & + & 3x_1 & - & 1 & \geq & 0 \quad (5)\\
 & & -x_3 & + & x_2 & - & 2x_1 & + & 5 & \geq & 0 \quad (6)
\end{array}
$$

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 0; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{lrcl}
(1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq - x_3 - 3x_2 - 3x_1 + 1 & (3) \\
(2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4)
\end{array}$$

Level 3

$$\begin{array}{lrcl}
(5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6)
\end{array}$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 0; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \le x_4 \qquad x_4 \le -x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$

$$(2) \quad -2x_3 - x_2 - 3 \le x_4 \qquad x_4 \le 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

Level 3

$$(5) \quad -3x_1 + 1 \le x_3 \qquad x_3 \le x_2 - 2x_1 + 5 \quad (6)$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 0; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \le x_4 \qquad x_4 \le -x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$
$$(2) \quad -2x_3 - x_2 - 3 \le x_4 \qquad x_4 \le 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

Level 3

$$(5) \quad -3x_1 + 1 \le x_3 \qquad x_3 \le x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5]$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{llll} (1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 & (3) \\ (2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \end{array}$$

Level 3

$$\begin{array}{llll} (5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \end{array}$$

$$x_3 \in [1;5] \; x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \leq x_4 \qquad x_4 \leq - x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$

$$(2) \quad -2x_3 - x_2 - 3 \leq x_4 \qquad x_4 \leq 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

$$x_4 \in [3; -3]$$

Level 3

$$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5] \; x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$
\begin{array}{lrcl}
(1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 & (3) \\
(2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \\
 & & x_4 \in [3; -3] \quad \mathit{Conflict}! &
\end{array}
$$

Level 3

$$
\begin{array}{lrcl}
(5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \\
 & & x_3 \in [1; 5] \; x_3 = 4 & (AR)
\end{array}
$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$

Level 4

$$
\begin{array}{llll}
(1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 & (3) \\
(2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \\
 & & x_4 \in [3; -3] \quad \mathit{Conflict!} &
\end{array}
$$

Level 3

$$
\begin{array}{llll}
(5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \\
 & & x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 & (CR) \\
 & & x_3 \in [1; 5] \; x_3 = 4 & (AR)
\end{array}
$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{llcl} (1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 & (3) \\ (2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \end{array}$$

Level 3

$$\begin{array}{llcl} (5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \\ & & x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 & (CR) \\ & & x_3 \in [1; 2] & \end{array}$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 1; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{lrl}
(1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 \quad (3) \\
(2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 \quad (4)
\end{array}$$

Level 3

$$\begin{array}{lrl}
(5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 \quad (6) \\
 & & x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 \quad (CR) \\
 & & x_3 \in [1; 2] \; x_3 = 1 \quad (AR)
\end{array}$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 1; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{llll} (1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq - x_3 - 3x_2 - 3x_1 + 1 & (3) \\ (2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \\ & & x_4 \in [-3; 0] & \end{array}$$

Level 3

$$\begin{array}{llll} (5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \\ & & x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 & (CR) \\ & & x_3 \in [1; 2] \ x_3 = 1 & (AR) \end{array}$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 1; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \leq x_4 \qquad x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$

$$(2) \quad -2x_3 - x_2 - 3 \leq x_4 \qquad x_4 \leq 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

$$x_4 \in [-3; 0] \; x_4 = 0$$

Level 3

$$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 \quad (CR)$$

$$x_3 \in [1; 2] \; x_3 = 1 \quad (AR)$$

Level 2 is empty

Level 1 is empty

Conflict Resolution (Example)

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 1; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$\begin{array}{llll} (1) & 2x_3 - x_1 - 5 \leq x_4 & x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 & (3) \\ (2) & -2x_3 - x_2 - 3 \leq x_4 & x_4 \leq 2x_3 + 2x_2 + x_1 + 6 & (4) \\ & & x_4 \in [-3; 0] \;\; x_4 = 0 & \end{array}$$

Level 3

$$\begin{array}{llll} (5) & -3x_1 + 1 \leq x_3 & x_3 \leq x_2 - 2x_1 + 5 & (6) \\ & & x_3 \leq -x_2 - \frac{2}{3}x_1 + 2 & (CR) \\ & & x_3 \in [1; 2] \;\; x_3 = 1 & (AR) \end{array}$$

Level 2 is empty

Level 1 is empty

Satisfiable! $\sigma : \{x_4 \mapsto 0; x_3 \mapsto 1, x_2 \mapsto 0; x_1 \mapsto 0\}$

Transformation Rules

Assignment Refinement rule (AR)

(at the level k) is the following rule:

$$(S, \sigma) \Rightarrow (S, \sigma^v_{x_k}),$$

where

1. σ satisfies all constraints in S of the levels $0, \ldots, k-1$.
2. σ violates at least one constraint in S of the level k.
3. $\sigma^v_{x_k}$ satisfies all constraints in S of the level k.

Transformation Rules

k-conflict

$\mathbb{S} = (S, \sigma)$ - a state,

k - a positive integer.

$\mathbb{S}$ *contains a* k-conflict $(x_k + p \geq 0, -x_k + q \geq 0)$ if

1. $x_k + p \geq 0 \in S$ and $-x_k + q \geq 0 \in S$
2. $p\sigma + q\sigma < 0$.

Conflict Resolution rule (CR)

(at the level k) is the following rule:

$$(S, \sigma) \Rightarrow (S \cup \{p + q \geq 0\}, \sigma),$$

where (S, σ) contains a k-conflict $(x_k + p \geq 0, -x_k + q \geq 0)$.

Conflict Resolution

Conflict Resolution Algorithm (CRA) is correct and terminating.

Theorem

Given a set of constraints S,

- If CRA returns 'unsatisfiable', then S is unsatisfiable
- If CRA outputs an assignment σ, then σ is a solution of S

Properties of CRA

Redundancy

A CR-inference at a level k is redundant w.r.t. a state (S,σ):

If the conclusion of this inference is a consequence of constraints in $S_{<k}$.

Key Properties

- Every CR-inference performed by CRA is non-redundant;
- In particular, the same constraint will never be added twice to S;
- The CRA can be easily made incremental (important for SMT);
- Can easily generate explanations for unsatisfiability;
- See the next slide...

Properties of CRA

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \le x_4 \qquad x_4 \le -x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$

$$(2) \quad -2x_3 - x_2 - 3 \le x_4 \qquad x_4 \le 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

$$x_4 \in [3; -3] \quad \mathit{Conflict!}$$

Level 3

$$(5) \quad -3x_1 + 1 \le x_3 \qquad x_3 \le x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5] \; x_3 = 4 \quad (AR)$$

Level 2 is empty

Level 1 is empty

Properties of CRA

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

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Properties of CRA

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$$x_3 \in [1; 5] \; x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Properties of CRA

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$$x_3 \in [1; 5] \;\; x_3 = 4 \qquad (AR)$$

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Properties of CRA

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

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$$x_4 \in [3; -3]$$

Level 3

$$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5] \ x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Properties of CRA

$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$

$(1), (4) \Rightarrow \quad x_2 + x_1 \geq -\frac{11}{2}$

Level 4

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$x_4 \in [3; -3]$

Level 3

$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$

$x_3 \in [1; 5] \; x_3 = 4 \qquad (AR)$

Level 2 is empty

Level 1 is empty

Properties of CRA

$$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$$

$$(1), (4) \Rightarrow \quad x_2 + x_1 \geq -\frac{11}{2}$$

Level 4

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$$x_4 \in [3; -3]$$

Level 3

$$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5] \ x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Properties of CRA

$x_4 \succ x_3 \succ x_2 \succ x_1; \quad \sigma : \{x_4 \mapsto 0; x_3 \mapsto 4; x_2 \mapsto 0; x_1 \mapsto 0\}$

$(1),(4) \Rightarrow \quad x_2 + x_1 \geq -\frac{11}{2} \qquad (5),(6) \Rightarrow \quad x_2 + x_1 \geq -4$

Level 4

$$(1) \quad 2x_3 - x_1 - 5 \leq x_4 \qquad x_4 \leq -x_3 - 3x_2 - 3x_1 + 1 \quad (3)$$

$$(2) \quad -2x_3 - x_2 - 3 \leq x_4 \qquad x_4 \leq 2x_3 + 2x_2 + x_1 + 6 \quad (4)$$

$$x_4 \in [3; -3]$$

Level 3

$$(5) \quad -3x_1 + 1 \leq x_3 \qquad x_3 \leq x_2 - 2x_1 + 5 \quad (6)$$

$$x_3 \in [1; 5] \;\; x_3 = 4 \qquad (AR)$$

Level 2 is empty

Level 1 is empty

Outline

Implementation

Linear constraints of the form
$q \diamond 0$, where $\diamond \in \{\geq, >, =, \neq\}$

Key parameters fore fine-tuning the CRA algorithm:

1. strategies for selecting conflicts,
2. strategies for selecting values in the assignment refinement rule,
3. order on variables.

Outline

Random Problems

Linux laptop, CPU 2.8GHz and memory 4Gb. Two sets of benchmarks:

1. Randomly generated
2. Extracted from real-life SMT problems
   using our Hard Reality tool (HRT)

4000 problems vars 3-12 (unsat/sat)				
	CRA	CVC3	FM	Ch
timeout (20s)	0/0	11/9	790/329	149/10
av. time (s)	0/0	0/0	0.4/0.1	0.6/0.1

400 problems vars 13-22 (unsat/sat)				
	CRA	CVC3	FM	Ch
timeout (20s)	5/2	21/33	183/144	155/65
av. time (s)	0.2/0.3	0/0	0.1/0.5	1.9/0.6

Table: Randomly Generated Problems

Hard Reality Problems

304 problems (unsat)				
	CRA	CVC3	FM	Ch
timeout (60s)	1	4	44	42
av. time	0.2	0.13	0.1	0.12

Table: Hard Reality Problems

400 problems vars 13-22 (unsat/sat)				
	faster	same	av. time	timeout (20s)
Barcelogic	28/29	146/167	0.04/0	0/0
CRA	23/7	146/167	0.2/0.3	5/2
400 problems vars 23-32 (unsat/sat)				
Barcelogic	110/67	31/88	0.25/1.0	0/0
CRA	63/41	31/88	0.7/1.6	60/37

Table: CRA vs Barcelogic

Outline

Conclusions

- New algorithm for solving systems of linear constraints CRA;
- Works good for both satisfiable and unsatisfiable problems;
- Is blocking redundant inferences;
- Is orders of magnitude better than the Fourier-Motzkin method and Chernikov algorithm;
- Shows good potential when compared to the Simplex algorithm;
- Can be easily made incremental and easily generate explanations for unsatisfiability (important for SMT).

Future Work

- Optimised implementation;
- Integration into SMT solving;
- Solving linear systems over the integers.

Thank you!

Any questions?