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The bisection width and the isoperimetric number of arrays☆

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Abstract

We prove that the *bisection width*, $bw(A^d)$, of a d-dimensional array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$, is given by $bw(A^d) = \sum_{i=e}^{d} K_i$ where e is the largest index for which k_e is even (if it exists, $e = 1$ otherwise) and $K_i = k_{i-1}k_{i-2}\cdots k_1$. We also show that the *edge-isoperimetric number* $i(A^d)$ is given by $i(A^d) = 1/\lfloor k_d/2 \rfloor$. Furthermore, a bisection and an isoperimetric set are constructed.

Keywords: Product graph; Array; Isoperimetric number; Bisection width; Extremal set; Hamming graph

1. Introduction

In this paper, we provide exact formulae for the *bisection width* and the *isoperimetric number* of d-dimensional arrays and specify the corresponding bisection and isoperimetric sets. We shall first give the necessary definitions and terminology.

A d-dimensional array A^d is a graph with $k_1 \times k_2 \times \cdots \times k_d$ vertices, $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$, each having a unique label $l = \langle l_1, l_2, \ldots, l_d \rangle$ where $0 \leqslant l_i \leqslant k_i - 1$. There is an edge between two vertices if their labels differ in exactly one dimension *and* the difference in that dimension is exactly one. A d-dimensional array can also be characterized as

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the Cartesian product of d *path graphs* of different sizes, i.e. $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ where P_k is a path graph (chain) on k vertices. The *Cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (u, v) and (u', v') are adjacent if and only if u is adjacent to u' in G and $v = v'$, or v is adjacent to v' in H and $u = u'$. The constituent graphs G and H are called *factors*.

Given a graph G and a subset X of its vertices, let ∂X denote the *edge-boundary* of X: the set of edges which connect vertices in X with vertices in $V(G) \setminus X$. The *edge-isoperimetric number*, or simply the *isoperimetric number*, of G is defined as

$$i(G) = \min_{1 \leqslant |X| \leqslant |V(G)|/2} \frac{|\partial X|}{|X|}. \tag{1}$$

That is, the set of vertices of G is partitioned into two nonempty sets and the ratio of the number of edges between the two parts and the number of vertices in the smaller one is minimized over all such partitions. As examples of isoperimetric numbers:

- $i(K_k) = \lceil k/2 \rceil$ for the complete graph K_k with k vertices,
- $i(P_k) = 1/\lfloor k/2 \rfloor$ for the path P_k with k vertices,
- $i(C_k) = 2/\lfloor k/2 \rfloor$ for the cycle C_k with k vertices.

A subset X of vertices which achieves the minimum ratio in (1) is called an *isoperimetric set*. We refer the reader to Mohar [7] or Chung [5] for a discussion of basic results and various interesting properties of $i(G)$, and to Bezrukov [4] for a comprehensive survey of this and related problems.

The isoperimetric number is closely related to the notion of *bisection width* $bw(G)$ of a graph G, which is the minimum number of edges that must be removed from the graph in order to split $V(G)$ into two *equal sized* (within one if the number of vertices in G is odd) subsets. That is,

$$bw(G) = \min_{|X| = \lfloor |V(G)|/2 \rfloor} |\partial X|,$$

where $X \subset V(G)$. As examples, $bw(P_k) = 1$ and $bw(C_k) = 2$, and

$$bw(K_k) = \begin{cases} \dfrac{k^2}{4} & \text{if } k \text{ is even,} \\ \dfrac{k^2 - 1}{4} & \text{if } k \text{ is odd.} \end{cases}$$

In this paper, we prove the following general result for the bisection width of arrays.

Theorem 1. *Given an array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ with $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$, let e be the largest index for which k_e is even. Set $e = 1$ if each factor has odd size. Then,*

$$bw(A^d) = \sum_{i=e}^{d} K_i, \tag{2}$$

where $K_i = k_{i-1}k_{i-2} \cdots k_1$ for $2 \leqslant i \leqslant d$ with $K_1 = 1$.

We also prove the following formula for the isoperimetric number of arbitrary arrays.

Theorem 2. *Given an array* $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ *with* $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$,

$$i(A^d) = \frac{1}{\lfloor k_d/2 \rfloor}. \tag{3}$$

Furthermore, we specify the subsets achieving the values on the right-hand sides of the formulae given in these two theorems.

2. Preliminaries

The formulae in (2) and (3) generalize what is currently known for only special cases of arrays. We summarize these results next by starting with the most restricted and moving towards the more general forms of arrays. First of all, the above results hold trivially for the one-dimensional array, i.e. the path graph. When all k_i's in $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ are equal, the resulting graph is called a d-dimensional k-ary array and denoted by A^d_k. It was shown by Leighton [6] that $bw(A^d_k) = k^{d-1}$ when k is even, and by Nakano [8] that $bw(A^d_k) = (k^d - 1)/(k-1)$ when k is odd. Azizoğlu and Eğecioğlu showed in [1] that $i(A^d_k) = i(P_k)$. Furthermore, given $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ with $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$, they proved in [2] that $i(A^d) = i(P_{k_d})$ and $bw(A^d) = k_1 k_2 \cdots k_{d-1}$ provided that k_d is even.

These results are obtained by using proof methods which usually depend on the parity of the size of the largest factor in the given array. Even though the techniques applied are similar in certain ways, none of the known techniques generalize to arbitrary arrays. The proof argument in these restricted cases involves embedding a special type of graph (such as a complete graph or a Hamming graph) of the same size into the array and showing that the boundary of a set of vertices of particular size cannot be smaller than a certain value, using the extremal sets minimizing the boundary in the embedded graph. An *extremal set* of a graph for a given number m is a collection of m vertices with minimum number of boundary edges (or maximum number of spanned edges) among all m-vertex subsets of the graph. Unfortunately, these techniques cannot be extended to prove the general case. The reason for this is that, contrary to the restricted cases above, extremal sets minimizing the boundary in the embedded Hamming graphs do not correspond to isoperimetric sets of arrays in general.

In this paper, we aim to unify the proofs of the special cases and fill in the gaps currently existent in the literature. To this end, we make use of a new construct called *extremal sets minimizing dimension-normalized boundary* in Hamming graphs. These extremal sets also form a nested family, but the edges are assigned weights as a function of the dimension in which they live. The correct weights with respect to which the extremal sets turn out to be nested, make the proof possible in the general case. To make things precise, some definitions and terminology are in order.

A d-dimensional Hamming graph H^d is the Cartesian product of d *complete graphs* of various sizes, i.e. $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ where K_k is a complete graph on k vertices. A Hamming graph H^d is similar to an array in that each vertex in H^d also has a label $l = \langle l_1, l_2, \ldots, l_d \rangle$ where $0 \leqslant l_i \leqslant k_i - 1$. There is an edge between two vertices if their labels differ in exactly one component (unlike arrays, however, the difference in that component does not have to be one). Next we formally define the *dimension-normalized boundary* in Hamming graphs.

Definition 1. Given a Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ and a subset X of its vertices, *the dimension-normalized boundary $B(X)$ of X* is defined by

$$B(X) = \frac{|\partial_1 X|}{c_1} + \frac{|\partial_2 X|}{c_2} + \cdots + \frac{|\partial_d X|}{c_d}, \tag{4}$$

where $\partial_i X$ is the set of boundary edges along dimension i and

$$c_i = \begin{cases} k_i^2 & \text{if } k_i \text{ is even,} \\ k_i^2 - 1 & \text{if } k_i \text{ is odd} \end{cases}$$

for $1 \leqslant i \leqslant d$.

Azizoğlu and Eğecioğlu [3] proved that the set of first m vertices in reverse-lexicographic order minimizes $B(X)$ in (4). The definition of the *reverse-lexicographic order* is as follows: Assuming $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$ in the given Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$, vertex $x = \langle x_1, \ldots, x_d \rangle$ precedes vertex $y = \langle y_1, \ldots, y_d \rangle$ in reverse-lexicographic order iff there exists an index i such that $x_d = y_d$, $x_{d-1} = y_{d-1}, \ldots, x_{i+1} = y_{i+1}$ and $x_i < y_i$ holds. In other words, we move in the direction of the *next smallest* dimension starting at the vertex labeled $\langle 0, 0, \ldots, 0 \rangle$. Specifically, Azizoğlu and Eğecioğlu proved the following theorem in [3].

Theorem 3. *Given a d-dimensional Hamming graph H^d, let X be any m-vertex subset of $V(H^d)$ and $\bar{X}$ be the set of first m vertices of H^d in reverse-lexicographic order. Then, $B(\bar{X}) \leqslant B(X)$.*

We refer the reader to [3] for proof of this theorem and a discussion of extremal sets minimizing dimension-normalized boundary, as well as their relationship with other types of extremal sets in Hamming graphs.

The proof techniques used to prove Theorems 1 and 2 are similar: To get the lower bounds, a Hamming graph of the same size is embedded into the array. Then, by using Theorem 3, we argue that the dimension-normalized boundary of the corresponding sets in the Hamming graph cannot be smaller than a certain value. The upper bounds are proved by specifying the subsets which achieve these lower bounds.

In the special cases mentioned above, once the isoperimetric number of the array is known then it is trivial to obtain a lower bound for the bisection width as well, using the fact that

$$\frac{bw(G)}{\lfloor \frac{|V(G)|}{2} \rfloor} \geqslant i(G)$$

for a graph G. However, in the general case, this is not always tight. For instance, $bw(P_9 \times P_7 \times P_4) = 7 \times 4 + 4 + 1 = 33$ and $i(P_9 \times P_7 \times P_4) = \frac{1}{4}$ according to Theorems 1 and 2. Since $9 \times 7 \times \frac{4}{2} = 126$, $\frac{1}{4}$ is not a tight lower bound to the ratio $\frac{33}{126}$. Fortunately, however, the lower bound for the bisection width in the general case can be obtained by using extremal sets minimizing dimension-normalized boundary of Hamming graphs.

The outline of the rest of this paper is as follows. First, we describe the standard embedding of a Hamming graph into the corresponding array. Following this, in Section 4, we make use of Theorem 3 to prove Theorem 1 on the bisection width of arrays, and describe the structure of the corresponding bisection. The proof of Theorem 2 on the edge-isoperimetric number of general arrays and the characterization of the corresponding isoperimetric sets are given in Section 5.

3. Embedding a Hamming graph into an array

Given an array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$, we embed the Hamming graph $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ into A^d in a straightforward manner: The vertex $\langle v_1, \ldots, v_d \rangle$ of H^d is identified with the corresponding vertex $\langle v_1, \ldots, v_d \rangle$ in A^d. The edge $(\langle v_1, \ldots, v_i, \ldots, v_d \rangle, \langle v_1, \ldots, v_i + r, \ldots, v_d \rangle)$ in dimension i of H^d is embedded into A^d through the path

$$\langle v_1, \ldots, v_i, \ldots, v_d \rangle \to \langle v_1, \ldots, v_i + 1, \ldots, v_d \rangle \to \cdots \to \langle v_1, \ldots, v_i + r, \ldots, v_d \rangle.$$

We let c_i be the *congestion of the embedding along dimension i*. That is, c_i is the maximum number of edges of the Hamming graph routed through any edge in the ith dimension of A^d. It is easy to see that

$$c_i = \begin{cases} \dfrac{k_i^2}{4} & \text{if } k_i \text{ is even,} \\[2ex] \dfrac{k_i^2 - 1}{4} & \text{if } k_i \text{ is odd.} \end{cases}$$

4. The bisection width of arrays

For a given array $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$, the proof of Theorem 1 is composed of two parts:

(i) proving the lower bound

$$bw(A^d) \geqslant K_e + K_{e+1} + \cdots + K_d, \tag{5}$$

where $K_i = k_{i-1}k_{i-2} \cdots k_1$ for $2 \leqslant i \leqslant d$, with $K_1 = 1$, and e is the largest index for which k_e is even ($e = 1$ if all k_i are odd), and

(ii) describing a bisection that actually achieves this lower bound.

4.1. The lower bound

We use the embedding of $H^d = K_{k_1} \times K_{k_2} \times \cdots \times K_{k_d}$ into $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ described in the previous section. Let X be any subset of $V(A^d)$ with $|X| = \lfloor V(A^d)/2 \rfloor$. Using the fact that $|\partial X| = |\partial_1 X| + |\partial_2 X| + \cdots + |\partial_d X|$ where $\partial_i X$ is the subset of boundary edges of X in dimension i, together with $|\partial_i X| \geqslant |\partial_i X'|/c_i$ where X' is the set of vertices of H^d corresponding to X via the embedding, we find that

$$|\partial X| \geqslant \frac{|\partial_1 X'|}{c_1} + \frac{|\partial_2 X'|}{c_2} + \cdots + \frac{|\partial_d X'|}{c_d}.$$

Let $\bar{X}$ be the set of first $\lfloor V(A^d)/2 \rfloor$ vertices of H^d in reverse-lexicographic order. Using Theorem 3, we have

$$\frac{|\partial_1 X'|}{c_1} + \cdots + \frac{|\partial_d X'|}{c_d} \geqslant \frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d}.$$

Now to prove the lower bound (5), it suffices to show

$$\frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d} = K_e + K_{e+1} + \cdots + K_d. \tag{6}$$

When k_d is even $\bar{X}$ contains all vertices labeled $\langle l_1, l_2, \ldots, l_{d-1}, p\rangle$ where $0 \leqslant l_i \leqslant k_i - 1$ for $1 \leqslant i \leqslant d-1$ and $0 \leqslant p \leqslant k_d/2 - 1$. When k_d is odd, however, $\bar{X}$ includes all vertices with labels $\langle l_1, l_2, \ldots, l_{d-1}, p\rangle$ where $0 \leqslant l_i \leqslant k_i - 1$ for $1 \leqslant i \leqslant d-1$ and $0 \leqslant p \leqslant (k_d - 1)/2 - 1$ (there are $\frac{1}{2}(k_d - 1)k_{d-1}k_{d-2}\cdots k_1$ of these) *plus* the first $|\bar{X}| - \frac{1}{2}(k_d - 1)k_1 \cdots k_{d-2}k_{d-1}$ vertices in reverse lexicographic order (call this $\bar{x}$) of the subgraph H^{d-1} containing all vertices with labels $\langle l_1, l_2, \ldots, l_{d-1}, (k_d - 1)/2\rangle$ where $0 \leqslant l_i \leqslant k_i - 1$ for $1 \leqslant i \leqslant d-1$. Note that $|\bar{x}| = \lfloor |V(H^{d-1})|/2 \rfloor$. Thus, the foregoing argument applies to $\bar{x}$ and H^{d-1} verbatim as it does to $\bar{X}$ and H^d.

As a matter of fact, the recursive structure also suggests an easy way of calculating $|\partial_i \bar{X}|/c_i$. When k_d is even, $|\partial_d \bar{X}| = (k_d/2)k_{d-1}k_{d-2}\cdots k_1(k_d/2)$ and $|\partial_d \bar{X}|/c_d = k_{d-1}k_{d-2}\cdots k_1$. Note that in this case $|\partial_i \bar{X}| = 0$, $0 \leqslant i \leqslant d-1$. If k_d is odd then

$$|\partial_d \bar{X}| = \frac{k_d + 1}{2}|\bar{x}|\frac{k_d - 1}{2} + \frac{k_d - 1}{2}(k_1 \cdots k_{d-2}k_{d-1} - |\bar{x}|)\frac{k_d + 1}{2}$$

and $|\partial_d \bar{X}|/c_d = K_d$. $|\partial_{d-1} \bar{X}|$ can be computed similarly using $\bar{x}$ and H^{d-1} and so on. This ultimately gives the desired lower bound.

4.2. A bisection in A^d

Let $X = \bar{X}$ be the first $\lfloor |V(A^d)|/2 \rfloor$ vertices of A^d in reverse-lexicographic order. Then X and its complement form a bisection of A^d. The number of edges from X to its complement is $|\partial X| = |\partial_1 X| + |\partial_2 X| + \cdots + |\partial_d X|$. We can easily compute the individual $|\partial_i X|$ starting with $|\partial_d X|$ and using the recursive structure of X. That is, $|\partial_d X| = k_{d-1}k_{d-2}\cdots k_1$ and $|\partial_{d-1} X| = k_{d-2}k_{d-3}\cdots k_1$ and so on, down to $|\partial_e X| = k_{e-1}k_{e-2}\cdots k_1$ where k_e is the size of the first (i.e. largest, since we are processing from the dth dimension down) even factor encountered. If each k_i is odd, then this count goes all the way down to $|\partial_1 X| = 1$. This proves Theorem 1.

5. The isoperimetric number of arrays

The structure of the proof of Theorem 2 is similar to that of Theorem 1, namely the lower bound is established first, and then an isoperimetric set achieving this lower bound is constructed. As before, we let $A^d = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_d}$ be a given array where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_d$.

5.1. The lower bound

In order to prove the lower bound, we use the embedding of H^d into A^d as before. Similar to the bisection argument, let X be a set of vertices in A^d where $|X| \leqslant |V(A^d)|/2$ and let X' be the corresponding set of vertices in H^d. Then, by the embedding, we have

$$|\partial X| = |\partial_1 X| + \cdots + |\partial_d X| \geqslant \frac{|\partial_1 X'|}{c_1} + \cdots + \frac{|\partial_d X'|}{c_d}$$

and by Theorem 3, we have

$$\frac{|\partial_1 X'|}{c_1} + \cdots + \frac{|\partial_d X'|}{c_d} \geqslant \frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d},$$

where $\bar{X}$ is the set of first $|X|$ vertices in reverse-lexicographic order in H^d. Thus to prove the lower bound, it suffices to prove

$$\frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d} \geqslant \begin{cases} \dfrac{2}{k_d}|\bar{X}| & \text{if } k_d \text{ is even,} \\ \dfrac{2}{k_d - 1}|\bar{X}| & \text{if } k_d \text{ is odd} \end{cases} \tag{7}$$

for all $1 \leqslant |\bar{X}| \leqslant |V(A^d)|/2$.

We use induction on d to prove (7). The base case $d=1$ with $A^1 = P_{k_1}$ is true since $|\partial_1 \bar{X}| = |\bar{X}|(k_1 - |\bar{X}|)$ and when $k_1 = |V(A^1)|$ is even,

$$\frac{|\bar{X}|(k_1 - |\bar{X}|)}{k_1^2/4} \geqslant \frac{2}{k_1}|\bar{X}|$$

holds whenever $|\bar{X}| \leqslant k_1/2$ as desired. Likewise, if k_1 is odd then we have

$$\frac{|\bar{X}|(k_1 - |\bar{X}|)}{(k_1^2 - 1)/4} \geqslant \frac{2}{k_1 - 1}|\bar{X}|,$$

which is true whenever $|\bar{X}| \leqslant (k_1 - 1)/2$, as desired.

Now, we assume that the claim holds for d, and we proceed to prove

$$\frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d} + \frac{|\partial_{d+1} \bar{X}|}{c_{d+1}} \geqslant \begin{cases} \dfrac{2}{k_{d+1}}|\bar{X}| & \text{if } k_{d+1} \text{ is even,} \\ \dfrac{2}{k_{d+1} - 1}|\bar{X}| & \text{if } k_{d+1} \text{ is odd,} \end{cases} \tag{8}$$

whenever $k_d \leqslant k_{d+1}$ and $1 \leqslant |\bar{X}| \leqslant k_1 \cdots k_d k_{d+1}/2$. Note that in this range $|\bar{X}|$ can be written as $|\bar{X}| = a_{d+1}k_1k_2\cdots k_d + \cdots + a_2k_1 + a_1$ for $0 \leqslant a_i \leqslant k_i - 1$. We have the following four cases to consider in order to apply the induction hypothesis.

Case 1: k_{d+1} is even and $1 \leqslant |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d/2$,

Case 2: k_{d+1} is even and $k_1 \cdots k_d/2 < |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d - 1$,

Case 3: k_{d+1} is odd and $1 \leqslant |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d/2$, *Case* 4: k_{d+1} is odd and $k_1 \cdots k_d/2 < |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d - 1$.

We remark that it is possible to have $|\bar{X}| - a_{d+1}k_1 \cdots k_d = 0$, which is *not* covered by these cases. But then $|\partial_i \bar{X}| = 0$ for $1 \leqslant i \leqslant d-1$ and $|\partial_d \bar{X}| = a_{d+1}(k_{d+1} - a_{d+1})k_1 \cdots k_d$. Since $|\bar{X}| = a_{d+1}k_1 \cdots k_d$, thus inequality (8) becomes

$$\frac{a_{d+1}(k_{d+1} - a_{d+1})k_1 \cdots k_d}{c_d} \geqslant \frac{2}{k_{d+1}} a_{d+1}k_1 \cdots k_d,$$

which holds since $a_{d+1} \leqslant \lfloor k_{d+1}/2 \rfloor$ because of the assumption $|X| \leqslant |V(A^d)|/2$.

We now handle each one of the four cases above in turn.

Case 1 (*k_{d+1} is even and* $1 \leqslant |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d/2$): We are required to prove

$$\frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d} + \frac{|\partial_{d+1} \bar{X}|}{c_{d+1}} \geqslant \frac{2}{k_{d+1}} |\bar{X}|,$$

whenever $k_1 \leqslant \cdots \leqslant k_d \leqslant k_{d+1}$ and $1 \leqslant |\bar{X}| \leqslant k_1 \cdots k_{d+1}/2$. By the induction hypothesis, we have

$$\frac{|\partial_1 \bar{X}|}{c_1} + \cdots + \frac{|\partial_d \bar{X}|}{c_d} \geqslant \frac{1}{\lfloor k_d/2 \rfloor} (|\bar{X}| - a_{d+1}k_1 \cdots k_d).$$

Thus, it suffices to prove

$$\frac{1}{\lfloor k_d/2 \rfloor} (|\bar{X}| - a_{d+1}k_1 \cdots k_d) + \frac{|\partial_{d+1} \bar{X}|}{k_{d+1}^2/4} \geqslant \frac{2}{k_{d+1}} |\bar{X}|. \tag{9}$$

Multiplying both sides by $k_{d+1}^2/4$ in (9) and using

$$\frac{k_{d+1}^2}{4} \frac{1}{\lfloor k_d/2 \rfloor} \geqslant \frac{k_{d+1}}{2}$$

we may equivalently prove

$$\frac{k_{d+1}}{2} (|\bar{X}| - a_{d+1}k_1 \cdots k_d) + |\partial_{d+1} \bar{X}| \geqslant \frac{k_{d+1}}{2} |\bar{X}|. \tag{10}$$

Observe that $|\partial_{d+1} \bar{X}| = a_{d+1}(a_{d+1} + 1)k_1 \cdots k_d + (k_{d+1} - 2a_{d+1} - 1)|\bar{X}|$ (see Azizoğlu and Eğecioğlu [1]). Thus, after substituting this value in (10) and factorizing, it is sufficient to prove

$$a_{d+1}k_1 \cdots k_d \geqslant (2|\bar{X}| - a_{d+1}k_1 \cdots k_d)\ (2a_{d+1} - k_{d+1} + 1).$$

Since $|\bar{X}| \leqslant k_1 \cdots k_{d+1}/2$, we must have $a_{d+1} \leqslant k_{d+1}/2 - 1$. Therefore, the inequality above holds, since $2|\bar{X}| - a_{d+1}k_1 \cdots k_d \geqslant 0$ and $2a_{d+1} - k_{d+1} + 1 \leqslant -1$.

Case 2 (*k_{d+1} is even and* $k_1 \cdots k_d/2 < |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d - 1$): Applying the induction hypothesis to the complement of the set of vertices in $\bar{X}$ in the only $(d-1)$-dimensional Hamming graph properly intersecting X, we are required to prove

$$\frac{k_{d+1}}{2} (k_1 \cdots k_d - (|\bar{X}| - a_{d+1}k_1 \cdots k_d)) + |\partial_{d+1} \bar{X}| \geqslant \frac{k_{d+1}}{2} |\bar{X}|.$$

Substituting in the value of $|\partial_{d+1} \bar{X}|$ as in the previous case and factorizing, it suffices to prove

$$k_1 \cdots k_d \left(a_{d+1} + \frac{k_{d+1}}{2} \right) (a_{d+1} + 1) \geqslant (2a_{d+1} + 1)|\bar{X}|. \tag{11}$$

Note that, for this case, we must have $a_{d+1} \leqslant k_{d+1}/2-2$ in order for $|\bar{X}| \leqslant k_1 \cdots k_{d+1}/2$. Thus, we shall show

$$k_1 \cdots k_d \left(a_{d+1} + \frac{k_{d+1}}{2} \right) (a_{d+1}+1) \geqslant 2|\bar{X}|(a_{d+1}+1)$$

or equivalently,

$$k_1 \cdots k_d \left(a_{d+1} + \frac{k_{d+1}}{2} \right) \geqslant 2|\bar{X}|, \tag{12}$$

which obviously implies (11). Using $|\bar{X}| = a_{d+1}k_1 \cdots k_d + \cdots + a_1$ and the fact that $|\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d - 1$ (12) holds if

$$\frac{k_{d+1} \cdots k_1}{2} \geqslant a_{d+1}k_1 \cdots k_d + 2k_1 \cdots k_d - 2. \tag{13}$$

Using $a_{d+1} \leqslant k_{d+1}/2 - 2$ and multiplying both sides of (13) by 2, we get that (11) holds if

$$k_{d+1} \cdots k_1 \geqslant k_{d+1} \cdots k_1 - 4k_1 \cdots k_d + 4k_1 \cdots k_d - 4,$$

which is obvious.

Case 3 (k_{d+1} is odd and $1 \leqslant |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d/2$): In this case we are required to prove

$$\frac{k_{d+1}+1}{2}(|\bar{X}| - a_{d+1}k_1 \cdots k_d) + |\partial_{d+1}\bar{X}| \geqslant \frac{k_{d+1}+1}{2}|\bar{X}|.$$

Again, after substituting in the value of $|\partial_{d+1}\bar{X}|$ and factorizing, the above inequality is equivalent to

$$(2|\bar{X}| - a_{d+1}k_1 \cdots k_d)\ (k_{d+1} - 2a_{d+1} - 1) \geqslant 0.$$

It is evident that $2|\bar{X}| - a_{d+1}k_1 \cdots k_d \geqslant 0$. Also note that when k_{d+1} is odd we have $a_{d+1} \leqslant k_{d+1} - 1/2$. Thus, $k_{d+1} - 2a_{d+1} - 1 \geqslant 0$ and the inequality holds.

Case 4 (k_{d+1} is odd and $k_1 \cdots k_d/2 < |\bar{X}| - a_{d+1}k_1 \cdots k_d \leqslant k_1 \cdots k_d - 1$): Finally, we are required to prove

$$\frac{k_{d+1}+1}{2}(k_1 \cdots k_d - (|\bar{X}| - a_{d+1}k_1 \cdots k_d)) + |\partial_{d+1}\bar{X}| \geqslant \frac{k_{d+1}+1}{2}|\bar{X}|.$$

This is equivalent to proving

$$k_1 \cdots k_d \left(a_{d+1} + \frac{k_{d+1}+1}{2} \right) \geqslant 2|\bar{X}|. \tag{14}$$

Using $|\bar{X}| = a_{d+1}k_1 \cdots k_d + \cdots + a_1$ and $k_1 \cdots k_d/2 < |\bar{X}| - a_{d+1}k_1 \cdots k_d$, inequality (14) is valid if

$$k_1 \cdots k_d(k_{d+1}+1) \geqslant 2a_{d+1}k_1 \cdots k_d + 4(k_1 \cdots k_d - 1). \tag{15}$$

Note that a_{d+1} can be at most $k_{d+1} - 3/2$. Thus, inequality (15) above holds since

$$\begin{aligned} k_1 \cdots k_d(k_{d+1}+1) &\geqslant 2\,\frac{k_{d+1}-3}{2}k_1 \cdots k_d + 4k_1 \cdots k_d - 4 \\ &\geqslant 2a_{d+1}k_1 \cdots k_d + 4k_1 \cdots k_d - 4. \end{aligned}$$

5.2. An isoperimetric set in A^d

Let X be the set of first $\lfloor k_d/2 \rfloor k_1 \cdots k_{d-1}$ vertices of A^d in reverse-lexicographic order. We claim that X is an isoperimetric set. To see this, note that $|\partial_i X| = 0$ for $1 \leqslant i \leqslant d-1$ and $|\partial_d X| = k_1 \cdots k_{d-1}$. Thus, we have

$$\frac{|\partial X|}{|X|} = \frac{k_1 \cdots k_{d-1}}{\lfloor k_d/2 \rfloor k_1 \cdots k_{d-1}} = \frac{1}{\lfloor k_d/2 \rfloor}$$

as desired.

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